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On Fuglede-Putnam property and orthogonality for derivations induced by hyponormal operators

David Kagali, Benard Okelo*, and Julia Owino

Abstract. The orthogonality of derivations induced by operators is an area with various applications in light of ever-dynamic technological advances. There are different types of orthogonality, and interesting results have emerged in which operators satisfying given conditions are chosen to establish Range-Kernel orthogonality. However, most of the results have focused on one type of orthogonality called the Birkhoff orthogonality. We have also herein considered the Birkhoff concept of orthogonality. Researchers have repeatedly posed the following question: Could there be a possibility for studying other types of orthogonality with respect to the range and the kernel of derivations apart from the Birkhoff orthogonality? In this note, we establish orthogonality conditions for derivations when implemented by hyponormal operators underthe Fuglede-Putnam property.

Keywords: Derivation, FP-Property, Orthogonality, Hyponormality

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1. Introduction

Functional analysis and classical analysis of mathematical physics are the major approaches for the advancement of the theory of hyponormal operators. Concepts are defined using abstract conditions and treated in terms of axioms. Some functional models exist which enrich the theory with advanced structures [8]. They

*Corresponding author

borrow some key spectral properties from normal ones, and several properties have emerged as a result of these previous developments [5].

Authors of [6] used approximate proper vectors to study hyponormal operators. For a non-empty spectrum of a hyponormal operator, it is made to rely on the elementary case of self-adjoint operators. In [7], it showed that a quasi-normal operator is subnormal and that a hyponormal operator is paranormal. The authors of [15] gave properties of paranormal and hyponormal operators. Conditions on generalizations of concepts for these operators were given. The work of [4] generalized backward extension for subnormal weighted shifts to subnormal operators. Later, the research of [9] came up with a structure theorem for p-paranormal operators. Another approach by [13] brought the idea of a square hyponormal operator to light. Spectral properties of these operators were studied. When z and w are distinct eigenvalues of T and a, $b \in H$ are eigen vectors, then $\langle a, b \rangle = 0$. Again [11] showed that T has a hyponormal property if a function f exists and is continuous on the set.

The authors of [12] defined a new class named D-hyponormal and D-quasi-hyponormal operators. Their basic properties were outlined while [14] looked at an (n, m)-hyponormal operator and its equivalence to operator S under isometry. The concept of unitary quasi-equivalence was introduced. The work of [15] introduced (m, k)-quasi-hyponormal operators on Hilbert spaces, which is an extension of classes of parahyponormal and k-quasi-parahyponormal operators. Matrix representation for these operators was presented. In [16], the work showed that hyponormal operators satisfy various types of Weyl's and Browder's theorems.

Due to the growth of operator and quantum theories, derivations became a key tool in analysing concepts in spaces, particularly Hilbert spaces. It is known that derivations can be inner or generalized [17], and most of their properties are discussed in [19]. Their norms are well established by [18] as well as their derivation ranges. The Kernels of these derivations were further expounded by [20], who looked at the structural properties of elementary operators, while [21] later studied the generalized derivations and their numerical ranges. Further, [24] gave an intensive study on Kernels of generalized derivations.

Commutants which intersect with derivation ranges that are closed include the normal operators as seen in the work of [23] on normal derivations, isometries, cyclic subnormal operators and Jordan operators. The study of [22] asserted that when a range of a derivation which is closed intersects with the kernel of the same derivation, the result is zero if the operators acting on them are hyponormal operators. It was shown by [25] that the adjoint of the kernel of a derivation is zero if the polynomial that acts has the Fuglede-Putnam property. The result was later extended to $\delta_{T,S}$. It has been our key interest to study these derivations under the properties and structures of hyponormal operators.

The work of [26] found that the norm closure of the range of inner derivations always contains lower triangular compact operators. It has been noted that derivations on commutative Von-neuman algebras are implemented by bounded operators, and also done to non-self-adjoint commutative algebras by similarity. In [27], the study considered the spatiality of derivations characterized on $\star$-algebras using positive linear functionals. Authors in [46] investigated properties of inner derivations strictly implemented by norm-attainable operators and determined their norms. and introduced norms while [28] used formulae to approximate norms of $\delta_{T,S}$. It was seen that $\delta_{T,S}$ is bounded. Its compactness was also shown whenever T and S are compact.

The Fuglede-Putnam theorem relates the properties of other operators to normal operators. Fuglede noted that, for a bounded S and N being normal, bounded or unbounded and S commutes with $T_i's$ for instance $ST_i = T_iS$, then S commutes with any function of N [29] In the case two operators are non-bounded, commutativity is not defined and several authors studied asymptotic Fuglede theorem. Other researchers like Seth [2] studied Fuglede-Putnam theorems for various classes of operators.

