Mathematical Analysis and its Contemporary Applications
Volume 7, Issue 4, 2025, 1–11
doi: 10.30495/maca.2025.2066672.1144
ISSN 2716-9898

Solving nonlinear integral equation using fuzzy F-interpolative Berinde weak contraction

Shobha Jain*, Shishir Jain, and Stojan Radenović

Abstract. This study presents a contraction referred to as fuzzy F-interpolative Berinde weak contraction, achieved by integrating two primary contractions, namely F-contraction and Berinde weak contraction, using a F-function within a fuzzy metric space. Utilizing this contraction, we have established a fixed point theorem applicable to self-mappings. To illustrate the implications of our results, we investigate the existence of solutions for nonlinear integral equations. An example has been devised to substantiate the established result.

Keywords: Fuzzy metric space, fixed point, F-contraction, fuzzy F-Interpolative Berinde weak contraction

2020 Mathematics Subject Classification: Primary: 47H10; Secondary: 54H25

---

1. Introduction

In fixed point theory, the sufficient condition for the existence and uniqueness of a fixed point of self mapping is guaranteed by Banach's contraction principle. This principle can be briefly articulated in a metric space context as follows:

Let (X, d) be a complete metric space and let T be a self mapping on X and if there exists a positive constant $K < 1$ such that

$$d(T(x), T(y)) \leq K d(x, y), \quad \text{for all} \quad x, y \in X.$$

---

*Corresponding author

Then T has a unique fixed point $x \in X$ such that $T(x) = x$.
Due to its extensive applications across various disciplines and its straightforward nature, numerous authors have sought to generalize the contractive condition. One notable attempt was made by Berinde in 2004, who introduced the concept of weak contraction.

Definition 1.1. [2] Let (X, d) be a metric space. A map $T : X \to X$ is called weak contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that $x, y \in X$

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Later on in year 2012 Wardowski initiated the F-contraction as follows:

Definition 1.2. [11] Let (X, d) be a metric space. A mapping $T : X \to X$, will be called an F-contraction, if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \text{ implies } \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : R^+ \to R$ be a mapping satisfying.

(a) F is strictly increasing, that is, for all $\alpha, \beta \in R^+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;

(b) For each sequence $\{\alpha_n\}_{n \in N}$ of positive numbers, $\lim_{n \to +\infty} \alpha_n = 0$ if and only if $\lim_{n \to +\infty} F(\alpha_n) = -\infty$.

(c) There exists $\lambda \in (0, 1)$ such that $\lim_{\lambda \to 0^+} \alpha^\lambda F(\alpha) = 0$.

In [9] Karapinar used interpolation technique and proved the following

Theorem 1.1. *Let (X, d) be a complete metric spaces and $T : X \to X$ be interpolative Kannan contraction mapping, i.e.*

$$d(Tx, Ty) \leq \lambda(d(x, Tx))^\alpha (d(y, Ty))^{(1-\alpha)}$$

for all $x, y \in X - Fix(T)$ where $Fix(T) = \{x \in X, Tx = x\}, \lambda \in [0, 1)$ and $\alpha \in (0, 1)$. Then T has a unique fixed point.

Drawing inspiration from the research conducted by Karapinar [9], we present the concept of Fuzzy F-interpolative Berinde weak contraction. This is achieved by merging F-contraction with Berinde weak contraction through an interpolate method within the context of fuzzy metric spaces, while considering only conditions (a) and (b) of the F-function. Furthermore, we demonstrate the existence of solutions to nonlinear integral equations in fuzzy metric spaces. The paper also features an example that illustrates the validity of our findings.

2. Preliminaries

In this section we give definitions of known notions as well as some known results from fuzzy metric spaces.

Definition 2.1. [10] A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (t-norm for short) if $*$ is continuous and satisfies the following conditions:

(i) $*$ is commutative and associative, i.e. $a * b = b * a$ and $a * (b * c) = (a * b) * c$, for all $a, b, c \in [0,1]$;

(ii) $1 * a = a$, for all $a \in [0,1]$;

(iii) $a * c \leq b * d$, for $a \leq b, c \leq d$ for $a, b, c, d \in [0,1]$.

Well known examples of t-norm are, the minimum t-norm $*_m$, $a *_m b = \min\{a, b\}$ written as $*_m$ and product t-norm $*$, $a * b = ab$.

Definition 2.2. [3] A fuzzy metric space is an ordered triple $(\Upsilon, M, *)$ such that Υ is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $\Upsilon \times \Upsilon \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in \Upsilon$ and $t, s > 0$;

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV5) $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is continuous.

