



Fixed point theorems of some mappings via interpolation in 2-Banach spaces

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ABSTRACT. In this article, we introduce weak contractive mappings and interpolative Kannan-type contraction mappings on a 2-Banach space. In particular, we discuss the existence and uniqueness of a fixed point of such mappings in a 2-Banach space. However, we define interpolative Reich-Rus-Ćirić type contraction mappings and interpolative Hardy-Rogers type contraction mappings, Kannan-Ćirić type contraction mappings and interpolative Kannan-Meir-Keeler type contractions on a 2-Banach space. In particular, we prove the existence of a fixed point of such mappings in a 2-Banach space.

Keywords: Fixed point theorems, 2-Banach spaces, closed and bounded sets

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1. Introduction

The Banach contraction mapping principle is one of the interesting results in functional analysis. Moreover, it was applied in several branches of mathematics, which is given by

Theorem 1.1. *Let (\mathcal{F}, d) be a complete metric space and $S : \mathcal{F} \longrightarrow \mathcal{F}$ be a mapping with*

$$d(Sz, Sf) \leq kd(z, f) \tag{1}$$

where $k \in (0, 1)$ and $z, f \in \mathcal{F}$, hence S has a unique fixed point.

There are many generalizations of this principle, see [1, 4, 20, 23]. The fixed point theory played a crucial role in functional analysis. Moreover, it is used in many branches of science such as biology, chemistry, economics, engineering and computer science. In a 2-Banach space framework, Gähler [8] initiated the study of 2-normed spaces. Recently, White [24] initiated and studied the 2-Banach spaces.

Recently, the authors [12] demonstrated that a contraction mapping has a one fixed point in bounded and closed subsets of a 2-Banach space. However, Kir and Kiziltunc [18] demonstrated several results on fixed points in 2-Banach spaces.

In this paper, we demonstrate a fixed point theorem for weakly contractive mappings in 2-Banach spaces, which is a generalization of the Banach contraction mapping principle. On the other hand, we introduce interpolative Kannan type contraction mappings, interpolative Reich-Rus-Ćirić type contraction mappings, interpolative Hardy-Rogers type contraction mappings, Kannan-Ćirić type contraction mappings and interpolative Kannan-Meir-Keeler type contractions on 2-Banach spaces. In particular, we prove the existence of a fixed point of such mappings in 2-Banach spaces.

2. Preliminaries

We begin with preliminaries:

Definition 2.1 ([8]). Let \mathcal{X} be a real vector space with $\dim \mathcal{X} \geq 2$. A 2-norm on \mathcal{X} is a function $\|\cdot\| : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ such that

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
- (ii) For all $x, y \in \mathcal{X}$, $\|x, y\| = \|y, x\|$.
- (iii) For any $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$, $\|\lambda x, y\| = |\lambda| \|x, y\|$.
- (iv) For each $x, y, z \in \mathcal{X}$, $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a 2-normed space.

Definition 2.2 ([6]). Let \mathcal{X} be a 2-normed space. A sequence $\{x_n\} \subset \mathcal{X}$ is said to be a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_n - x_m, y\| = 0$ for any $y \in \mathcal{X}$.

Definition 2.3 ([24]). Let \mathcal{X} be a 2-normed space. A sequence $\{x_n\} \subset \mathcal{X}$ converges in \mathcal{X} if there exists an element $x \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$. If $\{x_n\}$ converges to x , we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 2.4 ([24]). A 2-normed space in which every Cauchy sequence converges will be called a 2-Banach space.

Definition 2.5 ([12]). Let \mathcal{X} be a 2-normed space then the mapping $S : \mathcal{X} \rightarrow \mathcal{X}$ is said to be a contraction if there exists $k \in (0, 1)$ such that

$$\|Sx - Sy, z\| \leq k \|x - y, z\| \quad (2)$$

for all $x, y, z \in \mathcal{X}$.