Berberian obtained the Fuglede-Putnam theorem when an operator and the adjoint of another operator are hyponormal, and another operator is of Hilbert-Schmidt class. Later, [1] proved that X is of Hilbert Schmindt class, then $TX = XS$ implies $T^\star X = XS^\star$. This also applies when $T, S^\star \in B(H)$ are injective, and X is any operator as seen in [32]. Some authors also showed that Berberian's result can be reached without any condition on X. The work of [10] on derivative ranges helped us identify gaps that exist in the study of derivative ranges and also helped in the study of the relationship between Fuglede-Putnam properties and range-kernel orthogonality of derivations inheriting properties of hyponormal operators. Generalizations of this theorem have been important to our study because they bridged between normal operators and hyponormal operators, making their analysis simpler [35]. Therefore, we extended the same to the derivations they induce.

In [33] proved that for the pair $(T, S^\star)$ satisfying FP-property and C is in the kernel of $\delta_{T,S}$ and C is in $B(H)$, then $\|\delta_{T,S} + C\| \geq \|C\|$, $T^2X = XS^2$ and $T^3X = XS^3$ while [34] showed that is normal and unitarily equivalent if it is quasi-hyponormal and hyponormal. In [36], the author extended the FP-theorem on class (p, w)-hyponormal operators, while [3] later looked at the FP-property for N-class $A(k)$ operators. Range-Kernel results for generalized derivations induced by them were also given, and [37] provided a complete characterization of left-symmetric points for strong Birkhoff orthogonality. In [38], the authors obtained a relationship between two functionals and the existence of b-Birkhoff orthogonal elements in two-normed linear spaces. The study by [39] generalized properties of j-orthogonality and its relationship with metric projection on smooth uniform complex complete countable normed spaces. The j-orthogonal complement was also defined and characterized.

Some authors introduced the concept of orthogonality in Banach spaces, which generalizes the usual orthogonality in Hilbert spaces. This property is extremely important as it suggests that we could use the eigenvectors of an eigenvalue equation as a set of basis elements for this Hilbert space. In [40], they characterized inner derivations with orthogonality for normal operators and established the range kernel orthogonality for inner derivations in the sense of [43]. In normed spaces [42] established that Birkhoff orthogonality implies best approximation and vice versa, and thus the significance of this sense of orthogonality. Range Properties of derivations have been established in [41]. Of key interest, we dug deeper and considered derivations as projections. This laid the basis for orthogonality relations. Also, orthogonality can be of many forms apart from the aforementioned. Therefore, we have made an effort to link this study to other forms of orthogonality.

Operators in $R(\delta_T)$ (where R denotes the range and δ_T is an inner derivation induced by T) are significant and have been used by [44] to establish operators in $R(\delta_T)\bigcap T^{'}$ ($T^{'}$ denotes commutant of T) to be nilpotent if $P(T)$ is normal, isometric or co-isometric for some polynomial P. Hyponormal operators are a generalization of many other classes. Range-kernel orthogonality conditions for such large classes have been established by minimisation procedures. For instance, by using the power norm inequality and by compactness properties of hyponormal operators, [45]established approximation results for paranormal operators, which in turn have been used to establish orthogonality.

In the current study, we investigate derivations which are induced on hyponormal operators. Hyponormal operators have various properties ranging from their spectrum, boundedness, invertibility and density. These have been studied in [47]. Similarly, derivations induced on these operators have such properties. The derivations can be inner or generalized. This was checked in [48], while derivations having ranges were seen in [50]. Orthogonality can be established on these derivations [51]. Similarly, in [30], it was expounded to linear functionals and in [52], range-kernel orthogonality of these derivations was studied.

A detailed summary of known results on relations between different orthogonality concepts [53], their properties such as symmetry [54], homogeneity [31] and additivity(see [56]-[55] and the references therein) were given and proved about range and kernel orthogonality for elementary operators concerning unitary invariant norms [60]. Again they proved a necessary condition for $\|\delta_{T,S}+C\|_P \geq \|C\|_P$ where (T,S) have the property $\ker \delta_{T,S}|_{\ell_P} \subseteq \ker \delta_{T^\star,S^\star}|_{\ell_P}$ and the property $X, C \in \ell_P$ (Schatten p-class, $1 < p < \alpha$).

In [62], the author introduced the notion of orthogonal mapping in isosceles orthogonal space. Stability of orthogonally constant mapping was established, and also characterised Birkhoff-James orthogonality on $C^\star$-algebras. Again, they studied the domain set and dual of the target space as well as orthogonality with regard to operator norm and numerical radius norm.

In [61], they studied isosceles and Birkhoff-James orthogonalities of positive linear operators. They explored the relationship between these two types of orthogonality and noted that Birkhoff-James orthogonality has everything about smooth norms in reflexive Banach spaces. This is also true in computing the dimensions of underlying normed spaces, as illustrated by [61] characterized right symmetric and left symmetric operators with regard to Birkhoff-James orthogonality.

In [63] also showed preservation strongly by stating that linear operators on normed spaces with reverse orthogonality, and also characterized conditions for orthogonality, and also considered Hermite-Hadamard type of unitary Carlsson's orthogonality to characterize real inner product spaces. Later, they brought in the notion of orthogonal sets called Birkhoff orthogonal sets.

