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Some fixed point theorems in orbitally complete dq-metric space

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Abstract. This paper aims to obtain some new fixed point theorems in orbitally complete dislocated quasi-metric spaces for continuous self-mapping. This study opens new paths for research in dislocated quasi-metric spaces and enhances the ongoing progression of fixed point theory.

Keywords: Fixed point theorem, Orbitally complete dislocated quasi metric spaces, Dislocated quasi metric spaces

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1. Introduction

Banach [3] established a renowned fixed point theorem for contraction mappings in complete metric spaces. Widely known as the Banach Contraction Principle (BCP). It has many applications in differential equations, integral equations, engineering, etc. Rhoades [11] studied various contractive conditions that extend the classical Banach Contraction Principle, and Rhoades proposed numerous generalizations of contractive mappings, offering conditions under which fixed points exist, even when the mappings are not strict contractions. His work introduced new tools, such as partial ordering and altering distance functions, to study fixed point results more broadly. Khan et al. [8] introduced a new class of fixed point problems involving a control function, referred to as the altering distance function. This concept

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has since been extended to various forms, including two-variable, three-variable, multivalued, and fuzzy mappings. Zeyada et al [14] introduced the notion of a dislocated quasi-metric space and generalized the result of Hitzler and Seda in such spaces. Aage and Salunke [2] established additional fixed point results in dislocated and dislocated quasi-metric spaces. Samet et al. [13] introduced the concepts of α-admissible and $\alpha-\psi$-contractive mappings and established several fixed point theorems for these mappings. Their results are closely related to certain fixed point theorems in ordered metric spaces.

2. Preliminaries

In this section, we present some fundamental definitions and essential concepts that form the foundation of our study. These preliminaries will be useful in developing the results discussed in subsequent sections.

Definition 2.1. Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping satisfies the conditions:

(i) $d(x, x) = 0$ for all $x \in X$,
(ii) $d(x, y) = d(y, x) = 0 \implies x = y$,
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfies conditions (i), (ii), (iii) and (iv), it is called a metric on X. If d satisfies conditions (ii) and (iv), then it is referred to as a dislocated quasi-metric on X. The non-empty set X together with a dq-metric d is called a dislocated quasi-metric space, and it is denoted by (X, d).

Definition 2.2. [14] A sequence $\{x_n\}$ in dq-metric space (X, d) is called Cauchy if for given $\epsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that $\forall m, n \geq n_0$ implies $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$. i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$.

Definition 2.3. [14] A sequence $\{x_n\}$ in dislocated quasi-converges to x if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case x is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.

Lemma 2.1. *[14] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.*

Definition 2.4. [14] A dq-metric space (X, d) is considered complete if every Cauchy sequence within it is dq-convergent in X.

Definition 2.5. [14] Let (M, d_x) and (N, d_y) be a dq-metric spaces and $g : M \to N$ be a mapping. We say that g is continuous at $a \in M$ if each d_x-quasi convergent sequence $\{a_m\}$ to a in M, then $\{g(a_m)\}$ is d_y-quasi convergent to $g(a)$ in N.

Lemma 2.2. *[14] Let (X, d) be a complete dq-metric space and $f : X \to X$ be a contraction function, then $\{f^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.*

Definition 2.6. [14] Let (X, d) be a complete dq-metric space and let $f : X \to X$ be a continuous contraction function, then f has a unique fixed point.

Definition 2.7. [1] Let $f : X \to X$ be a mapping in a dq-metric space (X, d). The orbit of f at the point $x \in X$ is the set $O(x) = \{x, fx, f^2x, ...\}$. The closure of an orbit denoted by $\overline{O(x)}$, is the set of all $x \in X$ for which there exists a sequence in $O(x)$ that converges to x.

An orbit $O(x)$ of x in X is known as dq-bounded if there exists a positive integer k such that $d(x, y) \leq k$, $\forall x, y \in O(X)$ then constant k is called dq-bound. (X, d) is called f-orbitally complete if every Cauchy sequence in $O(x)$ converges to the point in X.

Remark 2.8. In a space (X, d), if the sequence $\{x_n\}$ converges to $x \in X$ (*i.e.* $\{x_n\} \to x$) and $\{x_{n_k}\}$ is sub-sequence of $\{x_n\}$ then $\{x_{n_k}\} \to x$ as $k \to \infty$.

Definition 2.9. [1] A mapping $f : X \to X$ in a dq- metric space (X, d) is called an orbitally contraction mapping if $\exists x_0 \in X$ and $\alpha \in [0, 1)$ such that

$$d(fa, fb) \leq \alpha d(a, b),$$

for all $a, b \in \overline{O(x_0)}$.

Definition 2.10. [1] A mapping $f : X \to X$ in a dq-metric space (X, d) is called an orbitally-Kannan mapping if $\exists x_0 \in X$ and $\alpha \in [0, \frac{1}{2})$ such that

$$d(fa, fb) \leq \alpha\{d(a, fa) + d(b, fb)\},$$

for all $a, b \in \overline{O(x_0)}$.

Lemma 2.3. *[1] Let (X, d) be a dq-metric space and $f : X \to X$ be an orbitally contraction mapping. If $x_0 \in X$ with $O(x_0)$ is bounded, then $\{f^n(x)\}$ is a Cauchy sequence for any x in $O(x_0)$.*

Theorem 2.4. *[1] Let $f : X \to X$ be a continuous orbitally contraction mapping in f-orbitally complete space (X, d). If there exists $x_0 \in X$ such that $O(x_0)$ is bounded, then f has a unique fixed point in $\overline{O(x_0)}$.*

Definition 2.11. [8] A function $\phi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function, if it satisfies the following conditions:

(i) $\phi(0) = 0$,
(ii) ϕ is continuous and monotonically non-decreasing.

3. Main Results

In this section, we establish several fundamental theorems within the framework of dislocated quasi-metric spaces. These results play a pivotal role in the development of our main findings and serve as a foundation for further extensions and generalizations. The theorems are presented and proved as follows.