Note that in view of condition (GV2) we have $M(x, x, t) = 1$, for all $x \in \Upsilon$ and $t > 0$ and $M(x, y, t) < 1$, for all $x \neq y$ and $t > 0$.

Definition 2.3. [3] A sequence $\{w_n\}$ in a fuzzy metric space $(\Upsilon, M, *)$is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon \in (0, 1)$ and each $t > 0$ there exists an $n_0 \in N$, such that $M(w_n, w_m, t) > 1 - \epsilon$, for all $n, m \geq n_0$ or equivalently, $\lim_{n,m \to +\infty} M(w_n, w_m, t) = 1$, for all $t > 0$.

Lemma 2.1. *[3] Let* $(\Upsilon, M, *)$ *be a fuzzy metric space. Then* $M(p, q, .)$ *is nondecreasing for all* $p, q \in \Upsilon$.

Theorem 2.2. *Let* $(\Upsilon, M, *)$ *be a fuzzy metric space. A sequence* $\{w_n\}_{n \in N}$ *in* Υ *converges to* $w \in \Upsilon$ *if* $\lim_{n \to +\infty} M(w_n, w, t) = 1$, *for any* $t > 0$.

Definition 2.4. [3] $(\Upsilon, M, *)$ (or simply Υ) is complete if every Cauchy sequence is convergent.

Lemma 2.3. *[6] Let* $(\Upsilon, M, *)$ *be a fuzzy metric space and let* $\{w_n\}$ *be a sequence in* Υ *such that*

$$\lim_{t \to 0^+} M(w_n, w_{n+1}, t) > 0, \; n \in \mathbf{N}, \tag{1}$$

and

$$\lim_{n\to+\infty} M(w_n, w_{n+1}, t) = 1,\ t > 0. \tag{2}$$

If $\{w_n\}$ *is not a Cauchy sequence in* $(\Upsilon, M, *)$*, then there exist* $\varepsilon \in (0,1)$*,* $t_0 > 0$*, and sequences of positive integers* $\{n_k\}$*,* $\{m_k\}$*,* $n_k > m_k > k$*,* $k \in \mathbf{N}$*, such that the following sequences*

$$\{M(w_{m_k}, w_{n_k}, t_0)\}, \{M(w_{m_k}, w_{n_k+1}, t_0)\}, \{M(w_{m_k-1}, w_{n_k}, t_0)\},$$

$$\{M(w_{m_k-1}, w_{n_k+1}, t_0)\}, \{M(w_{m_k+1}, w_{n_k+1}, t_0)\}$$

tend to $1-\varepsilon$*, as* $k \to +\infty$*.*

3. Fuzzy F-Interpolative Berinde weak contraction

This section starts with the definition of the Fuzzy F-Interpolative Berinde weak contraction, and we will take into account the F-function from [11], specifically under conditions (a) and (b) only.

Definition 3.1. A mapping $\Lambda : \Upsilon \to \Upsilon$, is called to be fuzzy F-Interpolative Berinde weak contractive, if there exists $\tau > 0, 0 < \mu < 1$ such that for all $u, v \in \Upsilon, u, v \in \Upsilon - Fix(\Lambda)$

$$M(\Lambda u, \Lambda v, t) > 0 \ \ implies \ \ \tau + F\left(\frac{1}{M(\Lambda u, \Lambda v, t)} - 1\right) \leq F\left(\begin{array}{c} \left(\frac{1}{M(u,v,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(u,\Lambda u,t)} - 1\right)^{(1-\mu)} \end{array}\right) \tag{3}$$

Example 3.2. [11]

- $F(w) = \ln w, for \ \ w > 0;$
- $F(w) = \ln w + w, for \ \ w > 0;$
- $F(w) = \frac{-1}{\sqrt{w}}, for \ \ w > 0.$

Remark 3.3. Every fuzzy F-Interpolative Bertin weak contraction is Interpolative Berinde weak contraction in the sense of Gregori and Sapena [4]
Taking $F(w) = \ln w$ in equation (3) we get

$$M(\Lambda u, \Lambda v, t) > 0 \ \ implies \ \ \frac{1}{M(\Lambda u, \Lambda v, t)} - 1 \leq e^{-\tau} \left\{ \begin{array}{c} \left(\frac{1}{M(u,v,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(u,\Lambda u,t)} - 1\right)^{(1-\mu)} \end{array} \right\}. \tag{4}$$

which distinctly illustrates Berinde's weak contraction.