Many authors have also considered and defined a new orthogonal geometric constant $\Omega(X)$ based on the parallelogram law and isosceles orthogonality. For normed spaces, a constant value is equal to one if the norm can be induced by inner product, and it has been noted that orthogonality is preserved in Krein spaces [64]. Four types of orthogonalities were discussed in the context of Krein spaces. For instance, [65] considered approximate symmetry and established its connection with some properties of the space X.

The work of [66] established range-kernel orthogonality for derivations implemented by hyponormal operators in the sense of Birkhoff was a cornerstone for our study. They recommended that further studies on this area should place emphasis on other forms of orthogonality different from Birkhoff's. They also advised that if further research could be done on adjoints of derivations induced by hyponormal operators [67]. Therefore, in our study, we have been focusing on various forms of orthogonality in relation to range and kernel for derivations and the Fuglede-Putnam property. To change the direction of this study, we also considered how hyponormal operators induce properties in derivations as a later considered implementation. To actualize our study, the following basic concepts have been useful for laying a firm foundation in our course.

2. Preliminaries

We provide preliminary concepts that are key to the study.

Definition 2.1. [58, Definition 2.2] Two maps on a vector space are considered orthogonal with regard to a specific inner product if their action on any pair of vectors a and b in that space results in the inner product of their images being zero, expressed as $\langle Ta, Sb\rangle = 0$.

Definition 2.2. [59, Definition 3.1] A mapping $S \in B(H)$ is hyponormal if $(SS^\star)^p - (S^\star S)^p \geq 0$ and $p = 1$.

Definition 2.3. [9, Definition 1.4] A mapping D is an inner derivation if it satisfies the Leibnitz rule for all vectors in its domain, that is, $D(\phi\phi) = (D\phi)\phi + \phi(D\phi)$ where ϕ are vectors in the domain of D and all vector ϕ in the Hilbert space H.

Definition 2.4. [6, Definition 1.2.1] T has Fuglede-Putnam property if $SC = CS$, then $TSC = SCT$ for some self-adjoint maps C and S. It suggests that if two self-adjoint operators commute (product and order do not matter), then when one of these operators is multiplied by a third self-adjoint operator, the order of multiplication still does not matter.

3. Main Results

We consider the Fuglede-Putnam (FP) property on derivations and establish orthogonality conditions for derivations induced by hyponormal operators satisfying the Fuglede-Putnam property. These results utilizes matrices, adjoints, geometric intuition, set theories and various mathematical techniques.

Proposition 3.1. *Suppose that $C, D \in G_H(H)$, then the derivation $\delta_{C,D}$ is bounded from above for all $Y \in G_H(H)$.*

Proof. Let C, D and Y be induced by c_n, d_n respectively and f_n be arbitrary elements of $B(H)$. By definition of δ, we have for $[\|\epsilon_n f_n\|^2] = 1$ and $\|\epsilon_n f_n\| \leq 1$ that $\|\delta_{C,D}\|^2 = \|\epsilon_n(c_n f_n - f_n d_n)\|^2 \leq \|\epsilon_n c_n f_n\|^2 + \|\epsilon_n f_n d_n\|^2 \leq [\epsilon_n|c_n|^2 + \epsilon_n|d_n|^2][\|\epsilon_n f_n\|^2]$ $= [\epsilon_n|c_n|^2 + \epsilon_n|d_n|^2]\ [\epsilon_n|c_n| + \epsilon_n|d_n|]$. Taking the supremum of both sides of the inequality gives us $\|\delta_{C,D}(Y)\| \leq [\epsilon_n|c_n|^2]^{\frac{1}{2}} + [\epsilon_n|d_n|^2]^{\frac{1}{2}}$. □

Next, we give the FP-property in a more general context.

Proposition 3.2. *Every derivation is bounded from below*

Proof. By definition of $\delta_{C,D}$, we see that $\delta_{C,D}(Y) = c_n - d_n$ for the bases c_n and d_n of C and D respectively with $\epsilon_n\|f_n\|^2 = 1$. Since c_n and d_n are bounded, from the definition of $\delta_{C,D}(Y)$, we have $\|\delta_{C,D}(f_n)\|^2 = \|\epsilon_n(c_n f_n - f_n d_n)\|^2 \geq \epsilon_n\|c_n f_n\|^2 - \epsilon_n\|f_n d_n\|^2 = [\epsilon_n|c_n|^2 - \epsilon_n|d_n|^2]\|f_n\|^2$ Since the difference of finite summation of c_n and d_n is also bounded, and clearly, $\|\delta_n\| \geq [\epsilon_n|c_n|^2] - [\epsilon_n|d_n|^2]^{\frac{1}{2}}$. □

Lemma 3.3. *Let $\delta_C : G_H(H) \to G_H^H$ defined by $\delta_C(Y) = CY - YC$ be of finite rank. Then δ_C is a derivation and $\delta_C = \delta_D$ if and only if $D = C - \lambda I$ for all $\lambda \in \mathbb{C}$ and $D \in G_H(H)$.*