Theorem 3.1. *Let $f : X \to X$ be a continuous mapping in f-orbitally complete dq-metric space (X, d) satisfies*

$$d(fx, fy) \leq \frac{h}{2}\left[d(x, fx) + d(y, fy)\right] \tag{1}$$

for all $x, y \in X$ with $x \neq y$, where $0 \leq h < \frac{1}{2}$. If there exists $x_0 \in X$ such that $O(x_0)$ is bounded, then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. Choose any $x_0 \in X$(fixed). We construct a sequence $x_n = f^n(x_0) = f(x_{n-1})$ within $O(x_0)$. Consider,

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d\left(f(x_{n-1}), f(x_n)\right) \\ &\leq \frac{h}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right], \\ &\leq \frac{h}{2} d(x_{n-1}, x_n) + \frac{h}{2} d(x_n, x_{n+1}). \end{aligned}$$

It implies that,

$$d\left(x_n, x_{n+1}\right) \leq \frac{h}{2-h}\, d\left(x_{n-1}, x_n\right).$$

That is,

$$d\left(x_n, x_{n+1}\right) \leq k\, d\left(x_{n-1}, x_n\right), \quad \text{where } k = \frac{h}{2-h}. \tag{2}$$

Similarly,

$$d\left(x_{n-1}, x_n\right) \leq \frac{h}{2-h}\, d\left(x_{n-2}, x_{n-1}\right).$$

That is,

$$d\left(x_{n-1}, x_n\right) \leq k\, d\left(x_{n-2}, x_{n-1}\right), \quad \text{where } k = \tfrac{h}{2-h}. \tag{3}$$

Using (2) and (3), we get

$$d\left(x_n, x_{n+1}\right) \leq k^2\, d\left(x_{n-2}, x_{n-1}\right).$$

Continuing in this way, we get

$$d\left(x_n, x_{n+1}\right) \leq k^n\, d\left(x_0, x_1\right). \tag{4}$$

Therefore, $k^n \to 0$ as $n \to \infty$. Thus, $d\left(x_n, x_{n+1}\right) \to 0$ as $n \to \infty$. Now, we will show $\{f^n(x_0)\}$ is Cauchy sequence. i.e. $\{x_n\}$ is Cauchy sequence. For any sufficiently large natural number $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ with $m, n \geq N$. Without loss of generality, we may assume that $m < n$. Consider

$$\begin{aligned} d\left(x_m, x_n\right) &\leq d\left(x_m, x_{m+1}\right) + d\left(x_{m+1}, x_{m+2}\right) + \cdots + d\left(x_{n-1}, x_n\right) \\ &\leq k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_0, x_1) \\ &= \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) d(x_0, x_1) \\ &\leq \left(k^m + k^{m+1} + \cdots + k^{n-1} + k^n + \cdots\right) d(x_0, x_1) \\ &= k^m \left(1 + k + \cdots + k^{n-m} + \cdots\right) d(x_0, x_1) \\ &= \frac{k^m}{1-k} d(x_0, x_1). \end{aligned}$$

That is,

$$d\left(x_m, x_n\right) \leq \frac{k^m}{1-k}\, d(x_0, x_1).$$

Since $k < 1$, $k^m \to 0$ as $m \to \infty$. Thus, $d\left(x_m, x_n\right) \to 0$ as $m, n \to \infty$. Hence $\{x_n\} = \{f^n(x_0)\}$ is Cauchy in $O(x_0)$. Since X is f-orbitally complete, $\{f^n(x_0)\}$ dq-converges to some $q \in \overline{O(x_0)}$. By continuity of f and since $\{f^{n+1}(x_0)\}$ is a subsequence of $\{f^n(x_0)\}$, we obtain

$$f(q) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{(n+1)}(x_0)) = q.$$

Thus, q is a fixed point of f and $q \in \overline{O(x_0)}$. Now, suppose $p \in \overline{O(x_0)}$ is another fixed point of f. Then $fp = p$ and $fq = q$. Consider

$$\begin{aligned} d(p,p) &= d(fp, fp) \\ &\leq \frac{h}{2}[d(p, fp) + d(p, fp)] \\ &= \frac{h}{2}[d(p,p) + d(p,p)] \\ &= h\, d(p,p). \end{aligned}$$

It gives that, $d(p,p) \leq h\, d(p,p)$. Since $0 \leq h < 1$ (in particular $1 - h > 0$) and $d(p,p) \geq 0$, the inequality $d(p,p) \leq h\, d(p,p)$ forces $d(p,p) = 0$. Similarly, we can show $d(q,q) = 0$. Now,

$$\begin{aligned} d(p,q) &= d(fp, fq) \\ &\leq \frac{h}{2}[d(p, fp) + d(q, fq)] \\ &= \frac{h}{2}[d(p,p) + d(q,q)] \\ &= 0. \end{aligned}$$

It gives that, $d(p,q) \leq 0$. But $d(p,q) \geq 0$. Thus, we have $d(p,q) = 0$. Similarly, we can show $d(q,p) = 0$. Therefore $d(p,q) = d(q,p) = 0$. Hence $p = q$. Therefore f has a unique fixed point in $\overline{O(x_0)}$. □

Theorem 3.2. *Let $f : X \to X$ be a continuous mapping in f-orbitally complete dq-metric space (X,d). If there exists $x_0 \in X$ such that $O(x_0)$ is bounded and satisfies for all $x, y \in X$,*

$$d\left(fx, fy\right) \leq h \max\left\{d\left(x, fx\right), d\left(y, fy\right)\right\},$$

where $0 \leq h < 1$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. Choose any $x_0 \in X$(fixed). We construct a sequence $x_n = f^n(x_0) = f(x_{n-1})$ within $O(x_0)$. Consider,

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d\left(f(x_{n-1}), f(x_n)\right) \\ &\leq h \max\left\{d\left(x_{n-1}, f(x_{n-1})\right), d\left(x_n, f(x_n)\right)\right\} \\ &= h \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \end{aligned}$$

That is,

$$d\left(x_n, x_{n+1}\right) \leq h \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \tag{5}$$

Case 1: If $\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} = d(x_n, x_{n+1})$. From inequality (5), we get

$$d(x_n, x_{n+1}) \leq h d(x_n, x_{n+1}),$$

which is contradiction, as $h \in [0, 1)$.