Gregori and Sapena in [4] defined the fuzzy contractive mappings as follows: Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called a fuzzy contractive mapping if there exists $k \in (0,1)$ such that

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq k\left(\frac{1}{M(x,y,t)} - 1\right),$$

for all $x, y \in X$.

Remark 3.4. Every fuzzy F-Interpolative Bertin weak contraction is fuzzy contractive if we take $\mu = 1, \tau = 1$ them equation (3) becomes

$$\frac{1}{M(\Lambda u,\Lambda v,t)} - 1 \leq K\left(\frac{1}{M(u,v,t)} - 1\right)$$

which clearly demonstrates fuzzy contraction with $K = e^{-1}$.

4. Main results

Our first new result is the next

Theorem 4.1. *Let $(\Upsilon, M, *)$ be a complete fuzzy metric space such that $\lim_{t\to 0+} M(u,v,t) > 0, u, v \in \Upsilon$ and Λ be a fuzzy F interpolative Berinde weak contractive self mapping. Then Λ has a fixed point in Υ.*

Proof. Let $z_0 \in \Upsilon$, be any arbitrary point. Define sequence $\{z_n\}$, by $\Lambda z_m = z_{m+1}$, for $m = 0, 1, 2, \ldots$. For existence of fixed point of map Λ following cases arise.

Case I Suppose $z_m = z_{m+1}$, for some $m \in N$. As $\Lambda z_m = z_{m+1}$, we have $\Lambda z_m = z_m$. In this case z_m becomes the fixed point of the map Λ. So we can assume the consecutive terms of the sequence $\{z_n\}$ are distinct.

Again, to see the existence of fixed point in other cases, we first show that all the terms of the sequence $\{z_m\}$ are distinct.

Case II Suppose $z_m = z_k$, for some $k > (m+1)$, then as all the consecutive terms of the sequence $\{z_m\}$ are distinct we assert that $z_{m+1} = z_{k+1}$. Now

$$z_{m+1} = \Lambda z_m = \Lambda z_k = z_{k+1}.$$

That is, $z_{m+1} = z_{k+1}$. Taking $u = z_{m+1}$ and $v = z_k$ in equation (3), we have

$$\tau + F\left(\frac{1}{M(\Lambda z_k,\Lambda z_{m+1},t)} - 1\right) \leq F\left(\begin{array}{c}\left(\frac{1}{M(z_{m+1},z_k,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_{m+1},\Lambda z_{m+1},t)} - 1\right)^{1-\mu}\end{array}\right).$$

Using $z_k = z_m$ and $z_{k+1} = z_{m+1}$ we get

$$\tau + F\left(\frac{1}{M(z_{m+2},z_{m+1},t)} - 1\right) \leq F\left(\begin{array}{c}\left(\frac{1}{M(z_{m+1},z_m,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_{m+1},z_{m+2},t)} - 1\right)^{1-\mu}\end{array}\right).$$

As $\tau > 0$ we have

$$F\left(\frac{1}{M(z_{m+2}, z_{m+1}, t)} - 1\right) < F\left(\begin{array}{c}\left(\frac{1}{M(z_{m+1}, z_m, t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_{m+1}, z_{m+2}, t)} - 1\right)^{1-\mu}\end{array}\right).$$

As F is strictly increasing we have

$$\left(\frac{1}{M(z_{m+2}, z_{m+1}, t)} - 1\right) < \left(\frac{1}{M(z_{m+1}, z_m, t)} - 1\right)^{\mu}\left(\frac{1}{M(z_{m+1}, z_{m+2}, t)} - 1\right)^{1-\mu}.$$

Thus

$$M(z_m, z_{m+1}, t) < M(z_{m+1}, z_{m+2}, t). \tag{5}$$

So

$$M(z_m, z_{m+1}, t) < M(z_{m+1}, z_{m+2}, t) < M(z_{m+2}, z_{m+3}, t) < ... < M(z_k, z_{k+1}, t).$$

That is, $M(z_m, z_{m+1}, t) < M(z_k, z_{k+1}, t) = M(z_m, z_{m+1}, t)$, which is not possible. So this case does not arise.

Hence, we conclude that $z_m \neq z_k$ for distinct $m, k \in \mathbf{N}$. Thus the elements of the sequence $\{z_m\}$ are distinct. Now we show that the sequence $\{z_m\}$ is M-Cauchy in Υ.