Proof. From the definition, we have;

$$\delta_C(XY) = CXY - XYC. \tag{1}$$

$$\delta_D(XY) = DXY - XYD \tag{2}$$

subtracting Equation (2) from Equation (1) we have:

$$\delta_C(XY) - \delta_D(XY) = CXY - DXY - XYC + XYD.$$

Then,

$$(\delta_C - \delta_D)(XY) = (C - D)XY - XY(C - D).$$

This implies

$$(\delta_C - \delta_D)(XY) = \delta_{C-D}(XY).$$

Thus,

$$\delta_C - \delta_D = \delta_{C-D}$$

which is a derivation. The converse is true, that is if δ is a derivation in $G_H(H)$ then their exists $C \in G_H(H)$ such that $\delta = \delta_C$.

Conversely, suppose for $C, D \in G_H(H)$, we have $\delta_C = \delta_D$, then this implies

$$\delta_C - \delta_D = \delta_{C-D} = 0.$$

Hence for all $Y \in G_H(H)$, we have

$$\delta_{C-D}(Y) = (C - D)Y - Y(C - D) = 0$$

this implies that

$$(C - D)Y = Y(C - D).$$

Setting $C - D = E$. We have $EY = YE$ implying $E = \lambda I$ thus $C - D = \lambda I$ and so, $D = C - \lambda I$. On the other hand, if $D = C - \lambda I$, then by applying derivation on both sides, we have $\delta_D(Y) = \delta_{C-\lambda I}(Y)$. Thus,

$$DY - YD = (C - \lambda I)Y - Y(C - \lambda I).$$

This implies

$$DY - YD = CY - \lambda Y - YC + Y\lambda,$$

and so,

$$DY - YD = CY - YC.$$

This means that

$$\delta_D = \delta_C. \tag{3}$$

□

The identities $\delta_C + \delta_D = \delta_{C+D}$, $\delta_C\delta_D - \delta_D\delta_C = \delta_{CD-DC}$ shows that the sum and lie product of two inner derivations is a derivation. However, the product $\delta_C\delta_D$ is a derivation only in trivial cases.

Theorem 3.4. *Let $\delta_C : G_H(H) \longrightarrow G_H(H)$ defined by $\delta_C(Y) = CY - YC$ be of finite rank. Then, $\delta_C(Y) = CY - YC$ is linear and bounded.*

PROOF. For linearity, let $X, Y \in G_H(H)$, then for scalars $\alpha, \beta \in C$. We have

$$\begin{aligned}\delta_C(\alpha X + \beta Y) =& C(\alpha X + \beta Y) - (\alpha X + \beta Y)C \\ =& \alpha CX - \alpha XC + \beta CY - \beta YC \\ =& \alpha(CX - XC) + \beta(CY - YC) \\ =& \alpha\delta_C + \beta\delta_C(Y). \end{aligned} \tag{4}$$

Hence δ_C is linear. But a derivation is a linear map $\delta : G_H(H)$ satisfying the Leibniz rule:

$$\delta(XY) = \delta(X)Y + X\delta(Y) \tag{5}$$

and is $\delta : G_H(H) \longrightarrow G_H(H)$ is a derivation, then their exists $C \in G_H$ such that $\delta = \delta_C$. Thus

$$\delta_C(XY) = \delta_C(X)Y + X\delta_C(Y)$$

$$\Rightarrow \delta_C(XY) = (CX - XC)Y + X(CY - YC).$$

But C is finite implying existence of $I \in G_H(H)$ such that

$$\|CX - XC - I\| \geq I$$

and

$$\|CY - YC - I\| \geq I$$

and hence from line 1, we have

$$\|\delta(XY)\| \leq \|CX - XC - I\|\|Y\| + \|X\|\|CY - YC - I\| \Rightarrow \|\delta_C(XY)\| \leq \|Y\| + \|X\|.$$

Thus their exists a positive integer $n \in N$ such that

$$\|\delta(XY)\| \leq n. \tag{6}$$

□

Corollary 3.5. *Let C be a hyponormal operator and D be a normal operator. Then $\delta(C) = \|D\|$ if $\delta(C) = \|D\|$.*

PROOF. Suppose that $\delta(C) = \|D\|$. We have $[y_n]_n$ in H with $\|y_n\| = 1$ for each n and such that $< Cy_n, y_n >\rightarrow 0$ and $\|Cy_n\| \rightarrow \|C\|$ as $n \rightarrow \alpha$. So $< DRy_n, Ry_n >\rightarrow 0$ as $n \rightarrow \alpha$. Moreover, since $\|Ry_n\| \rightarrow 1$ and $\|DRy_n\| \rightarrow \|D\|$ as $n \rightarrow \alpha$. Also, $\delta(D) = \|D\|$. Conversely, suppose that $\delta(D) = \|D\|$. We have $R_D = \|D\|$. Since $\sigma(D) \subseteq \sigma(C)$, then $N_D \leq R(C)$. We obtain $\|C\| \leq R_C$. Hence $\delta(C) = \|C\|$ which completes the proof. □

Boundedness is of great importance for both theoretical and practical reasons. It helps in the continuity and extension of derivations. These assist in approximating and applying them to dense subspaces. Boundedness preserves spectral properties more predictably. It helps in the stability of systems under perturbations. Next, we show that hyponormal operators are S-universal.

