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On sufficient conditions for some classes of multivalent functions

Pranay Goswami*, Sanjeev Kumar Shukla, and Sanjay Kumar Bansal

Abstract. In this paper, we investigate two distinct classes of multivalent functions and establish sufficient conditions for a multivalent function to belong to these classes. The results presented here extend and unify existing criteria related to the starlikeness and convexity of multivalently analytic functions. By generalizing earlier findings, our work provides a broader framework for analyzing geometric properties and inclusion relationships within subclasses of analytic multivalent functions.

Keywords: Multivalent Functions, Starlike Functions, Convex Functions

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1. Introduction and Definitions

Multivalent functions, a generalization of univalent functions, play a significant role in complex analysis, particularly within geometric function theory. These functions, which are analytic in a domain and possess multiple zeros or poles, are classified into various subclasses based on geometric properties such as starlikeness and convexity. A function is described as starlike if it transforms a domain into a star-shaped region centered around a specific point, usually the origin. In contrast, a function is considered convex if its image forms a convex domain. Classification of

*Corresponding author

multivalent functions into starlike and convex classes has been extensively studied due to their fundamental applications in conformal mapping theory, the derivation of sharp coefficient bounds, growth and distortion estimates, and subordination theory.

The systematic study of starlike and convex multivalent functions dates back to the pioneering work of Goodman [3], who extended the concept of univalence to p-valent functions. Goodman provided analytic criteria for starlikeness and convexity in terms of the behavior of the function's derivative within the unit open disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Subsequent contributions by Uralegaddi and Somanatha [10] refined these conditions, introducing subordination-based characterizations for p-valent starlike and convex functions. These criteria link the geometric properties of the image domains to analytic conditions involving the function and its derivatives. One classical direction in this area involves analyzing starlike and convex multivalent functions. Early and influential work in this domain, such as that by Jack [5], set important precedents for later developments. Further advancements were made by Aouf [1, 2], who focused on multivalent functions characterized by negative coefficients and examined their behavior using differential operators. Other notable contributions include the work of Nunokawa and collaborators [7, 8], who explored multivalent function theory through sharp inequalities and geometric interpretations. The use of integral operators and sufficient conditions for univalence, explored extensively by Owa and his co-authors [9], has opened further avenues for characterizing various subclasses of analytic functions.

This paper builds upon the foundational work of S.K. Bansal et al. (2012) [4], which investigated the starlikeness and convexity properties of multivalent analytic functions.

Definition 1.1. Suppose $\mathcal{A}_r(\eta)$ represents the collection of functions expressed as

$$\xi(\mathfrak{z}) = \mathfrak{z}^p + \sum_{j=r+\eta}^{\infty} a_j \mathfrak{z}^j, \quad (r \in \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1}$$

that are analytic on the open unit disk. A special case where $r = 1$ yields

$$\mathcal{A}_1(\eta) = \mathcal{A}(\eta).$$

Definition 1.2. A function $\xi(z)$ from the set $\mathcal{A}_r(\eta)$ is said to be starlike of order γ in $\mathbb{U}$ if

$$\operatorname{Re}\left\{\frac{\mathfrak{z}\xi'(z)}{\xi(z)}\right\} > \gamma, \quad (\mathfrak{z} \in \mathbb{U},\ 0 \le \gamma < r).$$

Remark 1.3. We denote the class of all functions in **Definition 1.2** by $\mathfrak{S}_r^*(\eta, \gamma)$ in particular, $\mathfrak{S}_1^*(\eta, 0) = \mathfrak{S}^*(\eta, 0)$ and $\mathfrak{S}_1^*(1, 0) =: \mathfrak{S}^*$

Definition 1.4. A function $\xi(\mathfrak{z}) \in \mathcal{A}_r(\eta)$ is called convex of order γ in $\mathbb{U}$, if

$$\operatorname{Re}\left\{1 + \frac{\mathfrak{z}\xi''(\mathfrak{z})}{\xi'(\mathfrak{z})}\right\} > \gamma, \quad (\mathfrak{z} \in \mathbb{U},\ 0 \le \gamma < r).$$

Remark 1.5. Let us denote the class of all functions in **Definition 1.3** by $\mathfrak{C}_r(\eta,\gamma)$ in particular, $\mathfrak{C}_1(\eta,0)=\mathfrak{C}(\eta,0)$.