Step 1 In this step we prove that $\lim_{m\to+\infty} M(z_m, z_{m+1}, t) = 1$, for all $t > 0$, and $m \in \mathbf{N}$. From equation (5) we have

$$M(z_m, z_{m+1}, t) < M(z_{m+1}, z_{m+2}, t), \text{ for all } t > 0.$$

Thus, $\{M(z_m, z_{m+1}, t)\}$, for each $t > 0$, is an strictly increasing sequence, which is bounded above by 1. As $\tau > 0$ from condition (3) with $u = z_m, v = z_{m+1}$, for all $t > 0$ we have

$$\begin{aligned} F\left(\frac{1}{M(z_{m+1}, z_{m+2}, t)} - 1\right) &\leq F\left(\frac{1}{M(z_m, z_{m+1}, t)} - 1\right) - \tau \\ &\leq F\left(\frac{1}{M(z_{m-1}, z_m, t)} - 1\right) - 2\tau \\ &\vdots \\ &\leq F\left(\frac{1}{M(z_0, z_1, t)} - 1\right) - m\tau. \end{aligned}$$

Taking limit as $m \to +\infty$ we obtain

$$\lim_{m\to+\infty} F\left(\frac{1}{M(z_{m+1}, z_{m+2}, t)} - 1\right) \leq F\left(\frac{1}{M(z_0, z_1, t)} - 1\right) - \lim_{m\to+\infty} m\tau = -\infty.$$

Using condition (b) of F-function we obtain

$$\lim_{m\to+\infty}\left(\frac{1}{M(z_{m+1}, z_m, t)} - 1\right) = 0,$$

which yield

$$\lim_{m\to+\infty} M(z_m, z_{m+1}, t) = 1, \text{ for all } t > 0. \tag{6}$$

Step 2 We assert that $\{z_m\}$ is a Cauchy sequence in Υ. Suppose this is not true then using Lemma 2.3 and equations (1) and (6) we have that there exist $\varepsilon \in (0,1)$, $t_0 > 0$ and sequences $\{z_{m_k}\}$ and $\{z_{n_k}\}$ such that $\lim_{k\to+\infty} M(z_{m_k+1}, z_{n_k+1}, t_0) = 1-\varepsilon$. By (3) we have

$$\tau + F\left(\frac{1}{M(\Lambda z_{m_k}, \Lambda z_{n_k}, t_0)} - 1\right) \leq F\left\{\begin{array}{c} \left(\frac{1}{M(z_{m_k}, z_{n_k}, t_0)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_{m_k}, \Lambda z_{m_k}, t_0)} - 1\right)^{1-\mu} \end{array}\right\}.$$

As $\tau > 0$ we have

$$F\left(\frac{1}{M(z_{m_k+1}, z_{n_k+1}, t_0)} - 1\right) < F\left\{\begin{array}{c} \left(\frac{1}{M(z_{m_k}, z_{n_k}, t_0)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_{m_k}, z_{m_k+1}, t_0)} - 1\right)^{1-\mu} \end{array}\right\}.$$

By condition (a) we obtain

$$\left(\frac{1}{M(z_{m_k+1}, z_{n_k+1}, t_0)} - 1\right) < \left(\frac{1}{M(z_{m_k}, z_{n_k}, t_0)} - 1\right)^{\mu} \left(\frac{1}{M(z_{m_k}, z_{m_k+1}, t_0)} - 1\right)^{1-\mu}.$$

Taking limit as $k \to +\infty$ we have

$$\lim_{k\to+\infty}\left(\frac{1}{M(z_{m_k+1}, z_{n_k+1}, t_0)} - 1\right) \leq \lim_{k\to+\infty}\left\{\begin{array}{c} \left(\frac{1}{M(z_{m_k}, z_{n_k}, t_0)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_{m_k}, z_{m_k+1}, t_0)} - 1\right)^{1-\mu} \end{array}\right\}.$$

that is

$$\lim_{k\to+\infty}\left(\frac{1}{M(z_{m_k+1}, z_{n_k+1}, t_0)} - 1\right) \leq \left\{\begin{array}{c} \lim_{k\to+\infty}\left(\frac{1}{M(z_{m_k}, z_{n_k}, t_0)} - 1\right)^{\mu} \\ \lim_{k\to+\infty}\left(\frac{1}{M(z_{m_k}, z_{m_k+1}, t_0)} - 1\right)^{1-\mu} \end{array}\right\}.$$