Proposition 3.6. *Hyponormality implies S-universality, that is,* $diam(\sigma(C)) = 2R_C$.

Proof. Since S-universality and hyponormality are preserved under translations, we may assume that $\delta(C) = \|C\|$ and hence $\delta(D) = \|D\|$. Suppose that T is S-universal. By theorem above, we have; $\|\delta_{2,C}\| = \|\delta_C\| = 2\|C\| = 2\|D\| = \|\delta_D\|$. Consider $[y_n]_n$ in $C_2(H)$ with $\|y_n\|_2 = 1$ for which $\|Cy_n - y_nC\| \to 2\|C\|$ as $n \to \alpha$. Since $\|Cy_n - y_nC\|_2 \leq \|Cy_n\|_2 + \|y_nC\|_2 \leq \|C\| + \|y_nC\|_2 \leq 2\|C\|$. We deduce that, $\|Cy_n\|_2 \to \|C\|$. Similarly, we get $\|Y_nC\|_2 \to \|C\|$. Now, from the identity $\|Cy_n - y_nC\|_2^2 = \|Cy_n\|_2^2 + \|y_nC\|_2^2 - 2R(< Cy_n, y_nC >)$, we conclude that $-R(< Cy_n, y_nC >) \to \|C\|^2$ as $n \to \alpha$ where R denotes the real part. Consider the operator $Ry_nR^\star \in L(H)$. Since $y_n \in C_2(H)$ and $\|y_n\| = 1$, then $Ry_nR^\star \in C_2(K)$ and $\|Ry_nR^\star\|_2 \leq 1$. Furthermore, $< NRy_nR^\star, Ry_nR^\star D >= tr(DRy_nR^\star(Ry_nR^\star D)^\star)$ $=< Cy_n, y_nC >$. Hence $R(< DRy_nR^\star, Ry_nR^\star D >) \to -\|D\|^2$ as $n \to \alpha$. Since $|R(< DRy_nR^\star, Ry_nR^\star D >)| \leq \|DRy_nR^\star\|^2\|Ry_nR^\star D\|_2 \leq \|D\|^2$, and so $\|DRy_nR^\star\|_2 \to \|D\|$, $\|\|_2 \to \|Ry_nR^\star D\|_2 \to \|D\|$ as $S \to \alpha$. Whence we infer $\|\delta_{2,D}(Ry_nR^\star)\|_2 \to 2\|D\|$ as $n \to \alpha$. That is $\|\delta_{2,D}\| = 2\|D\|$. Since D is normal, it is guaranteed that $diam(\sigma(D)) = \|\delta_2(D)\|$. On the other hand, we see that $diam(\sigma(D)) \leq diam(\sigma(C)) \leq \|\delta_{2,D}\| \leq 2\|D\|$. Therefore, $diam(\sigma(C)) = 2\|D\| = 2\|C\| = 2R_C$. The sufficient condition follows trivially. □

Remark 3.1. Let $C \in L(H)$ be hyponormal, it follows that $\bar{W}(C) = Co(\sigma(C))$ where Co is the convex hull. $R_C = Inf[\|C - \beta\| : \beta \in \mathbb{C}]$.

Drazin invertibility is key in spectral decomposition, solving operator equations when exact inverses do not exist and handles operators that are invertible up to a nilpotent part. Hyponormal operators are isoloid. δ_{CD} retains the property in cases in which $C, D^\star$ are Hyponormal.

Theorem 3.7. δ_{CD} *is isoloid.*

Proof. If $\lambda \in iso\sigma(\delta_{CD})$, then $0 \in iso\sigma(\delta_{CD} - \lambda)$. Now $\lim_{n\to\infty} \|(\delta_{CD} - \lambda)^n x_{11}\|^{\frac{1}{n}} = 0$. The operator C_1 and $D_1 + \lambda$ in $\delta_{C_1(D_1+\lambda)} = \delta_{C_1D_1} - \lambda$ being normal,

$$\lim_{n\to\infty} \|(\delta_{C_1}(D_1 + \lambda)^n x_{11}\|^{\frac{1}{n}} \leftrightarrow (\delta_{C_1}(D_1 + \lambda)^n x_{11} = 0).$$

That is if and only if 0 is an eigenvalue of $\delta_{C_1}(D_1+\lambda)$. Now $(\delta_{CD}-\lambda)x_{11}\oplus 0 = 0$. In the case in $\lambda \neq -1$, then

$$P(B(H)) = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & 0 \end{pmatrix} \in B(H) : (\triangle_{CD} - \lambda)x = \phi_{ID}(x)$$

or

$$\phi_{CD}(x) = 0.$$

Thus, if $0 \in iso\sigma(\delta_{CD})$ then $P(B(H)) = \delta_{CD}^{-1}(0)$. So, $\delta_{CD} = \delta_{C_0D_0} \oplus \delta_{C_1D_1}$ where $\delta_{C_0D_0}$ is nilpotent and $\delta_{C_1D_1}$ is invertible. □

Remark 3.2. Operators satisfying isoloid property are normal, hyponormal, subnormal and compact which are key properties that build strong adjoint relations necessary for FPP to be fulfilled in derivations.