Definition 1.6. Consider the function $\xi(\mathfrak{z})=\mathfrak{z}^r+\sum_{j=r+\eta}^{\infty}a_j\mathfrak{z}^j$, for any $\tau\in\mathbb{N}\cup 0$ let us denote τ^{th} derivative of $\xi(\mathfrak{z})$ by $\mathcal{D}^\tau\xi(\mathfrak{z})$, and define it as $\mathcal{D}^0\xi(\mathfrak{z})=\xi(\mathfrak{z})$, and for any $\tau>1$

$$\mathcal{D}^\tau\xi(\mathfrak{z})=\frac{r!}{(r-\tau)!}\mathfrak{z}^{r-\tau}+\sum_{j=r+\eta}^{\infty}\frac{j!}{(j-\tau)!}a_j\,\mathfrak{z}^{j-\tau}. \tag{2}$$

Definition 1.7. Let $\mathfrak{S}_r^*(\tau,\eta,\gamma)$ denote the class of functions in $\mathcal{A}_r(\eta)$ such that

$$\operatorname{Re}\left\{\frac{\mathfrak{z}\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^\tau\xi(\mathfrak{z})}\right\}>\gamma,\quad (|\mathfrak{z}|<1,\ 0\le\gamma<r-\tau).$$

Definition 1.8. Let $\mathfrak{C}_r(\tau,\eta,\gamma)$ denote the class of functions $\xi(\mathfrak{z})\in\mathcal{A}_r(\eta)$ such that

$$\operatorname{Re}\left\{1+\frac{\mathfrak{z}\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^\tau\xi(\mathfrak{z})}\right\}>\gamma,\quad (|\mathfrak{z}|<1,\ 0\le\gamma<r-\tau).$$

Remark 1.9. The classes $\mathfrak{S}_r^*(\eta,\gamma)$ and $\mathfrak{C}_r(\eta,\gamma)$ shall become particular cases of our class

$$\mathfrak{S}_r^*(0,\eta,\gamma)=\mathfrak{S}_r^*(\eta,\gamma)\quad\text{and}\quad\mathfrak{C}_r(1,\eta,\gamma)=\mathfrak{C}_r(\eta,\gamma).$$

2. Condition for Starlikeness

To prove our results, we shall employ the following lemma given by Mocanu [6]

Lemma 2.1. *If* $\xi(\mathfrak{z})\in\mathcal{A}(\eta)$ *satisfies condition*

$$|\xi'(\mathfrak{z})-1|<\frac{\eta+1}{\sqrt{(\eta+1)^2+1}},\quad (|\mathfrak{z}|<1,\ \eta\in\mathbb{N}),$$

then $\xi(\mathfrak{z})\in\mathfrak{S}^*(\eta,0)$.

Theorem 2.2. *Let* $\xi(\mathfrak{z})\in\mathcal{A}_r(\eta)$, *and suppose that*

$$u(\mathfrak{z})=\left(\frac{(r-\tau)!\,\mathcal{D}^\tau\xi(\mathfrak{z})}{r!\,\mathfrak{z}}\right)^{\frac{1}{r-\tau-\gamma}}\times\left[\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^\tau\xi(\mathfrak{z})}\mathfrak{z}^{\frac{1-\gamma}{r-\tau-\gamma}}-\gamma\mathfrak{z}^{\frac{1-r+\tau}{r-\tau-\gamma}}\right], \tag{3}$$

satisfies the inequality

$$\frac{|u(\mathfrak{z})-(r-\tau-\gamma)|}{r-\tau-\gamma}<\frac{\eta+1}{\sqrt{(\eta+1)^2+1}},\quad \mathfrak{z}\in\mathbb{U},\ \eta\in\mathbb{N},\ 0\le\gamma<r-\tau.$$

Then $\xi(\mathfrak{z})\in\mathfrak{S}_r^*(\tau,\eta,\gamma)$.