Case 2: If $\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} = d(x_{n-1}, x_n)$. From inequality (5), we get

$$d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n). \tag{6}$$

Similarly,

$$d(x_{n-1}, x_n) \leq h d(x_{n-2}, x_{n-1}). \tag{7}$$

Using inequality (6) and (7), we get

$$d(x_n, x_{n+1}) \leq h^2 d(x_{n-2}, x_{n-1}).$$

Continuing in this way, we get

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1). \tag{8}$$

Since $0 \leq h < 1$, therefore $\lim\limits_{n\to\infty} h^n \to 0$, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \leq \lim_{n\to\infty} h^n d(x_0, x_1) = 0.$$

Now, we will show $\{f^n(x_0)\}$ is Cauchy sequence. i.e. $\{x_n\}$ is Cauchy sequence. For any sufficiently large natural number $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ with $m, n \geq N$. Without loss of generality, we may assume that $m < n$. Consider

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &\leq h^m d(x_0, x_1) + h^{m+1} d(x_0, x_1) + h^{m+2} d(x_0, x_1) + \cdots + h^{n-1} d(x_0, x_1) \\ &= h^m (1 + h + h^2 + \cdots h^{n-m-1}) d(x_0, x_1) \\ &\leq h^m (1 + h + h^2 + \cdots) d(x_0, x_1), \quad \text{(since } h \geq 0) \\ &= \frac{h^m}{1-h} d(x_0, x_1). \end{aligned}$$

Thus,

$$d(x_m, x_n) \leq \frac{h^m}{1-h} d(x_0, x_1).$$

As $h \in [0, 1)$ and as $m, n \to \infty$, we have $d(x_m, x_n) \to 0$. Hence, $\{x_n\} = \{f^n(x_0)\}$ is Cauchy in $O(x_0)$. But, X is f-orbitally complete, so $\{f^n(x_0)\}$ is dq-converges to q, for some $q \in \overline{O(x_0)}$. Thus, we have

$$f(q) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{(n+1)}(x_0)) = q,$$

since f is continuous, and $\{f^{n+1}(x_0)\}$ is a subsequence of $\{f^n(x_0)\}$, it follows that $f(q) = q$; that is, q is a fixed point of f. Suppose p is another fixed point of f. Consider,

$$\begin{aligned} d(p, p) &= d(fp, fp) \\ &\leq h \max\{d(p, fp), d(p, fp)\} \\ &= h d(p, p). \end{aligned}$$

It implies that, $d(p, p) \leq h d(p, p)$, as $h \in [0, 1)$, we get $d(p, p) = 0$. Similarly, we can show $d(q, q) = 0$. Now,

$$\begin{aligned} d(p, q) &= d(fp, fq) \\ &\leq h \max\{d(p, fp), d(q, fq)\} \\ &= h \max\{d(p, p), d(q, q)\}. \end{aligned}$$

It implies that, $d(p, q) \leq h \max\{d(p, p), d(q, q)\}$. This gives us that $d(p, q) \leq 0$. Since, $d(p, q) \geq 0$. Therefore $d(p, q) = 0$. Also

$$\begin{aligned} d(q, p) &= d(fq, fp) \\ &\leq h \max\{d(q, fq), d(p, fp)\} \\ &= h \max\{d(q, q), d(p, p)\}. \end{aligned}$$

That is, $d(p,q) \leq \max\{d(q,q), d(p,p)\}$ this gives that $d(q,p) \leq 0$ and hence, $d(q,p) = 0$. Therefore, $d(p,q) = d(q,p) = 0 \implies p = q$. Hence, f has a unique fixed point in $\overline{O(x_0)}$. □

Theorem 3.3. *Let $f : X \to X$ be a continuous mapping in f-orbitally complete dq-metric space (X,d) satisfying*

$$d(fx, fy) \leq h \max\{d(x, fy), d(y, fx)\} \tag{9}$$

for all $x, y \in X$ with $x \neq y$, where $0 \leq h < 1$. If exists $x_0 \in X$ such that $O(x_0)$ is bounded, then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. Choose any $x_0 \in x$ (fixed). Since $f : X \to X$, so $fx_0 \in X$. Therefore, we construct a sequence $x_n = f^n(x_0)$, $n \geq 1$. Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq h \max \ \{d(x_{n-1}, fx_n), d(x_n, fx_{n-1})\} \\ &= h \max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}. \end{aligned}$$

Case 1. If $\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} = d(x_n, x_n)$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq hd(x_n, x_n) \\ &\leq h\{d(x_n, x_{n+1}) + d(x_{n+1}, x_n)\}. \end{aligned}$$

It implies that,

$$d(x_n, x_{n+1}) \leq \frac{h}{1-h} d(x_{n+1}, x_n).$$

That is,

$$d(x_n, x_{n+1}) \leq kd(x_{n+1}, x_n), \quad \text{where } k = \frac{h}{1-h}. \tag{10}$$

Similarly, we can show,

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n+1}). \tag{11}$$

From (10) and (11), we have

$$d(x_n, x_{n+1}) \leq k^2 d(x_n, x_{n+1}).$$

It is a contradiction, since $k < 1$.
Case 2. If $\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} = d(x_{n-1}, x_{n+1})$. Then,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq hd(x_{n-1}, x_{n+1}) \\ &\leq h\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \end{aligned}$$

It implies that,

$$d(x_n, x_{n+1}) \leq \frac{h}{1-h} d(x_{n-1}, x_n), \quad \text{where } k = \frac{h}{1-h} < 1.$$

Thus,

$$d\left(x_n, x_{n+1}\right) \leq kd\left(x_{n-1}, x_n\right).$$

Similarly, we can show

$$d\left(x_{n-1}, x_n\right) \leq kd\left(x_{n-2}, x_{n-1}\right).$$

Using above two inequalities, we have

$$d\left(x_n, x_{n+1}\right) \leq k^2 d\left(x_{n-2}, x_{n-1}\right).$$

Continuing in this way, we obtain

$$d\left(x_n, x_{n+1}\right) \leq k^n d\left(x_0, x_1\right), \quad \text{where } k < 1.$$

Since, $0 \leq k < 1$, therefore $\lim\limits_{n\to\infty} k^n \to 0$, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \leq \lim_{n\to\infty} k^n d(x_0, x_1) = 0.$$