Now we consider the Fuglede-Putnam property. We have seen that for derivations induced by hyponormal operators to satisfy the Fuglede-Putnam property, they should have properties like compactness, subnormality, boundedness and isoloid. Fuglede-Putnam property ensures adjoint relations are preserved. It ensures symmetry between an operator and its adjoint. In this section, we look at the combinations of different classes of hyponormal operators that give a strong adjoint relation. Classes which are near normality, like Subnormal operators, log-hyponormal operators and compact hyponormal operators have a strong adjoint relation. We start by giving a key relationship.

$$Hyponormal \subset M - hyponormal \subset dorminant$$

$$Hyponormal \subset p - hyponormal \subset w - hyponormal$$

When we study one class of these operators, we can generalize to the larger class of hyponormal operators. For an M-hyponormal operator C and for a p-hyponormal operator $D^\star$ in $B(H)$, then $\ker(\delta_{C,D}) \subset \ker(\delta_{C^\star,D^\star})$. This is because if an M-hyponormal operator is dominant, then the pair (C, D) satisfies FPP.

Proposition 3.8. *Let C be M-hyponormal and let $D^\star$ be w-hyponormal operators in $B(H)$. Then $\delta_{C,D}(X) = 0$ entails $\delta_{C^\star,D^\star} = 0$. Moreover, it satisfies the unitary property.*

Proof. Invariant subspaces for C and D are $\bar{r}an(X)$ and $(\ker(X))^\perp$ respectively because $\delta_{C,D}(X) = 0$. We can write $C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & 0 \\ D_2 & D_3 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : H_2 \to H_1$ under the decompositions $H = H_1 = \bar{r}an(X) \oplus ran(X)^\perp$, $H = H_2 = (\ker X)^\perp \oplus \ker X$. From $\delta_{C,D}(X) = 0$, we get $C_1X_1 = X_1D_1$ where C_1 is M-hyponormal and D_1 is w-hyponormal. Let $C_1X_1 = X_1|D_1^\star|U$. Multiplying the two sides of this equation at right by $|D_1^\star|^{\frac{1}{2}}$, we obtain $C_1(X_1|D_1^\star|^{\frac{1}{2}}) = X_1|D_1^\star|U|D_1^\star|^{\frac{1}{2}} = (X_1|D_1^\star|^{\frac{1}{2}}\bar{D}_1^\star)$. The Alugthe transform $\bar{D}_1^\star$ of $D_1^\star$ is semi-hyponormal. Hence the pair $(C, \bar{D}_1^\star)$ satisfies the FPP. Thus, the restriction $C_1|_{\bar{r}an(X_1|D_1^\star|^{\frac{1}{2}})}$ and $\bar{D}_1^\star|_{\ker(X_1|D_1^\star|^{\frac{1}{2}})^\perp}$ are equivalent normal operators. Since X_1 is a quasi-affinity and $|D_1^\star|^{\frac{1}{2}}$ is injective, $\bar{r}an(X_1|D_1^\star|^{\frac{1}{2}}) = \bar{r}anX_1 = \bar{r}anX$ and $\ker(X_1|D_1^\star|^{\frac{1}{2}}) = \ker X_1 = \ker X$. The operator

$D^\star$ and its restriction $D_1^\star$ on $(\ker X)^\perp$ is normal. Consequently, $\ker X$ reduces $D^\star$. Hence $D_2 = 0$. Similarly, C is M-hyponormal and its restriction C_1 on $\bar{r}anX$ is normal. Then $\bar{r}anX$ reduces T. Thus $T_2 = 0$. Since the pair (C_1, D_1) satisfies FPP, $C_1^\star X_1 = X_1 D_1^\star$. Finally $C^\star X = X D^\star$. □

Proposition 3.9. *The FP-Property holds for a (p,w)-hyponormal operator $C \in B(H)$ with $\ker C \subset \ker C^\star$ and a p-hyponormal operator $D^\star$.*