Proof. Let us consider

$$\xi(\mathfrak{z})=\mathfrak{z}^p+\sum_{j=r+\eta}^{\infty}a_j\mathfrak{z}^j.$$

Then using the **Definition 1.4** we have

$$\mathcal{D}^{\tau}\xi(\mathfrak{z}) = \frac{r!}{(r-\tau)!}\,\mathfrak{z}^{r-\tau} + \sum_{j=r+\eta}^{\infty} \frac{j!}{(j-\tau)!} a_j\,\mathfrak{z}^{j-\tau}.$$

This implies

$$\frac{(r-\tau)!\,\mathcal{D}^{\tau}\xi(\mathfrak{z})}{r!\,\mathfrak{z}^{\gamma}} = \mathfrak{z}^{r-\tau-\gamma} + \frac{(r-\tau)!}{r!} \sum_{j=r+\eta}^{\infty} \frac{j!}{(j-\tau)!} a_j\,\mathfrak{z}^{j-\tau-\gamma}.$$

After appropriate manipulation, we shall finally obtain the following

$$\left[\frac{(r-\tau)!\,\mathcal{D}^{\tau}\xi(\mathfrak{z})}{r!}\frac{1}{\mathfrak{z}^{\gamma}}\right]^{\frac{1}{r-\tau-\gamma}} = \mathfrak{z}\left[1 + \frac{(r-\tau)!}{r!} \sum_{j=r+\eta}^{\infty} \frac{j\,a_j}{(j-\tau)}\,\mathfrak{z}^{\,j-r}\right]^{\frac{1}{r-\tau-\gamma}}$$

Expanding in a series, we get

$$= \mathfrak{z}\Big[1 + \frac{(r-\tau)!}{r!(\,r-\tau-\gamma)} \sum_{j=r+\eta}^{\infty} \frac{j\,a_j}{(j-\tau)}\,\mathfrak{z}^{\,j-r} + \dots\Big].$$

Let us define

$$\mathfrak{F}(\mathfrak{z}) = \left[\frac{(r-\tau)!\,\mathcal{D}^{q}\xi(\mathfrak{z})}{r!}\frac{1}{\mathfrak{z}^{\gamma}}\right]^{\frac{1}{r-\tau-\gamma}} = \mathfrak{z} + \frac{(r-\tau)!}{r!\,\big(r-\tau-\gamma\big)} \sum_{j=r+\eta}^{\infty} \frac{j\,a_j}{(j-\tau)!}\,\mathfrak{z}^{\,j-r+1} + \cdots. \tag{4}$$

Differentiating logarithmically, we obtain

$$\frac{\mathfrak{F}'(\mathfrak{z})}{\mathfrak{F}(\mathfrak{z})} = \frac{1}{(r-\tau-\gamma)}\left[\frac{\mathcal{D}\big(\mathcal{D}^{\tau}\xi(\mathfrak{z})\,\mathfrak{z}^{-\gamma}\big)}{\mathcal{D}^{\tau}\xi(\mathfrak{z})\,\mathfrak{z}^{-\gamma}}\right]. \tag{5}$$