Now, we will show $\{f^n(x_0)\}$ is Cauchy sequence. i.e. $\{x_n\}$ is Cauchy sequence. For any sufficiently large natural number $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ with $m, n \geq N$. Without loss of generality, we may assume that $m < n$. Consider,

$$\begin{aligned} d\left(x_m, x_n\right) &\leq d\left(x_m, x_{m+1}\right) + d\left(x_{m+1}, x_{m+2}\right) + \cdots + d\left(x_{n-1}, x_n\right) \\ &\leq k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_0, x_1) \\ &= \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) d(x_0, x_1) \\ &\leq \left(k^m + k^{m+1} + \cdots + k^{n-1} + k^n + \cdots\right) d(x_0, x_1) \\ &= k^m \left(1 + k + \cdots + k^{n-m} + \cdots\right) d(x_0, x_1) \\ &= \frac{k^m}{1-k} d(x_0, x_1). \end{aligned}$$

That is,

$$d\left(x_m, x_n\right) \leq \frac{k^m}{1-k} d(x_0, x_1).$$

Since $k < 1$, $k^m \to 0$ as $m \to \infty$. Thus, $d\left(x_m, x_n\right) \to 0$ as $m, n \to \infty$. Hence $\{x_n\} = \{f^n(x_0)\}$ is Cauchy in $O(x_0)$. Since X is f-orbitally complete, $\{f^n(x_0)\}$ dq-converges to some $q \in \overline{O(x_0)}$. By continuity of f and since $\{f^{n+1}(x_0)\}$ is a subsequence of $\{f^n(x_0)\}$, we obtain

$$f(q) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{(n+1)}(x_0)) = q.$$

Thus, q is a fixed point of f and $q \in \overline{O(x_0)}$. Now, suppose $p \in \overline{O(x_0)}$ is another fixed point of f. Then $fp = p$ and $fq = q$. Consider

$$\begin{aligned} d(q,q) &= d(fq,fq) \\ &\leq h\max\{d(q,fq),d(q,fq)\} \\ \Rightarrow \quad d(q,q) &\leq hd(q,q) \end{aligned}$$

as $h \in [0,1)$, we get $d(q,q) \leq 0$. But $d(q,q) = 0$. Here $d(q,q) = 0$. Similarly, we can show $d(p,p) = 0$. Now consider,

$$\begin{aligned} d(p,q) &= d(fp,fq) \\ &\leq h\max\{d(p,fq),d(q,fp)\} \\ &= h\max\{d(p,q),d(q,p)\}. \end{aligned}$$

That is,

$$d(p,q) \leq h\max\{d(p,q),d(q,p)\}. \tag{12}$$

There are two cases.

Case 1: If $\max\{d(p,q),d(q,p)\} = d(q,p)$. From (12), we get

$$d(p,q) \leq hd(q,p).$$

Also,

$$\begin{aligned} d(q,p) &= d(fq,fp) \\ &\leq h\max\{d(q,fp),d(p,fq)\} \\ &= h\max\{d(q,p),d(p,q)\}. \end{aligned}$$

Thus, $d(q,p) \leq h\max\{d(q,p),d(p,q)\}$. As per assumption, we have

$$d(q,p) \leq hd(q,p) \tag{13}$$

It is a contradiction, since $h < 1$.

Case 2: If $\max\{d(p,q),d(q,p)\} = d(p,q)$. From inequality (12), we get

$$d(p,q) \leq hd(p,q).$$

As $0 \leq h < 1$, it implies that $d(p,q) \leq 0$. But $d(p,q) \geq 0$. Hence, $d(p,q) = 0$. Similarly, we can show $d(q,p) = 0$. Hence, $d(p,q) = d(q,p) = 0$ and hence $p = q$. Thus, p is a unique fixed point of f. □

Theorem 3.4. *Let $f : X \to X$ be a continuous mapping in f-orbitally complete dq-metric space (X,d) satisfying*

$$d(fx,fy) \leq h\max\{d(x,fx),d(y,fy),d(x,y))\} \tag{14}$$

for all $x,y \in X, x \neq y$, where $0 \leq h < 1$. If there exists $x_0 \in X$ such that $O(x_0)$ is bounded, then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. Choose any $x_0 \in X$ (fixed). Construct the sequence $x_n = f^n(x_0) = f(x_{n-1})$ within $O(x_0)$. Consider,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\leq h \max\{d(x_{n-1}, f(x_{n-1})), d(x_n, f(x_n)), d(x_{n-1}, x_n)\} \\ &= h \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= h \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{15}$$

Case 1: If

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}),$$

then, from inequality (15), we obtain

$$d(x_n, x_{n+1}) \leq h\, d(x_n, x_{n+1}),$$

which is a contradiction, since $0 \leq h < 1$.

Case 2: If

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n),$$

then, from (15), we have

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n). \tag{16}$$

Similarly,

$$d(x_{n-1}, x_n) \leq h\, d(x_{n-2}, x_{n-1}). \tag{17}$$

Using (16) and (17), we get

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n) \leq h^2\, d(x_{n-2}, x_{n-1}).$$

Continuing in this way, we obtain

$$d(x_n, x_{n+1}) \leq h^n\, d(x_0, x_1). \tag{18}$$

Since $h < 1$, $h^n \to 0$ as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we will show $\{f^n(x_0)\}$ is Cauchy sequence. i.e. $\{x_n\}$ is Cauchy sequence. For any sufficiently large natural number $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ with $m, n \geq N$. Without loss of generality, we may assume that $m < n$. Consider,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1) \left(h^m + h^{m+1} + \cdots + h^{n-1}\right) \\ &\leq d(x_0, x_1) \left(h^m + h^{m+1} + \cdots + h^{n-1} + h^n + \cdots\right) \\ &= \frac{h^m}{1-h}\, d(x_0, x_1). \end{aligned}$$