Theorem 2.1 ([12]). *Let \mathcal{X} be a 2-normed space and \mathcal{F} be a nonempty closed and bounded subset of \mathcal{X} . Let $S : \mathcal{F} \rightarrow \mathcal{F}$ be a contraction, then S has a unique fixed point.*

Definition 2.6 ([5]). A mapping S on a 2-normed space \mathcal{X} is called a Meir-Keeler contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,

$$\varepsilon \leq \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon. \quad (3)$$

Definition 2.7 ([5]). A mapping S on a 2-normed space \mathcal{X} is called a Ćirić contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,

$$\varepsilon < \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon. \quad (4)$$

Theorem 2.2 ([5]). *Let \mathcal{X} be a 2-Banach space and let S be a Meir-Keeler contraction on \mathcal{X} , then S has a unique fixed point in \mathcal{X} .*

Theorem 2.3 ([5]). *Let \mathcal{X} be a 2-Banach space and let S be a Ćirić contraction on \mathcal{X} , then S has a unique fixed point in \mathcal{X} .*

Definition 2.8 ([5]). A mapping S on a 2-normed space \mathcal{X} is called a Hardy-Rogers contraction if S is a self-mapping on \mathcal{X} satisfying for each $x, y, z \in \mathcal{X}$,

$$\begin{aligned} \|Sx - Sy, z\| &\leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned} \quad (5)$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

Consider the following condition

$$\begin{aligned} x \neq y \implies \|Sx - Sy, z\| &< a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\|. \end{aligned} \quad (6)$$

Theorem 2.4 ([5]). *Let \mathcal{X} be a 2-normed space and S a self-mapping on \mathcal{X} satisfying for each $x, y, z \in \mathcal{X}$,*

$$\begin{aligned} \|Sx - Sy, z\| &\leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned} \quad (7)$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

- (i) *If \mathcal{X} is complete and $\beta < 1$, then S has a unique fixed point.*
- (ii) *If (5) is modified to condition (6) and in this case \mathcal{X} is compact, S is continuous and $\beta = 1$, then S has a unique fixed point.*

Theorem 2.5 ([5]). *Let \mathcal{X} be a 2-Banach space, a, b, c, e, f be monotonically decreasing functions from $[0, \infty)$ to $[0, 1)$ and let the sum of these five functions be less than 1. Assume that $S : \mathcal{X} \rightarrow \mathcal{X}$ satisfies condition (5) with $a = a(\|x - y, z\|), \dots, f = f(\|x - y, z\|)$ for each $x, y, z \in \mathcal{X}$. Then S has a unique fixed point.*

Theorem 2.6 ([5]). *Let \mathcal{X} be a 2-Banach space and $S_n : \mathcal{X} \rightarrow \mathcal{X}$, $n = 1, 2, \dots$ satisfy the conditions of Theorem 2.5 with the coefficients a, b, c, e, f . Let $S_n x_n = x_n$ and assume that $S_n \rightarrow S$ pointwise on \mathcal{X} . Then $x = \lim_{n \rightarrow \infty} x_n$ is the unique fixed point of S .*

Theorem 2.7 ([5]). *Let \mathcal{X} be a 2-Banach space and $S_n : \mathcal{X} \rightarrow \mathcal{X}$, $n = 1, 2, \dots$ be functions with at least one fixed point x_n , $n = 1, 2, \dots$. Let S satisfy the hypothesis of Theorem 2.5 and $S_n \rightarrow S$ uniformly on \mathcal{X} . Then $x = \lim_{n \rightarrow \infty} x_n$ is the unique fixed point of S .*

Ettayb [5] established a generalization of Banach's principle of contraction mappings in 2-Banach spaces as follows.

Theorem 2.8 ([5]). *Let \mathcal{X} be a 2-Banach space and \mathcal{F} be a nonempty closed and bounded subset of \mathcal{X} . If $S : \mathcal{F} \rightarrow \mathcal{F}$ is a continuous mapping such that S^k is a contraction for some $k \geq 1$, then S has a unique fixed point.*

3. Main results

In this section, we start with the following definition.

Definition 3.1. A mapping S on a 2-normed space \mathcal{X} is called a weak contractive if for all $x, y, z \in \mathcal{X}$,

$$\|Sx - Sy, z\| \leq \|x - y, z\| - \varphi(\|x - y, z\|) \quad (8)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Now, we present our main result.