Expanding further,

$$= \frac{1}{(r-\tau-\gamma)}\left[\frac{\mathfrak{z}^{\gamma}\,\mathcal{D}^{\tau+1}\xi(\mathfrak{z}) - \gamma\,\mathfrak{z}^{\gamma-1}\,\mathcal{D}^{\tau}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})\,\mathfrak{z}^{-\gamma}}\right]$$

$$= \frac{1}{(r-\tau-\gamma)}\left[\frac{\mathfrak{z}^{\gamma}\,\mathcal{D}^{\tau+1}\xi(\mathfrak{z}) - \gamma\,\mathfrak{z}^{\gamma-1}\,\mathcal{D}^{\tau}\xi(\mathfrak{z})}{\mathfrak{z}^{\gamma}\,\mathcal{D}^{\tau}\xi(\mathfrak{z})}\right].$$

Thus,

$$\frac{\mathfrak{F}'(\mathfrak{z})}{\mathfrak{F}(\mathfrak{z})} = \frac{1}{(r-\tau-\gamma)}\left[\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})} - \frac{\gamma}{\mathfrak{z}}\right]. \tag{6}$$

Rearranging,

$$\begin{aligned}
\mathfrak{F}'(\mathfrak{z}) =& \mathfrak{F}(\mathfrak{z})\frac{1}{(r-\tau-\gamma)}\left[\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}-\frac{\gamma}{\mathfrak{z}}\right]\\
=& \frac{1}{(r-\tau-\gamma)}\left[\frac{(r-\tau)!}{r!}\frac{\mathcal{D}^{\tau}\xi(\mathfrak{z})}{\mathfrak{z}^{\gamma}}\right]\times\left[\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}-\frac{\gamma}{\mathfrak{z}}\right]\\
=& \frac{1}{(r-\tau-\gamma)}\left(\frac{(r-\tau)!}{r!}\right)^{\frac{1}{r-\tau-\gamma}}\left(\frac{\mathcal{D}^{\tau}\xi(\mathfrak{z})}{\mathfrak{z}}\right)^{\frac{1}{r-\tau-\gamma}}\times\frac{1}{\mathfrak{z}^{\frac{\gamma-1}{r-\tau-\gamma}}}\left[\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}-\frac{\gamma}{\mathfrak{z}}\right]\\
=& \frac{1}{(r-\tau-\gamma)}\left(\frac{(r-\tau)!}{r!}\frac{\mathcal{D}^{\tau}\xi(\mathfrak{z})}{\mathfrak{z}}\right)^{\frac{1}{r-\tau-\gamma}}\times\left[\frac{\mathfrak{z}^{\frac{1-\gamma}{r-\tau-\gamma}}\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}-\gamma\mathfrak{z}^{\frac{1-r+\tau}{r-\tau-\gamma}}\right].
\end{aligned}$$

Thus,

$$\mathfrak{F}'(\mathfrak{z}) = \frac{1}{r-\tau-\gamma}\,u(\mathfrak{z}).$$

$$\begin{aligned}
|\mathfrak{F}'(\mathfrak{z})-1| &= \left|\frac{1}{r-\tau-\gamma}u(\mathfrak{z})-1\right|\\
&= \left|\frac{1}{r-\tau-\gamma}\left[u(\mathfrak{z})-r+\tau+\gamma\right]\right|.
\end{aligned}$$

By the given condition on $u(\mathfrak{z})$, we see that

$$|\mathfrak{F}'(\mathfrak{z})-1| < \frac{\eta+1}{\sqrt{(\eta+1)^2+1}}, \quad |\mathfrak{z}|<1, \eta\in\mathbb{N}. \tag{7}$$

Therefore, by the lemma, we obtain

$$\mathfrak{F}(\mathfrak{z})\in\mathfrak{S}^*(\eta,0).$$

Therefore

$$\mathrm{Re}\left(\frac{\mathfrak{z}\mathfrak{F}'(\mathfrak{z})}{\mathfrak{F}(\mathfrak{z})}\right)>0.$$

From equation 6,

$$\mathrm{Re}\left[\frac{1}{r-\tau-\gamma}\left(\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}-\frac{\gamma}{\mathfrak{z}}\right)\right]>0.$$