Thus

$$d(x_m, x_n) \le \frac{h^m}{1-h}\, d(x_0, x_1).$$

Clearly, $h^m \to 0$, as $m \to \infty$, because $0 \le h < 1$. Therefore,

$$\lim_{m\to\infty, n\to\infty} d(x_m, x_n) = 0.$$

Hence, $\{x_n\} = \{f^n(x_0)\}$ is Cauchy in $O(x_0)$. Hence, $\{x_n\} = \{f^n(x_0)\}$ is Cauchy in $O(x_0)$. But, X is f-orbitally complete, so $\{f^n(x_0)\}$ is dq-converges to q, for some $q \in \overline{O(x_0)}$. Thus, we have

$$f(q) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{(n+1)}(x_0)) = q.$$

Thus, q is a fixed point of f and $q \in \overline{O(x_0)}$. Suppose that $r \in \overline{O(x_0)}$ and r is another fixed point of $f \in \overline{O(x_0)}$. Now,

$$\begin{aligned} d(r,r) &= d(fr, fr) \\ &\le h \max\{d(r, fr), d(r, fr), d(r,r)\} \\ &= h \max\{d(r,r), d(r,r), d(r,r)\}, \\ \implies d(r,r) &\le h\, d(r,r), \end{aligned}$$

It implies that $d(r,r) \le 0$. But $d(r,r) \ge 0$. Hence $d(r,r) = 0$. Similarly, $d(q,q) = 0$. Now,

$$\begin{aligned} d(q,r) &= d(fq, fr) \\ &\le h \max\{d(q, fq), d(r, fr), d(q,r)\} \\ &= h \max\{d(q,q), d(r,r), d(q,r)\} \\ &= h \max\{0, d(q,r)\} \\ &= hd(q,r). \end{aligned}$$

As $h < 1$, the inequality $d(q,r) \le hd(q,r)$ implies $d(q,r) = 0$. Hence $q = r$. Hence f has unique fixed point in $\overline{O(x_0)}$. □

Theorem 3.5. *Let $f : X \to X$ be a continuous mapping in f-orbitally complete dislocated quasi-metric space (X, d) satisfies*

$$\psi(d(fx, fy)) \le \psi(d(x,y)) - \phi(d(x,y)), \tag{19}$$

where, $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous, non-decreasing functions with $\psi(t_1) \le \psi(t_2)$ implies that $t_1 \le t_2$ and $\psi(t) = 0 = \phi(t)$ iff $t = 0$. If there exists $x_0 \in X$ such that $O(x_0)$ is bounded, then f has a unique fixed point in $\overline{O(x_o)}$.

Proof. Choose any $x_0 \in X$(fixed), we construct the sequence $\{f^n(x_0)\}$ as $x_n = f^n(x_0) = f(x_{n-1})$ within $O(x_0)$. Here $x_1 = f(x_0), x_2 = f(x_1)$ and in general it can be written as $x_n = f(x_{n-1})$. Put $x = x_{n-1}$ and $y = x_n$ in the inequality (19), we get

$$\psi\left(d\left(x_n, x_{n+1}\right)\right) \leq \psi\left(d\left(x_{n-1}, x_n\right)\right) - \phi\left(d\left(x_{n-1}, x_n\right)\right). \tag{20}$$

As $\phi(t) \geq 0, \forall t \in \mathbb{R}^+ \implies \psi\left(d\left(x_n, x_{n+1}\right)\right) \leq \psi\left(d\left(x_{n-1}, x_n\right)\right)$. Therefore by using property of ψ, we get

$$d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right).$$

It shows that $\{d(x_n, x_{n+1})\}$ is monotonically decreasing. Moreover, every $d(x_n, x_{n+1}) \geq 0, \forall n \in \mathbb{N}$. Thus it is monotonically decreasing sequence, which is bounded below and hence it is convergent. Therefore, there is $u \in [0, \infty)$ such that $\{d(x_n, x_m)\} \to u$ as $n \to \infty$. If we assume $u > 0$ then letting $n \to \infty$ in (20), we get

$$\psi(u) \leq \psi(u) - \phi(u).$$

This implies

$$\psi(u) < \psi(u), \quad \text{since} \quad \phi(u) > 0.$$

Thus, it is contradiction. Hence, $u = 0$. Therefore, $d(x_n, x_{n+1})$ approaches to 0 as $n \to \infty$. Now, we have to show $\{f^n(x_0)\}$ is Cauchy sequence. i.e. to show $\{x_n\}$ is Cauchy sequence. Let us assume that $\{x_n\}$ is not Cauchy sequence. Therefore, for given $\epsilon > 0$, we can find subsequences $\{x_{p(k)}\}$ and $\{x_{q(k)}\}$ of $\{x_n\}$ with $q(k) > p(k) > k$ such that

$$d\left(x_{p(k)}, x_{q(k)}\right) \geq \epsilon \tag{21}$$

corresponding to $p(k)$, we can choose $q(k)$ in such a way that, it is the smallest integer with $q(k)$ and satisfy inequality (21), and we have

$$d\left(x_{p(k)}, x_{q(k)-1}\right) < \epsilon.$$

Also,

$$\begin{aligned}\epsilon &\leq d\left(x_{p(k)}, x_{q(k)}\right)\\ &\leq d\left(x_{p(k)}, x_{q(k)-1}\right) + d\left(x_{q(k)-1}, x_{q(k)}\right)\\ &< \epsilon + d\left(x_{q(k)-1}, x_{q(k)}\right),\end{aligned} \tag{22}$$

as $k \to \infty$ inequality (22) becomes,

$$\epsilon \leq \lim_{k\to\infty} d\left(x_{p(k)}, x_{q(k)}\right) \leq \epsilon.$$

Therefore,

$$\lim_{k\to\infty} d\left(x_{p(k)}, x_{q(k)}\right) = \epsilon.$$

Now,

$$d\left(x_{p(k)}, x_{q(k)}\right) \leq d\left(x_{p(k)}, x_{p(k)-1}\right) + d\left(x_{p(k)-1}, x_{q(k)-1}\right) + d\left(x_{q(k)-1}, x_{q(k)}\right) \tag{23}$$

$$d\left(x_{p(k)-1}, x_{q(k)-1}\right) \leq d\left(x_{p(k)-1}, x_{p(k)}\right) + d\left(x_{p(k)}, x_{q(k)}\right) + d\left(x_{q(k)}, x_{q(k)-1}\right), \tag{24}$$

as $k \to \infty$ and using $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ above inequality becomes,

$$\epsilon \le \lim_{k\to\infty} d\left(x_{p(k)-1}, x_{q(k)-1}\right) \le \epsilon.$$