Theorem 3.1. *Let \mathcal{X} be a 2-Banach space and $\mathcal{F} \subset \mathcal{X}$ is a nonempty, closed and bounded subset. If S is a weak contractive mapping on \mathcal{F} , then S has a unique fixed point.*

PROOF. Let $x_0 \in \mathcal{F}$ and $\{x_n\} \subset \mathcal{F}$ be given by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Hence for each $z \in \mathcal{F}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_{n-1} - Sx_n, z\| \\ &\leq \|x_{n-1} - x_n, z\| - \varphi(\|x_{n-1} - x_n, z\|) \\ &\leq \|x_{n-1} - x_n, z\| \end{aligned}$$

then the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing, hence there exists $d \in [0, \infty)$ such that for each $z \in \mathcal{F}$, $\|x_n - x_{n+1}, z\| \rightarrow d$ as $n \rightarrow \infty$. Take $x = x_{n-1}$ and $y = x_n$ in (8) and letting $n \rightarrow \infty$, we get

$$d \leq d - \varphi(d).$$

Hence $\varphi(d) = 0$ thus $d = 0$. So for any $z \in \mathcal{F}$,

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = 0. \quad (9)$$

Next, we demonstrate that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not Cauchy, hence there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that for any $z \in \mathcal{F}$,

$$\|x_{m(k)} - x_{n(k)}, z\| \geq \varepsilon. \quad (10)$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (10). Hence

$$\|x_{m(k)} - x_{n(k)-1}, z\| < \varepsilon. \quad (11)$$

Thus we get for each $z \in \mathcal{F}$,

$$\varepsilon \leq \|x_{m(k)} - x_{n(k)}, z\| \leq \|x_{m(k)} - x_{n(k)-1}, z\| + \|x_{n(k)-1} - x_{n(k)}, z\| < \varepsilon + \|x_{n(k)-1} - x_{n(k)}, z\|.$$

Utilizing (9) and letting $n \rightarrow \infty$, we obtain

$$\lim_{k \rightarrow \infty} \|x_{m(k)} - x_{n(k)}, z\| = \varepsilon. \quad (12)$$

Also,

$$\|x_{n(k)} - x_{m(k)}, z\| \leq \|x_{n(k)} - x_{n(k)-1}, z\| + \|x_{n(k)-1} - x_{m(k)-1}, z\| + \|x_{m(k)-1} - x_{m(k)}, z\|$$

and

$$\|x_{n(k)-1} - x_{m(k)-1}, z\| \leq \|x_{n(k)-1} - x_{n(k)}, z\| + \|x_{n(k)} - x_{m(k)}, z\| + \|x_{m(k)-1} - x_{m(k)}, z\|.$$

Taking $k \rightarrow \infty$ in the above two inequalities and utilizing (10),(12), we have

$$\lim_{k \rightarrow \infty} \|x_{n(k)-1} - x_{m(k)-1}, z\| = \varepsilon. \quad (13)$$

Putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (8) and utilizing (10), we get

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon) \quad (14)$$

which is a contradiction with $\varepsilon > 0$. This proves that $\{x_n\}$ is a Cauchy sequence. Since \mathcal{F} is complete, we get it is convergent to $w \in \mathcal{F}$ as $n \rightarrow \infty$. Substituting $x = x_{n-1}$ and $y = w$ in (8), we have

$$\|x_n - Sw, z\| \leq \|x_{n-1} - w, z\| - \varphi(\|x_{n-1} - w, z\|). \quad (15)$$

Letting $n \rightarrow \infty$, we get for each $z \in \mathcal{F}$,

$$\|w - Sw, z\| \leq \|w - w, z\| - \varphi(\|w - w, z\|) = 0 - \varphi(0) = 0. \quad (16)$$

Then $Sw = w$. To demonstrate the uniqueness of the fixed point, let us assume that w_1 and w_2 are two fixed points of S . Setting $x = w_1$ and $x = w_2$ in (8), we get for any $z \in \mathcal{F}$,

$$\begin{aligned} \|w_1 - w_2, z\| &= \|Sw_1 - Sw_2, z\| \\ &\leq \|w_1 - w_2, z\| - \varphi(\|w_1 - w_2, z\|), \end{aligned}$$

thus $\varphi(\|w_1 - w_2, z\|) = 0$ for each $z \in \mathcal{F}$. Hence $w_1 = w_2$. \square

In particular, if $\varphi(t) = kt$ where $k \in (0, 1)$ we obtain the result in Theorem 2.1. Now, we introduce an interpolative Kannan type contraction in a 2-normed space as follows.