Which Implies

$$\frac{1}{r-\tau-\gamma}\left[\mathrm{Re}\left(\frac{\mathcal{D}^{q\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}\right)-\gamma\right]>0.$$

Therefore,

$$\mathrm{Re}\left(\frac{\mathcal{D}^{\tau+1}\xi(\mathfrak{z})}{\mathcal{D}^{\tau}\xi(\mathfrak{z})}\right)>\gamma.$$

Hence

$$\xi(\mathfrak{z})\in\mathfrak{S}_r^*(\tau,\eta,\gamma).$$

□

Theorem 2.3. *Let $\xi(\mathfrak{z})$ be any function of the set $\mathcal{A}_r(\eta)$, and define the function*

$$\mathfrak{F}(\mathfrak{z}) = \left[\frac{(r-\tau)!\,\mathcal{D}^{\tau}\xi(\mathfrak{z}}{r!}\cdot\frac{1}{\mathfrak{z}^{\gamma}}\right]^{\frac{1}{r-\tau-\gamma}},$$

for $(|\mathfrak{z}| < 1)$ and $(0 \le \gamma < r-\tau)$. Suppose further that

$$|\mathfrak{F}''(\mathfrak{z}| \le \frac{\eta+1}{\sqrt{(\eta+1)^2+1}}, \quad \textit{for all } |\mathfrak{z}| < 1.$$

Then $\xi(\mathfrak{z}) \in \mathfrak{S}^*_r(\tau,\eta,\gamma)$.

Proof. We have

$$|\mathfrak{F}'(\mathfrak{z}) - 1| = \left|\int_0^{\mathfrak{z}} \mathfrak{F}''(\mathfrak{t})\,dt\right|.$$

Applying the triangle inequality,

$$\le \int_0^{|\mathfrak{z}|} |\mathfrak{F}''(\rho e^{i\theta})|\,d\rho.$$

Using the given condition on $|\mathfrak{F}''(\mathfrak{z})|$, we obtain

$$\le \frac{\eta+1}{\sqrt{(\eta+1)^2+1}}|\mathfrak{z}|.$$

Hence, we conclude that

$$|\mathfrak{F}'(\mathfrak{z}) - 1| \le \frac{\eta+1}{\sqrt{(\eta+1)^2+1}}|\mathfrak{z}|.$$

Therefore, $\mathfrak{F}(\mathfrak{z}) \in \mathfrak{S}^*(n,0)$, and as shown earlier, this implies $\xi(\mathfrak{z}) \in \mathfrak{S}^*_r(\tau,\eta,\gamma)$. □

Corollary 2.4. *If $\xi(\mathfrak{z})$ be any function of the set $\mathcal{A}_r(\eta)$. Suppose the function $u(\mathfrak{z})$ is given by (2.1) in Theorem 2.1 satisfies*

$$\frac{|2u(\mathfrak{z}) - 2r + 2\tau + 1|}{(2r-2\tau-1)} < \frac{\eta+1}{\sqrt{(\eta+1)^2+1}}, \quad , |\mathfrak{z}| < 1,\ \eta \in \mathbb{N}, \quad \frac{1}{2} < r-\tau,$$

then $\xi(\mathfrak{z}) \in \mathfrak{S}^*_p(\tau,\eta,1/2)$.

Corollary 2.5. *If $\xi(\mathfrak{z})$ be any function of the set $\mathcal{A}_r(\eta)$ and the function*

$$\mathfrak{F}(\mathfrak{z}) = \left[\frac{(r-\tau)!\,\mathcal{D}^{\tau}\xi(\mathfrak{z}}{r!}\,\frac{1}{\mathfrak{z}^{1/2}}\right]^{\frac{2}{2r-2\tau-1}}$$

(for $|\mathfrak{z}| < 1, 1/2 < r-\tau$) satisfies

$$|\mathfrak{F}''(\mathfrak{z})| \le \frac{\eta+1}{\sqrt{(\eta+1)^2+1}},$$

for $|\mathfrak{z}| < 1, \quad 1/2 < r-\tau$, then $\xi(\mathfrak{z}) \in \mathfrak{S}^*_r(\tau,\eta,1/2)$.