Therefore,

$$\lim_{k\to\infty} d\left(x_{p(k)-1}, x_{q(k)-1}\right) = \epsilon.$$

Put $x = x_{p(k)-1}$ and $y = x_{q(k)-1}$ in (19), we get

$$\psi\left(d\left(x_{p(k)}, x_{q(k)}\right)\right) \le \psi\left(d\left(x_{p(k)-1}, x_{q(k)-1}\right)\right) - \phi\left(d\left(x_{p(k)-1}, x_{q(k)-1}\right)\right),$$

as $k \to \infty$ above inequality becomes,

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon),$$

$\epsilon = 0$ which is contradiction. Hence, $\{x_n\}$ is Cauchy sequence. i.e. $\{f^n(x_0)\}$ is a Cauchy sequence in $O(x_0)$. As X is f-orbitally complete. So $f^n(x_0)$ is dq-converges to q for some $q \in \overline{O(x_0)}$. We have,

$$f(q) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{(n+1)}(x_0)) = q.$$

Since, f is continuous and $\{f^{n+1}(x_0)\}$ is a subsequence of $\{f^n(x_0)\}$. We have, $f(q) = q$ and $q \in \overline{O(x_0)}$. Let q_1 and q_2 be two fixed points of f i.e. $f(q_1) = q_1$ and $f(q_2) = q_2$. By using inequality (19), we get

$$\psi\left(d\left(f\left(q_1\right), f\left(q_2\right)\right)\right) \le \psi\left(d\left(q_1, q_2\right)\right) - \phi\left(d\left(q_1, q_2\right)\right)$$
$$\psi\left(d\left(q_1, q_2\right)\right) \le \psi\left(d\left(q_1, q_2\right)\right) - \phi\left(d\left(q_1, q_2\right)\right),$$

this gives us,

$$\psi\left(d\left(q_1, q_2\right)\right) \le \psi\left(d\left(q_1, q_2\right)\right).$$

Thus, $d(q_1, q_2) = 0$. Therefore, $d(q_1, q_2) = 0 \implies q_1 = q_2$. Hence, f has a unique fixed point. □

Theorem 3.6. *Let $f : X \to X$ be a continuous mapping in f-orbitally complete dislocated quasi-metric space (X, d) satisfies*

$$\psi\left(d\left(fx, fy\right)\right) \le \psi\left(M(x, y)\right) - \phi\left(M(x, y)\right), \tag{25}$$

where, $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\}$ and $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and non-decreasing functions with $\psi(t_1) \le \psi(t_2)$ implies that $t_1 \le t_2$ and $\psi(t) = 0 = \phi(t)$ iff $t = 0$. If there exists $x_0 \in X$ such that $O(x_0)$ is bounded, then f has a unique fixed point in $\overline{O(x_o)}$.

Proof. Choose any $x_0 \in X$(fixed), we construct the sequence $\{f^n(x_0)\}$ as $x_n = f^n(x_0) = f(x_{n-1})$ within $O(x_0)$. Here $x_1 = f(x_0), x_2 = f(x_1)$ and in general it can be written as $x_n = f(x_{n-1})$. Put $x = x_{n-1}$ and $y = x_n$ in the inequality (25), we get

$$\psi\left(d\left(x_n, x_{n+1}\right)\right) \le \psi\left(M\left(x_{n-1}, x_n\right)\right) - \phi\left(M\left(x_{n-1}, x_n\right)\right), \tag{26}$$

where,

$$\begin{aligned} M\left(x_{n-1}, x_n\right) &= \max\left\{d\left(x_{n-1}, x_n\right), d\left(x_{n-1}, f\left(x_{n-1}\right)\right), d\left(x_n, f\left(x_n\right)\right)\right\} \\ &= \max\left\{d\left(x_{n-1}, x_n\right), d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\right\} \\ &= \max\left\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\right\}, \end{aligned} \tag{27}$$

Case 1: If $\max\left\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\right\} = d\left(x_n, x_{n+1}\right)$. From inequality (27), we get

$$\psi\left(d\left(x_n, x_{n+1}\right)\right) \leq \psi\left(d\left(x_n, x_{n+1}\right)\right) - \phi\left(d\left(x_n, x_{n+1}\right)\right),$$

If $x_n = x_{n+1} \implies x_n = f(x_n)$. Then x_n is a fixed point of f. If $x_n \neq x_{n+1}$, $\phi\left(d\left(x_n, x_{n+1}\right)\right) > 0 \implies \psi\left(d\left(x_n, x_{n+1}\right)\right) < \psi\left(d\left(x_n, x_{n+1}\right)\right)$, which is contradiction.

Case 2: If $\max\left\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\right\} = d\left(x_{n-1}, x_n\right)$. From inequality (27), we get

$$\psi\left(d\left(x_n, x_{n+1}\right)\right) \leq \psi\left(d\left(x_{n-1}, x_n\right)\right) - \phi\left(d\left(x_{n-1}, x_n\right)\right). \tag{28}$$

As $\phi(t) \geq 0, \forall t \in [0, \infty) \implies \psi\left(d\left(x_n, x_{n+1}\right)\right) \leq \psi\left(d\left(x_{n-1}, x_n\right)\right)$. Therefore, using $\psi(t_1) \leq \psi(t_2) \implies t_1 \leq t_2$, we get

$$d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right),$$

this gives us that $\{d(x_n, x_{n+1})\}$ is monotonically decreasing. Moreover, every $d(x_n, x_{n+1}) \geq 0, \forall n \in \mathbb{N}$ and it is bounded below, hence it is convergent. Therefore, there is $u \in [0, \infty)$ such that $\{d(x_n, x_m)\} \to u$ as $n \to \infty$. If we assume $u > 0$ then letting $n \to \infty$ in (28), we get