Definition 3.2. A self-mapping S on a 2-normed space \mathcal{X} is called an interpolative Kannan type contraction if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\|Sx - Sy, z\| \leq \lambda[\|x - Sx, z\|]^\alpha \cdot [\|y - Sy, z\|]^{1-\alpha} \quad (17)$$

for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$ where $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$.

Theorem 3.2. Let \mathcal{X} be a 2-Banach space and let S be an interpolative Kannan type contraction on \mathcal{X} , then S has a fixed point in \mathcal{X} .

PROOF. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Without loss of generality, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Indeed, if there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms a fixed point. Hence

$$\|x_n - x_{n+1}, z\| > 0$$

for all $n \in \mathbb{N}_0$ and $z \in \mathcal{X}$. Putting $x = x_n$ and $y = x_{n-1}$ in (17), we obtain for all $z \in \mathcal{F}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\|x_n - Sx_n, z\|]^\alpha \cdot [\|x_{n-1} - Sx_{n-1}, z\|]^{1-\alpha} \\ &= \lambda[\|x_n - Sx_n, z\|]^\alpha \cdot [\|x_{n-1} - x_n, z\|]^{1-\alpha}, \end{aligned}$$

then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda \|x_{n-1} - x_n, z\|^{1-\alpha}. \quad (18)$$

Thus, we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = M$ for all $z \in \mathcal{X}$. We will indicate that $M = 0$. Indeed, by (18), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (19)$$

Letting $n \rightarrow \infty$ in (19), we obtain $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting $n, m \rightarrow \infty$, we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to some $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S . Hence for each $z \in \mathcal{X}$,

$$\|x_{n+1} - Sx, z\| = \|Sx_n - Sx, z\| \leq \lambda [\|x_n - Sx_n, z\|]^\alpha \cdot [\|y - Sy, z\|]^{1-\alpha}.$$

Taking $n \rightarrow \infty$, we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx = x$ thus x is a fixed point of S . □

Now, we introduce interpolative Reich-Rus-Ćirić type contractions in 2-normed spaces as follows.

Definition 3.3. A self-mapping S on a 2-normed space \mathcal{X} is called an interpolative Reich-Rus-Ćirić type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$,

$$\|Sx - Sy, z\| \leq \lambda [\|x - y, z\|]^\beta \cdot [\|x - Sx, z\|]^\alpha \cdot [\|y - Sy, z\|]^{1-\alpha-\beta}. \quad (20)$$

Theorem 3.3. Let \mathcal{X} be a 2-Banach space and let S be an interpolative Reich-Rus-Ćirić type contraction on \mathcal{X} , then S has a fixed point in \mathcal{X} .

PROOF. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. If there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms a fixed point. The proof is complete. Henceforth, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (20), we have for all

$z \in \mathcal{X}$,

$$\begin{aligned}
\|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\
&\leq \lambda[\|x_n - x_{n-1}, z\|]^\beta \cdot [\|x_n - Sx_n, z\|]^\alpha \cdot [\|x_{n-1} - Sx_{n-1}, z\|]^{1-\alpha-\beta} \\
&= \lambda[\|x_n - x_{n-1}, z\|]^\beta \cdot [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_{n-1} - x_n, z\|]^{1-\alpha-\beta} \\
&= \lambda[\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_{n-1} - x_n, z\|]^{1-\alpha}
\end{aligned} \tag{21}$$

then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda\|x_{n-1} - x_n, z\|^{1-\alpha}. \tag{22}$$

Hence we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = M$ for all $z \in \mathcal{X}$. We will indicate that $M > 0$. Indeed, by (22), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda\|x_{n-1} - x_n, z\| \leq \lambda^n\|x_0 - x_1, z\|. \tag{23}$$

Letting $n \rightarrow \infty$ in (23), we obtain $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned}
\|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\
&= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\
&\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\
&\leq \lambda^n\|x_0 - x_1, z\| + \lambda^{n+1}\|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1}\|x_0 - x_1, z\| \\
&= \lambda^n(1 + \lambda + \cdots + \lambda^{l-1})\|x_0 - x_1, z\| \\
&\leq \frac{\lambda^n}{1 - \lambda}\|x_0 - x_1, z\|.
\end{aligned}$$