3. Condition for Convexity

Theorem 3.1. *Let $\xi(\mathfrak{z})$ be any function of the set $\mathcal{A}_r(\eta)$, and define a function $v(\mathfrak{z})$ by*

$$v(\mathfrak{z}) = \left[\frac{(r-\tau)!}{r!} \cdot \frac{\left(D^{\tau}\xi(\mathfrak{z})\right)^{1-r+\tau-\gamma}}{\mathfrak{z}^{r-\tau}}\right]^{\frac{1}{r-\tau-\gamma}} \left[\mathfrak{z}\, D^{\tau+1}\xi(\mathfrak{z}) - \gamma\, D^{\tau}\xi(\mathfrak{z})\right]. \qquad (8)$$

If the function $v(z)$ satisfies the condition

$$|v(\mathfrak{z}) - r + \tau + \gamma| < \frac{(\eta+1)(r-\tau-\gamma)}{\sqrt{(\eta+1)^2+1}}, \quad |\mathfrak{z}| < 1, \quad 0 \le \gamma < r-\tau,$$

then

$$\xi(\mathfrak{z}) \in \mathfrak{C}_r(\tau, \eta, \gamma)$$

Proof. We know that

$$D^{\tau}\xi(\mathfrak{z}) = \frac{r!}{(r-\tau)!}\mathfrak{z}^{r-\tau} + \sum_{j=r+\eta}^{\infty} \frac{j!a_j}{(j-\tau)!}\mathfrak{z}^{j-\tau}.$$

Using this, we arrive at the equation

$$\left(\frac{(r-\tau)!}{r!}\frac{D^{\tau}\xi(\mathfrak{z})}{\mathfrak{z}^{r-\tau}}\right)^{\frac{1}{r-\tau-\gamma}} = 1 + \frac{1}{r-\tau-\gamma}\frac{(r-\tau)!}{r!}\sum_{j=r+\eta}^{\infty}\frac{j!}{(j-\tau)!}a_j\mathfrak{z}^{j-r}.$$

Integrating both sides, we get

$$\int_0^{\mathfrak{z}} \left(\frac{(r-\tau)!\, D^{\tau}\xi(t)}{r!\, t^{r-\tau}}\right)^{\frac{1}{r-\tau-\gamma}} dt = \mathfrak{z} + \frac{1}{r-\tau-\gamma} \cdot \frac{(r-\tau)!}{r!}\sum_{j=r+\eta}^{\infty}\frac{j!}{(j-\tau)!}\, a_j \cdot \frac{\mathfrak{z}^{j-r+1}}{j-r+1}.$$

Define the function $\mathfrak{H}(\mathfrak{z})$ as

$$\mathfrak{H}(\mathfrak{z}) = \int_0^{\mathfrak{z}} \left(\frac{(r-\tau)!D^{\tau}\xi(t)}{r!t^{r-\tau}}\right)^{\frac{1}{r-\tau-\gamma}} dt. \qquad (9)$$

Also, let $\mathfrak{G}(\mathfrak{z}) = \mathfrak{z}\mathfrak{H}'(\mathfrak{z})$. Differentiating $\mathfrak{G}(\mathfrak{z})$ logarithmically, we obtain

$$\frac{\mathfrak{G}'(\mathfrak{z})}{\mathfrak{G}(\mathfrak{z})} = \frac{1}{r-\tau-\gamma}\left[\frac{D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})} - \frac{\gamma}{\mathfrak{z}}\right]. \qquad (10)$$