Proof. Consider $H = H_2 = (\ker C)^\perp \oplus (\ker C)$, $H = H_2 = (\ker D^\star)^\perp \oplus (\ker D^\star)$. From equation $CX = XD$, we get $C_1 X_1 = X_1 D_1$ and $C_1 X_2 = X_3 D_1 = 0$. Since C_1 and D_1 are one to one, $X_2 = X_3 = 0$, C_1 is a one to one p–hyponormal operator. Let $C_1 = U|C_1|$ be the polar decomposition of C_1. Equation above can be written as $U|C_1|X_1 = X_1 D_1$. Multiplying the two sides of this equation on the left by $|C_1|^{\frac{1}{2}}$, we get $|C_1|^{\frac{1}{2}} U |C_1|^{\frac{1}{2}} |C_1|^{\frac{1}{2}} X_1 = |C_1|^{\frac{1}{2}} X_1 D_1$ So $\bar{C}_1(|C_1|^{\frac{1}{2}} X_1) = (|C_1|^{\frac{1}{2}} X_1) D_1$. The Alugthe transform $\bar{C}_1$ of C_1 is $\frac{p}{2} - hyponormal$ and $D_1^\star$ is p-hyponormal. The pair $(\bar{C}_1, D_1)$ satisfies the FPP. Thus, $\bar{C}_1^\star(|C_1|^{\frac{1}{2}} X_1) = (|C_1|^{\frac{1}{2}} X_1) D_1^\star$. Consequently, restrictions $\bar{C}_1|_{\bar{r}an(|C_1|^{\frac{1}{2}} X_1)}$ and $D_1|_{\ker(|C_1|^{\frac{1}{2}})^\perp}$ are unitarily equivalent normal operators. Since the operator $|C_1|^{\frac{1}{2}}$ and X_1 are one to one, the operator $|C_1|^{\frac{1}{2}} X_1$ so is. Thus, $(\ker(|C_1|^{\frac{1}{2}} X_1))^\perp = [0]^\perp = (\ker X_1)^\perp = (\ker X)^\perp$ and $\bar{r}an(|\bar{C}_1|^{\frac{1}{2}} X_1) = (\ker(|C_1|^{\frac{1}{2}} X_1))^\perp = [0]^\perp = \bar{r}an(X_1) = \bar{r}an(X)$. Thus, $\bar{C}_1$ is a normal operator. The operator C_1 so is. Therefore, $\bar{r}anX$ reduces C_1 and $(\ker X_1)^\perp$ reduces $D_1^\star$. Since C_1 is normal and $D_1^\star$ is p-hyponormal, the FPP holds for the pair (C_1, D_1). Thus, $C_1^\star X_1 = X_1 D_1^\star$ and then $C^\star X = X D^\star$. The converse is true since pair (C, D) satisfies FP-Property. □

Proposition 3.10. *$\delta_{C,D} \subset \delta_{C^\star, D^\star}$ for a (p,w)-hyponormal operator C with $\ker C \subset \ker C^\star$ and a log-hyponormal operator $D^\star$.*

Proof. Let the restriction $C|_M$ be log-hyponormal. This property helps us to prove this theorem in the sequel. Let $D_1 = U|D_1|$ be of D_1. $C_1 X_1 = X_1 |D_1^\star| U$. Multiplying the two sides of this equation on the right by $|D_1^\star|^{\frac{1}{2}}$, we get, $C_1(X_1|D_1^\star|^{\frac{1}{2}}) = (X_1|D_1^\star|^{\frac{1}{2}})|D_1^\star|^{\frac{1}{2}} U |D_1^\star|^{\frac{1}{2}} = (X_1|D_1^\star|^{\frac{1}{2}})\bar{D}_1^\star$ C_1 is p-w-hyponormal and the Alugthe transform $\bar{D}_1^\star$ of $D_1^\star$ is $\frac{1}{2}$-hyponormal.By theorem above, the FPP holds for the pair $(C_1, \bar{D}_1^\star)$. Hence $C_1^\star(X_1|D_1^\star|^{\frac{1}{2}}) = (X_1|D_1^\star|^{\frac{1}{2}})\bar{D}_1^\star$. Further more, $C_1|_{\bar{r}an((X_1|D_1^\star|^{\frac{1}{2}}))}$ and $\bar{D}_1^\star|_{((X_1|D_1^\star|^{\frac{1}{2}}))^{\frac{1}{2}}}$ are unitarily equivalent normal operators. Since $|D_1^\star|^{\frac{1}{2}}$ and X_1 are one to one, the operator $(X_1|D_1^\star|^{\frac{1}{2}})$ so is. □

Theorem 3.11. *Let $C^\star, D \in B(H)$ be injective then $XC^\star = D^\star X$ for some positive operator X.*

Proof. Let decompositions $H = (\ker X)^\perp \oplus \ker X$ and $K = \bar{r}anX \oplus (\bar{r}anX)^\perp$ be considered. Then we have the following matrix representations: $C = \begin{pmatrix} C_1 & 0 \\ C_2 & C_3 \end{pmatrix}$,

$D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$ where $C_1^\star$ is p-hyponormal, D_1 is injective (p,k)-quasi-hyponormal and X_1 is injective with dense range. Therefore, we have $X_1C_1x = XCx = DXx = D_1X_1x$ for $x \in (\ker X)^\perp$. That is, $X_1C_1 = D_1X_1$ and hence C_1 and D_1 are normal and $X_1C_1^\star = D_1^\star X_1$ by the FPP. $(\ker X)^\perp$ and $\bar{r}anX$ reduces $C^\star$ and D respectively. Hence, we obtain $XC^\star = D^\star X$. Therefore we recapture a generalized FPP for p-hyponormal operators. □

Incase $n = 2$, $\delta_{C,D}(Y) = 0$. This is also the case with normal and subnormal operators C and $D^\star$. It can also be concluded that $\delta_{C^\star,D^\star}(Y) = 0$ from the FPP if C and $D^\star$ are normal and in general cases. The theorem can be proved for generalized scalar operators.