Letting $n, m \rightarrow \infty$, we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to certain $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S . Hence for each $z \in \mathcal{X}$,

$$\begin{aligned}
\|x_{n+1} - Sx, z\| &= \|Sx_n - Sx, z\| \\
&\leq \lambda[\|x_n - x, z\|]^\beta \cdot [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x - Sx, z\|]^{1-\alpha-\beta}.
\end{aligned} \tag{24}$$

Letting $n \rightarrow \infty$ in (24), we obtain

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx = x$, thus x is a fixed point of S . □

We introduce interpolative Hardy-Rogers type contractions in 2-normed spaces as follows.

Definition 3.4. A self-mapping S on a 2-normed space \mathcal{X} is called an interpolative Hardy-Rogers type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$,

$$\begin{aligned} \|Sx - Sy, z\| &\leq \lambda[\|x - y, z\|]^\beta \cdot [\|x - Sx, z\|]^\alpha \cdot [\|y - Sy, z\|]^\gamma \\ &\quad \cdot \left[\frac{1}{2}(\|x - Sy, z\| + \|y - Sx, z\|)\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \quad (25)$$

Theorem 3.4. Let \mathcal{X} be a 2-Banach space and let S be an interpolative Hardy-Rogers type contraction on \mathcal{X} , then S has a fixed point in \mathcal{X} .

PROOF. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. If there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms a fixed point. The proof is complete. Henceforth, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Putting $x = x_n$ and $y = x_{n-1}$ in (25), we have for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\|x_n - x_{n-1}, z\|]^\beta \cdot [\|x_n - Sx_n, z\|]^\alpha \cdot [\|x_{n-1} - Sx_{n-1}, z\|]^\gamma \\ &\quad \cdot \left[\frac{1}{2}(\|x_n - x_n, z\| + \|x_{n-1} - x_{n+1}, z\|)\right]^{1-\alpha-\beta-\gamma} \\ &= \lambda[\|x_n - x_{n-1}, z\|]^\beta \cdot [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_{n-1} - x_n, z\|]^\gamma \\ &\quad \cdot \left[\frac{1}{2}(\|x_{n-1} - x_{n+1}, z\|)\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \quad (26)$$

If $\|x_{n-1} - x_n, z\| \leq \|x_n - x_{n+1}, z\|$, then from (26), we obtain

$$\|x_n - x_{n+1}, z\|^{\beta+\gamma} \leq \lambda \|x_{n+1} - x_n, z\|^{\beta+\gamma}$$

which is a contradiction. Then $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = M$ for all $z \in \mathcal{X}$. We will indicate that $M = 0$. Indeed, by (26), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (27)$$

Letting $n \rightarrow \infty$ in (27), we obtain $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting $n, m \rightarrow \infty$, we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to certain $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S . Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - Sx, z\| &= \|Sx_n - Sx, z\| \\ &\leq \lambda \|x_n - x, z\|^\beta \cdot \|x_n - Sx_n, z\|^\alpha \cdot \|x - Sx, z\|^\gamma \\ &\quad \cdot \left[\frac{1}{2}(\|x_n - Sx, z\| + \|x - Sx_n, z\|)\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \quad (28)$$

Letting $n \rightarrow \infty$ in (28), we obtain

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx = x$ thus x is a fixed point of S . \square

Now, we introduce an interpolative Kannan-Ćirić type contraction in a 2-normed space as follows.

Definition 3.5. A mapping S on a 2-normed space \mathcal{X} is called an interpolative Kannan-Ćirić type contraction if there exists $\alpha \in (0, 1)$ such that for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$, we have

(i) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < [\|x - Sx, z\|]^\alpha [\|y - Sy, z\|]^{1-\alpha} < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon, \quad (29)$$

and

(ii)

$$\|Sx - Sy, z\| < [\|x - Sx, z\|]^\alpha [\|y - Sy, z\|]^{1-\alpha}. \quad (30)$$

Theorem 3.5. Let \mathcal{X} be a 2-Banach space and let S be an interpolative Kannan-Ćirić type contraction on \mathcal{X} , then S has a fixed point in \mathcal{X} .