Using equation (6) we get

$$\lim_{k\to+\infty}\left(\frac{1}{M(z_{m_k+1}, z_{n_k+1}, t_0)} - 1\right) = 0,$$

which yields

$$\lim_{k\to+\infty} M(z_{m_k+1}, z_{n_k+1}, t_0) = 1 \tag{7}$$

which contradicts with assumption. So, the sequence $\{z_m\}$ is an M-Cauchy sequence in Υ which is M-complete. Therefore there exists $z \in \Upsilon$ such that

$$\{z_m\} \to z. \tag{8}$$

That is,

$$\{\Lambda z_m\} \to z. \tag{9}$$

Taking $u = z_m$ and $v = z$ in equation (3), we have

$$\tau + F\left(\frac{1}{M(\Lambda z_m, \Lambda z, t)} - 1\right) \leq F\left(\begin{array}{c}\left(\frac{1}{M(z_m,z,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_m,\Lambda z_m,t)} - 1\right)^{1-\mu}\end{array}\right).$$

As $\tau > 0$ we have

$$F\left(\frac{1}{M(\Lambda z_m, \Lambda z, t)} - 1\right) < F\left(\begin{array}{c}\left(\frac{1}{M(z_m,z,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(z_m,\Lambda z_m,t)} - 1\right)^{1-\mu}\end{array}\right).$$

By condition (a) of F-function we have

$$\frac{1}{M(\Lambda z_m, \Lambda z, t)} - 1 < \left(\frac{1}{M(z_m, z, t)} - 1\right)^{\mu}\left(\frac{1}{M(z_m, z_{m+1}, t)} - 1\right)^{1-\mu}.$$

Taking limit as $m \to +\infty$ we get

$$\frac{1}{M(z, \Lambda z, t)} - 1 \leq 0,$$

which implies $M(z, \Lambda z, t) = 1$ which gives $\Lambda z = z$. Thus z is a fixed point of the map Λ. □

Taking $F(w) = \ln w$ in equation (3) we obtain interpolative Berinde contraction in the sense of Gregori and Sapena [4] in the setting of fuzzy metric space.

Corollary 4.2. *Let Λ be a self-map on a M-complete fuzzy metric space $(\Upsilon, M, *)$ satisfying*

$$M(\Lambda u, \Lambda v, t) > 0 \text{ implies } \frac{1}{M(\Lambda u, \Lambda v, t)} - 1 \leq e^{-\tau}\left(\begin{array}{c}\left(\frac{1}{M(u,v,t)} - 1\right)^{\mu} \\ \left(\frac{1}{M(u,\Lambda u,t)} - 1\right)^{1-\mu}\end{array}\right),$$

for all $u, v \in \Upsilon$, for all $t > 0$, then the map Λ has a fixed point in Υ.

Example 4.1. (of Theorem 4.1) Let $\Upsilon = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...,\right\} \bigcup \{1\}$ and define a fuzzy set M on $\Upsilon \times \Upsilon \times (0, +\infty)$ by:

$$M(u, v, t) = \begin{cases} 1; & \text{if } u = v, \\ \min\{u, v\}, & \text{if } u \neq v. \end{cases} \text{ for all u, v} \in \Upsilon, t \in (0, +\infty).$$

Taking $F(s) = -1/\sqrt{s}$, for all $s \in [0, +\infty)$, and $a * b = \min\{a, b\}$, then $(\Upsilon, M, *)$ is a M-complete fuzzy metric space.

Define self mapping Λ on Υ by $\Lambda(u_n) = u_{n+1}$, for all $n \in N$ and $\Lambda 1 = 1$.Additionally, the sequence $u_n = \frac{n}{n+1}$ is an increasing sequence. When considering the fixed point of the mapping Λ two scenarios emerge.

Case 1 Assume $u = u_n$ and $v = u_{n+1}$ for some $n \in N$. Consequently, we have $M(u, v, t) = M(u_n, u_{n+1}, t) = u_n, M(\Lambda u, \Lambda v, t) = M(u_{n+1}, u_{n+2}, t) = u_{n+1}$ and $M(u, \Lambda u, t) = M(u_n, u_{n+1}, t) = u_n$.

Case 2 Assume $u = u_n$ and $v > u_{n+1}$ for some $n \in N$. Thus, we find that $M(u, v, t) = u_n, M(\Lambda u, \Lambda v, t) = u_{n+1}$ and $M(u, \Lambda u, t) = u_n$. Therefore, in both scenarios, the contractive condition is expressed as

$$\tau + F\left(\frac{1}{u_{n+1}} - 1\right) \leq F\left(\frac{1}{u_n} - 1\right),$$

i.e.

$$\tau + \frac{-1}{\sqrt{\frac{1}{u_{n+1}} - 1}} \leq \frac{-1}{\sqrt{\frac{1}{u_n} - 1}},$$

and is satisfied if we select a sufficiently small $\tau > o$, as the sequence is increasing.