Lemma 3.12. *Orthogonality via FP-property suffices for a derivation and its adjoint.*

Proof. Let $\bar{C} \in B(\bar{H})$ and $\bar{D} \in Q(\bar{K})$ be positive and normal and $\bar{H} \subset H$, $\bar{K} \subset K$,

$$\bar{C} = \begin{pmatrix} C & C_1 \\ 0 & C \end{pmatrix}$$

and

$$\bar{D} = \begin{pmatrix} D & 0 \\ D_1 & D_2 \end{pmatrix}.$$

Define $\bar{Y} : \bar{K} \to \bar{H}$ by

$$\bar{Y} = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$$

It can be seen that

$$\delta^n_{\bar{C},\bar{D}}(\bar{Y}) = \begin{pmatrix} \delta^n_{\bar{C},\bar{D}}(\bar{Y}) & 0 \\ 0 & 0 \end{pmatrix}$$

for all n, hence

$$\lim_n \|\delta^n_{\bar{C},\bar{D}}(\bar{Y})\|^{\frac{1}{n}} = 0.$$

The rest follow trivially. □

Now we characterize operators C, D and N for which the pair (C, D) has property $FPP(\delta(N))$ and establish a relationship between the $FPP(\delta(N))$-property of the pair (C, D) and the range-kernel orthogonality of the operator $\delta_{C,D}$.

Theorem 3.13. *Let C, D, $N \in B(H)$ where N has the polar decomposition $N = U|N|$. Then the pair $(C, D) \in FPP(\delta(N))$ and*

(i). $[C, |N^\star|] = 0$.
(ii). $[D, |N|] = 0$.
(iii). $\delta_{C,D}(U) = 0$.

PROOF. If $N \in \ker(\delta_{C,D})$ and $(C, D) \in FPP(\delta(N))$, then

$$\delta_{CD}(N) = 0 = \delta_{C^\star D^\star}(N) \tag{7}$$

and so let

$$D : \ker N (= \ker U) \to \ker N.$$

Hence

$$\delta_{CD}(U) = 0.$$

Since $\overline{r}anN$ reduces C (by (i)) and $\ker^\perp N$ reduces D by (ii), it follows from $\delta_{C,D}(N) = 0$ that $\delta_{C_1,D_1}(N) = 0$ where $C_1 = C|_{\overline{r}anN}$

$$D_1 = D|_{\ker^\perp N}$$

and the quasi-affinity

$$N_1 : \ker^\perp N \to \overline{r}anN$$

is

$$N_1 x = Nx.$$

Let N_1 have the polar decomposition

$$N_1 = U_1|N_1|$$

then U_1 is a unitary and $|N_1|$ is a quasi-affinity. Clearly

$$[D_1, |D_1|] = 0. \tag{8}$$

Hence, $S\delta_{C_1D_2}(N_1) = 0$ implies that $\delta_{C_1D_2}(U_1) = 0$, that is

$$D_1 = U_1^\star.$$

Thus,

$$D_1^\star|N_1| = |N_1|D_1^\star$$

implies

$$U_1^\star C_1^\star U_1|N_1| = |N_1|D_1^\star$$

or

$$\delta_{C_1^\star D_2^\star}(N_1) = 0.$$

This implies that

$$\delta_{C^\star D^\star}(N) = 0. \tag{9}$$

□

4. Conclusion

Characterizations involving orthogonality of derivations induced by operators are an area with various applications with regard to the ever-dynamic technological advances. There are different types of orthogonality. Interesting results have come up where operators possessing given conditions are chosen for Range-Kernel orthogonality to be established. However, most of the results have focused on one type of orthogonality called the Birkhoff orthogonality. We have also herein considered the Birkhoff concept of orthogonality. Researchers have repeatedly posed the following question: Could there be a possibility for studying other types of orthogonality with respect to the range and the kernel of derivations apart from the Birkhoff orthogonality? This problem has been partially solved herein.

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(Kagali) Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, Box 210-40601, Bondo, Kenya
Email address: kelvinotae@yahoo.com

(Okelo) Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, Box 210-40601, Bondo-Kenya
Email address: bnyaare@yahoo.com

(Owino) Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, Box 210-40601, Bondo-Kenya
Email address: faithpatience47@gmail.com