Thus,

$$\mathfrak{G}'(\mathfrak{z}) = \mathfrak{G}(\mathfrak{z})\frac{1}{r-\tau-\gamma}\left[\frac{D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})} - \frac{\gamma}{\mathfrak{z}}\right].$$

Rearranging, we obtain

$$\mathfrak{G}'(\mathfrak{z}) = \frac{1}{r-\tau-\gamma}v(\mathfrak{z})$$

and by the given condition on $v(\mathfrak{z})$,

$$|v(\mathfrak{z})-r+\tau+\gamma|<\frac{(\eta+1)(r-\tau-\gamma)}{\sqrt{(\eta+1)^2+1}},\quad |\mathfrak{z}|<1,\quad 0\leq\gamma<r-\tau,$$

which implies that

$$|\mathfrak{G}'(\mathfrak{z})-1|<\frac{(\eta+1)}{\sqrt{(\eta+1)^2+1}},|\mathfrak{z}|<1. \tag{11}$$

Therefore,

$$\mathfrak{G}(\mathfrak{z})\in\mathfrak{S}^*(\eta,0),$$

which implies

$$\text{Re}\left(\frac{\mathfrak{z}\mathfrak{G}'(\mathfrak{z})}{\mathfrak{G}(\mathfrak{z})}\right)>0.$$

Furthermore,

$$\mathfrak{G}'(\mathfrak{z})=\mathfrak{H}'(\mathfrak{z})+\mathfrak{z}\mathfrak{H}''(\mathfrak{z}),$$

which leads to

$$\frac{\mathfrak{z}\mathfrak{G}'(\mathfrak{z})}{\mathfrak{G}(\mathfrak{z})}=1+\frac{\mathfrak{z}\mathfrak{H}''(\mathfrak{z})}{\mathfrak{z}\mathfrak{H}'(\mathfrak{z})}. \tag{12}$$

Since

$$\text{Re}\left(1+\frac{\mathfrak{z}\mathfrak{H}''(\mathfrak{z})}{\mathfrak{H}'(\mathfrak{z})}\right)>0,$$

we conclude that

$$\mathfrak{H}(\mathfrak{z})\in\mathfrak{C}(\eta,0).$$

Also

$$\mathfrak{H}'(\mathfrak{z})=\left[\frac{\mathcal{D}^{\tau}\xi(\mathfrak{z})}{{}^{r}P_{\tau}\mathfrak{z}^{r-\tau}}\right]^{\frac{1}{r-\tau-\gamma}}.$$

Now differentiating $\mathfrak{H}'(\mathfrak{z})$ logarithmically, and adding 1 we get

$$\begin{aligned}1+\frac{\mathfrak{z}\mathfrak{H}''(\mathfrak{z})}{\mathfrak{H}'(\mathfrak{z})}&=\frac{1}{(r-\tau-\gamma)}\left[\frac{\mathfrak{z}D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})}-r+\tau\right]+1\\&=\frac{1}{(r-\tau-\gamma)}\left[\frac{\mathfrak{z}D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})}-r+\tau+r-\tau-\gamma\right]\\&=\frac{1}{(r-\tau-\gamma)}\left[\frac{\mathfrak{z}D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})}-\gamma\right].\end{aligned} \tag{13}$$

Since

$$\text{Re}\left(1+\frac{\mathfrak{z}\mathfrak{H}''(\mathfrak{z})}{\mathfrak{H}'(\mathfrak{z})}\right)>0.$$

Therefore

$$\text{with the condition}\quad 0\leq\gamma<p-q.$$

We get

$$\text{Re}\left(1+\frac{\mathfrak{z}D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})}\right)=1+\text{Re}\left(\frac{\mathfrak{z}D^{\tau+1}\xi(\mathfrak{z})}{D^{\tau}\xi(\mathfrak{z})}\right)>1+\gamma>\gamma. \tag{14}$$