$$\psi(u) \leq \psi(u) - \phi(u).$$

This implies,

$$\psi(u) < \psi(u), \quad \text{since} \quad \phi(u) > 0.$$

Thus, it is contradiction. Hence, $u = 0$. Hence, $d(x_n, x_{n+1})$ approaches 0 as $n \to \infty$. Now, we have to show that $\{f^n(x_0)\}$ is Cauchy sequence. i.e. to show $\{x_n\}$ is Cauchy sequence. Let us assume that $\{x_n\}$ is not Cauchy sequence. Therefore, for given $\epsilon > 0$, we can find subsequences $\left\{x_{p(k)}\right\}$ and $\left\{x_{q(k)}\right\}$ of $\{x_n\}$ with $q(k) > p(k) > k$ such that

$$d\left(x_{p(k)}, x_{q(k)}\right) \geq \epsilon \tag{29}$$

corresponding to $p(k)$, we can choose $q(k)$ in such a way that, it is the smallest integer with $q(k)$ which satisfy inequality (29) and we get

$$d\left(x_{p(k)}, x_{q(k)-1}\right) < \epsilon.$$

Also,

$$\begin{aligned}\epsilon &\leq d\left(x_{p(k)}, x_{q(k)}\right)\\ &\leq d\left(x_{p(k)}, x_{q(k)-1}\right) + d\left(x_{q(k)-1}, x_{q(k)}\right)\\ &< \epsilon + d\left(x_{q(k)-1}, x_{q(k)}\right), \end{aligned} \tag{30}$$

as $k \to \infty$ inequality (30) becomes,

$$\epsilon \leq \lim_{k\to\infty} d\left(x_{p(k)}, x_{q(k)}\right) \leq \epsilon.$$

Therefore,

$$\lim_{k\to\infty} d\left(x_{p(k)}, x_{q(k)}\right) = \epsilon.$$

Now,

$$d\left(x_{p(k)}, x_{q(k)}\right) \leq d\left(x_{p(k)}, x_{p(k)-1}\right) + d\left(x_{p(k)-1}, x_{q(k)-1}\right) + d\left(x_{q(k)-1}, x_{q(k)}\right) \tag{31}$$

$$d\left(x_{p(k)-1}, x_{q(k)-1}\right) \leq d\left(x_{p(k)-1}, x_{p(k)}\right) + d\left(x_{p(k)}, x_{q(k)}\right) + d\left(x_{q(k)}, x_{q(k)-1}\right), \tag{32}$$

as $k \to \infty$ and using $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ above inequality becomes,

$$\epsilon \leq \lim_{k\to\infty} d\left(x_{p(k)-1}, x_{q(k)-1}\right) \leq \epsilon.$$

Therefore,

$$\lim_{k\to\infty} d\left(x_{p(k)-1}, x_{q(k)-1}\right) = \epsilon.$$

Put $x = x_{p(k)-1}$ and $y = x_{q(k)-1}$ in (25), we get

$$\psi(d(x_{p(k)}, x_{q(k)})t) \leq \psi(M(x_{p(k)-1}, x_{q(k)-1})) - \phi(M(x_{p(k)-1}, x_{q(k)-1})), \tag{33}$$

where, $M(x_{p(k)-1}, x_{q(k)-1}) =$

$$\begin{aligned}&\max\{d(x_{p(k)-1}, x_{q(k)-1}), d(x_{p(k)-1}, f(x_{p(k)-1})), d(x_{q(k)-1}, f(x_{q(k)-1}))\}\\ &= \max\{d(x_{p(k)-1}, x_{q(k)-1}), d(x_{p(k)-1}, x_{p(k)}), d(x_{q(k)-1}, x_{q(k)})\},\end{aligned} \tag{34}$$

Case 1: If $\max(d(x_{p(k)-1}, x_{q(k)-1}), d(x_{p(k)-1}, x_{p(k)}), d(x_{q(k)-1}, x_{q(k)})) = d(x_{p(k)-1}, x_{q(k)-1})$. From inequality (34), we get

$$\psi(d(x_{p(k)}, x_{q(k)})) \leq \psi(d(x_{p(k)-1}, x_{q(k)-1})) - \phi(d(x_{p(k)-1}, x_{q(k)-1})), \tag{35}$$

as $k \to \infty$ inequality (35) becomes,

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is contradiction if $\epsilon > 0$, then $\epsilon = 0$.

Case 2: If $\max(d(x_{p(k)-1}, x_{q(k)-1}), d(x_{p(k)-1}, x_{p(k)}), d(x_{q(k)-1}, x_{q(k)})) = d(x_{p(k)-1}, x_{p(k)})$. From inequality (34), we get

$$\psi(d(x_{p(k)}, x_{q(k)})) \leq \psi(d(x_{p(k)}, x_{q(k)})) - \phi(d(x_{p(k)}, x_{q(k)})), \tag{36}$$

as $k \to \infty$ inequality (36) becomes,

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is contradiction if $\epsilon > 0$, then $\epsilon = 0$.

Case 3: If $\max(d(x_{p(k)-1}, x_{q(k)-1}), d(x_{p(k)-1}, x_{p(k)}), d(x_{q(k)-1}, x_{q(k)})) = d(x_{q(k)-1}, x_{q(k)})$. From inequality (34), we get

$$\psi(d(x_{p(k)}, x_{q(k)})) \leq \psi(d(x_{q(k)-1}, x_{q(k)})) - \phi(d(x_{q(k)-1}, x_{q(k)})), \tag{37}$$

as $k \to \infty$ inequality (37) becomes,

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is contradiction if $\epsilon > 0$, then $\epsilon = 0$. Hence $\{x_n\} = \{f^n(x_0)\}$ is Cauchy in $O(x_0)$. Since X is f-orbitally complete, $\{f^n(x_0)\}$ dq-converges to some $q \in \overline{O(x_0)}$. By continuity of f and since $\{f^{n+1}(x_0)\}$ is a subsequence of $\{f^n(x_0)\}$, we obtain

$$f(q) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{(n+1)}(x_0)) = q.$$

Thus, q is a fixed point of f and $q \in \overline{O(x_0)}$. Let q_1 and q_2 be two fixed points of f i.e. $f(q_1) = q_1$ and $f(q_2) = q_2$. By using inequality (25), we get

$$\begin{aligned} \psi(d(fq_1, fq_2)) &\leq \psi(M(q_1, q_2)) - \phi(M(q_1, q_2)) \\ \psi(d(q_1, q_2)) &\leq \psi(M(q_1, q_2)) - \phi(M(q_1, q_2)), \end{aligned} \tag{38}$$

where,

$$\begin{aligned} M(q_1, q_2) &= \max(d(q_1, q_2), d(q_1, f(q_1)), d(q_2, f(q_2))) \\ &= \max(d(q_1, q_2), d(q_1, q_1), d(q_2, q_2)), \end{aligned} \tag{39}$$