PROOF. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\} \subset \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}_0$. Putting $x = x_{n-1}$ and $y = x_n$ in (30), we have for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_{n-1} - Sx_n, z\| \\ &< [\|x_{n-1} - Sx_{n-1}, z\|]^\alpha [\|x_n - Sx_n, z\|]^{1-\alpha} \\ &= [\|x_{n-1} - x_n, z\|]^\alpha [\|x_n - x_{n+1}, z\|]^{1-\alpha}. \end{aligned} \quad (31)$$

Then $[\|x_n - x_{n+1}, z\|]^\alpha < [\|x_{n-1} - x_n, z\|]^\alpha$, thus $\{\|x_n - x_{n+1}, z\|\}$ is strictly decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} \|x_n - x_{n-1}, z\| = M$ for all $z \in \mathcal{X}$. We claim that $M = 0$. By monotonicity, we have

$$\|x_n - x_{n+1}, z\| > M \quad \text{for all } n \in \mathbb{N}_0 \text{ and } z \in \mathcal{X}. \quad (32)$$

Indeed, if we suppose that $M > 0$, we can find $N \in \mathbb{N}$ such that

$$M < \|x_n - x_{n+1}, z\| < M + \delta(M) \quad (33)$$

for all $n \geq N$. Since $M < \|x_n - x_{n+1}, z\| < [\|x_{n-1} - x_n, z\|]^\alpha [\|x_n - x_{n+1}, z\|]^{1-\alpha}$, it follows from (29) that $\|x_n - x_{n+1}, z\| \leq M$ for any $n \geq N$ which is a contradiction with (32). So $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $\varepsilon > 0$ be fixed and we can consider that $\delta(\varepsilon)$ can be choose such that $\delta(\varepsilon) < \varepsilon$. Since $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = 0$, we can find $N_1 \in \mathbb{N}$ such that $\|x_n - x_{n+1}, z\| < \frac{\varepsilon}{2}$ for $n \geq N_1$ and we claim that

$$\|x_n - x_{n+p}, z\| < \varepsilon \quad (34)$$

for all $p \in \mathbb{N}$. Of course, the inequality (34) holds for $p = 1$. Supposing that for some p , (34), we get for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+p+1}, z\| &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+p+1}, z\| \\ &= \|x_n - x_{n+1}, z\| + \|Sx_n - Sx_{n+p}, z\| \\ &< \|x_n - x_{n+1}, z\| + [\|x_n - x_{n+1}, z\|]^\alpha [\|x_{n+p} - x_{n+p+1}, z\|]^{1-\alpha} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{x_n\} \subset \mathcal{X}$ is Cauchy. Then $\{x_n\}$ converges to some $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S . Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - Sx, z\| &\leq \|x_{n+1} - x, z\| + \|x_{n+1} - Sx, z\| \\ &= \|x_{n+1} - Sx, z\| + \|Sx_n - Sx, z\| \\ &< \|x_n - Sx_n, z\|^\alpha [\|x - Sx, z\|]^{1-\alpha} \\ &= \|x_n - x_{n+1}, z\|^\alpha [\|x - Sx, z\|]^{1-\alpha}. \end{aligned} \quad (35)$$

Letting $n \rightarrow \infty$ in (35), we obtain

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Then $Sx = x$ thus x is a fixed point of S . \square

We introduce interpolative Kannan-Meir-Keeler type contractions in 2-normed spaces as follows.

Definition 3.6. A mapping S on a 2-normed space \mathcal{X} is called an interpolative Kannan-Meir-Keeler type contraction if there exists $\alpha \in (0, 1)$ such that for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$, we have

(i) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq [\|x - Sx, z\|]^\alpha [\|y - Sy, z\|]^{1-\alpha} < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon, \quad (36)$$

and

(ii)

$$\|Sx - Sy, z\| < [\|x - Sx, z\|]^\alpha [\|y - Sy, z\|]^{1-\alpha}. \quad (37)$$

We finish with the following theorem.

Theorem 3.6. *Let \mathcal{X} be a 2-Banach space and let S be an interpolative Kannan-Meir-Keeler type contraction on \mathcal{X} , then S has a fixed point in \mathcal{X} .*

PROOF. The proof is similar to the proof of Theorem 3.5. □

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