Therefore $\xi(\mathfrak{z}) \in \mathfrak{C}_r(\tau, \eta, \gamma)$ □

Theorem 3.2. *Consider the function*

$$\mathfrak{H}(\mathfrak{z}) = \int_0^{\mathfrak{z}} \left(\frac{(r-\tau)! D^\tau \xi(t)}{r! t^{r-\tau}} \right)^{\frac{1}{r-\tau-\gamma}} dt. \tag{15}$$

Suppose that the function $\mathfrak{H}(\mathfrak{z})$ satisfies the condition:

$$|\mathfrak{H}''(\mathfrak{z})| \leq \frac{(\eta+1)}{2\sqrt{(\eta+1)^2+1}}, \quad (|\mathfrak{z}| < 1, \quad 0 \leq \gamma < r-\tau). \tag{16}$$

Then $\xi(\mathfrak{z}) \in \mathfrak{C}_r(\tau, \eta, \gamma)$.

Proof. Consider function $\mathfrak{G}(\mathfrak{z}) = \mathfrak{z}\mathfrak{H}'(\mathfrak{z})$

$$\begin{aligned} \mathfrak{G}(\mathfrak{z}) &= \mathfrak{z}\left[1 + \frac{1}{(r-\tau-\gamma)} \frac{(r-\tau)!}{r!} \sum_{j=r+\eta}^{\infty} \frac{j!}{(j-\tau)!} a_j \mathfrak{z}^{j-r}\right] \\ &= \mathfrak{z} + \frac{1}{(r-\tau-\gamma)} \frac{(r-\tau)!}{r!} \sum_{j=r+\eta}^{\infty} \frac{j!}{(j-\tau)!} a_j . \mathfrak{z}^{j-r+1}. \end{aligned} \tag{17}$$

Therefore $\mathfrak{G}(\mathfrak{z}) \in \mathcal{A}(\eta)$. Also,

$$\begin{aligned} |\mathfrak{G}'(\mathfrak{z}) - 1| &= |\mathfrak{H}'(\mathfrak{z}) + \mathfrak{z}\mathfrak{H}''(\mathfrak{z}) - 1| \\ &\leq |\mathfrak{H}'(\mathfrak{z}) - 1| + |\mathfrak{z}\mathfrak{H}''(\mathfrak{z})| \\ &= \left| \int_0^{\mathfrak{z}} \mathfrak{H}''(t) dt \right| + |\mathfrak{z}\mathfrak{H}''(\mathfrak{z})| \\ &\leq \left(\frac{(\eta+1)}{2\sqrt{(\eta+1)^2+1}} + \frac{(\eta+1)}{2\sqrt{(\eta+1)^2+1}} \right) |\mathfrak{z}| \end{aligned}$$

or that $|\mathfrak{G}'(\mathfrak{z}) - 1| \leq \frac{(\eta+1)}{\sqrt{(\eta+1)^2+1}} |\mathfrak{z}|$.

Since $|\mathfrak{z}| < 1$, we have

$$|\mathfrak{G}'(\mathfrak{z}) - 1| \leq \frac{(\eta+1)}{\sqrt{(\eta+1)^2+1}}, \tag{18}$$

or $\mathfrak{G}(\mathfrak{z}) \in \mathfrak{S}^*(\eta, 0)$, which implies $\mathfrak{H}(\mathfrak{z}) \in \mathfrak{C}(\eta, 0)$. Therefore $\xi(\mathfrak{z}) \in \mathfrak{C}_r(\tau, \eta, \gamma)$. □

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(Goswami) School of Liberal Studies, Dr B. R. Ambedkar University, Delhi-11006, India
Email address: pranaygoswami83@gmail.com

(Shukla) School of Liberal Studies, Dr B. R. Ambedkar University, Delhi-11006, India
Email address: sanju1987shukla@gmail.com

(Bansal) Department of Mathematics, Jaipur Engineering College and Research Center, Rajasthan, Jaipur-302022, India
Email address: bansalindian@gmail.com