Case 1: If $\max(d(q_1, q_2), d(q_1, q_1), d(q_2, q_2)) = d(q_1, q_2)$. From inequality (38), we get

$$\psi(d(q_1, q_2)) \leq \psi(d(q_1, q_2)) - \phi(d(q_1, q_2)).$$

This gives us,

$$\psi(d(q_1, q_2)) < \psi(d(q_1, q_2)), \quad \text{as } \phi\left(d(q_1, q_2)\right) > 0,$$

which is contradiction unless $d(q_1, q_2) = 0$. Therefore, $d(q_1, q_2) = 0 \implies q_1 = q_2$.

Case 2: If $\max(d(q_1, q_2), d(q_1, q_1), d(q_2, q_2)) = d(q_1, q_1)$. From inequality (38), we get

$$\psi(d(q_1, q_2)) \leq \psi(d(q_1, q_1)) - \phi(d(q_1, q_1)), \tag{40}$$

as $d(q_1, q_2) \leq d(q_1, q_1)$ and ψ is non-decreasing, we have

$$\psi(d(q_1, q_2)) \leq \psi(d(q_1, q_1)). \tag{41}$$

Using (40) and (41), we obtain

$$0 \leq -\phi\left(d(q_1, q_1)\right)$$

This forces $\phi\left(d(q_1, q_1)\right) = 0$. Therefore, $d(q_1, q_1) = 0 \implies \psi(d(q_1, q_1)) = 0$. By substituting $\psi(d(q_1, q_1)) = 0$ into the inequality (40). It follows that $\psi(d(q_1, q_2)) = 0$ and hence $q_1 = q_2$.

Case 3: If $\max(d(q_1, q_2), d(q_1, q_1), d(q_2, q_2)) = d(q_2, q_2)$, then from inequality (38), we get

$$\psi\big(d(q_1, q_2)\big) \leq \psi\big(d(q_2, q_2)\big) - \phi\big(d(q_2, q_2)\big). \tag{42}$$

Since $d(q_1, q_2) \leq d(q_2, q_2)$ and ψ is non-decreasing, we have

$$\psi\big(d(q_1, q_2)\big) \leq \psi\big(d(q_2, q_2)\big). \tag{43}$$

Using (42) and (43), we obtain

$$0 \leq -\phi\big(d(q_2, q_2)\big).$$

This forces $\phi\big(d(q_2, q_2)\big) = 0$. Therefore, $d(q_2, q_2) = 0 \implies \psi\big(d(q_2, q_2)\big) = 0$. By substituting $\psi\big(d(q_2, q_2)\big) = 0$ into inequality (42), it follows that $\psi\big(d(q_1, q_2)\big) = 0$, and hence $q_1 = q_2$. Hence, f has a unique fixed point. □

4. Conclusion

In this paper, we have established several new fixed point theorems for continuous self-mappings in orbitally complete dislocated quasi-metric spaces. These results serve as significant generalizations of classical fixed-point theorems in asymmetric settings. The concept of orbital completeness in the context of dislocated quasi-metrics provides a novel framework for analyzing convergence and stability of mappings. Our findings not only enrich the existing body of fixed point theory but also highlight the potential of dislocated quasi-metric spaces for further mathematical investigation. These theorems may find applications in various areas such as nonlinear analysis, optimization, and computational mathematics. Future work may focus on extending these results to multivalued mappings or other generalized metric structures.

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References

[1] A. H. Alwan, *On fixed points in orbit in dislocated quasi-metric spaces*, AIP Conf. Proc., **2386** (2022), 060007.

[2] C. T. Aage and J. N. Salunke, *The results on fixed points in dislocated and dislocated quasi-metric space*, Appl. Math. Sci., **59**(2) (2008), 2941–2948.

[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[4] R. M. T. Bianchini, *Su un problema di s. Reich riguardante la teoria dei punti fissi*, Boll. Un. Mat. Ital., **5** (1972), 103–108.

[5] D. W. Boyd and J. S. W. Wong, *Nonlinear contractions*, Proc. Amer. Math. Soc., **20**(2) (1969), 458–464.

[6] P. N. Dutta and B. S. Chaudhary, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl., **2008** (2008), Article ID 406368, 8 pages.

[7] P. Hitzler and A. K. Seda, *Dislocated topologies*, J. Electr. Eng., **51**(12/s) (2000), 3–7.

[8] M. Khan, M. Swaleh, and S. Sessa, *Fixed point theorems by altering distance between the points*, Bull. Aust. Math. Soc., **30**(1) (1984), 1–9.

[9] N. R. More and S. G. Dapke, *On fixed point theorems in dislocated quasi metric spaces*, Int. J. Sci. Res. Sci. Engin. Technol., **12**(4) (2025), 185–189.

[10] B. E. Rhoades, *A comparison of various definitions of contraction mappings*, Trans. Amer. Math. Soc., **226** (1997), 257–289.

[11] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. Theory Meth. Appl., **47**(4) (2001), 2683–2693.

[12] V. M. Sehgal, *On fixed and periodic points for a class of mappings*, J. London Math. Soc., **5**(2) (1972), 95–96.

[13] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorems for $\alpha-\psi$ contractive type mappings*, Nonlinear Anal. Theory Methods Appl., **75**(4) (2012), 2154–2165.

[14] F. M. Zeyada, G. H. Hassan, and M. A. Ahmed, *A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces*, Arab. J. Sci. Eng., **31** (2005), 111–114.

[15] Q. Zhang and Y. Song, *Fixed point theory for generalized ψ-weak contractions*, Appl. Math. Lett., **22**(1) (2009), 75–78.

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