Thus all the conditions of Theorem 4.1 holds. Therefore, by Theorem 4.1 the map Λ has a fixed point in Υ.

5. Application to integral equation

In this section by applying Corollary 4.2, we determine the solution of nonlinear integral equations. For this, we need specified conditions in the frame of our result. The nonlinear integral equation is

$$w(t) = \psi(t) + \int_0^t F(t, s, w(s))ds, \text{t} \in [0, L], \tag{10}$$

where $F : [0, L] \times [0, L] \times R \to R$. Let $\Upsilon = C([0, L], R)$ be the set of all continuous function, $w : [0, L] \to R$ together with the sup norm

$$\|w\| = \max_{t \in [0,L]} |w(t)|.$$

Let $\Lambda : \Upsilon \to \Upsilon$ be a mapping defined as

$$\Lambda(w(t)) = \psi(t) + \int_0^t F(t, s, w(s))ds, \text{for all w} \in X, \tag{11}$$

and d is a metric on Υ defined by

$$d(w, v) = \sup_{t \in [0,L]} e^{-t}|w(t) - v(t)|.$$

Then (Υ, d) is a complete fuzzy metric space if we take $M(w, v, t) = \frac{t}{t+d(w,v)}$, for all $w, v \in \Upsilon$ and $t > 0$ with product norm. Now $w \in \Upsilon$ is a solution of equation (10) if and only if $w \in \Upsilon$ is a solution of equation (11).

Theorem 5.1. *Consider the self mapping Λ and equation (11) with continuous function F satisfying*

$$|F(t, s, \Lambda w(s) - F(t, s, \Lambda v(s)| \leq e^{-\tau}|w(s) - v(s)|^{\mu}|\Lambda w(s) - w(s)|^{1-\mu} \tag{12}$$

for some $\tau > 0$ *and* $0 < \mu < 1$*. Then the equation (11) has a solution in* Υ*.*

Proof.

$$\begin{aligned}
|\Lambda w(t) - \Lambda v(t)| &= \left| \int_0^t (F(t,s,w(s)) - F(t,s,v(s))\, ds \right| \\
&\leq \int_0^t |F(t,s,w(s)) - F(t,s,v(s)|ds \\
&\leq \int_0^t e^{-\tau} |w(s) - v(s)|^{\mu} |\Lambda w(s) - w(s)|^{1-\mu} ds \text{ (using(12))} \\
&= e^{-\tau} \int_0^t (e^{-s}|w(s) - v(s)|)^{\mu} (e^{-s}|\Lambda w(s) - w(s)|)^{1-\mu} e^s ds \\
&\leq e^{-\tau} \int_0^t (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu} e^s ds \\
&= e^{-\tau} (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu} \int_0^t e^s ds \\
&= e^{-\tau} (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu} e^t.
\end{aligned}$$

That is,

$$e^{-t} |\Lambda w(t) - \Lambda v(t)| \leq e^{-\tau} (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu},$$

so that

$$\sup_{t \in [0,L]} e^{-t} |\Lambda w(t) - \Lambda v(t)| \leq e^{-\tau} (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu},$$

or

$$d(\Lambda w, \Lambda v) \leq e^{-\tau} (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu}. \tag{13}$$

Also, we have

$$\begin{aligned}
\frac{1}{M(\Lambda w, \Lambda v, t)} - 1 &= \frac{d(\Lambda w, \Lambda v)}{t} \\
&\leq \frac{e^{-\tau} (d(w,v))^{\mu} (d(\Lambda w, w))^{1-\mu}}{t} \text{ (using(13))} \\
&= e^{-\tau} \left(\frac{d(w,v)}{t} \right)^{\mu} \left(\frac{d(\Lambda w, w)}{t} \right)^{1-\mu} \\
&= e^{-\tau} \left(\frac{1}{M(w,v,t)} - 1 \right)^{\mu} \left(\frac{1}{M(\Lambda w, w, t)} - 1 \right)^{(1-\mu)}
\end{aligned}$$

for all $w, v \in S, t > 0$. So, by Corollary 4.3 the map Λ has a fixed point in Υ. Hence by corollary 4.3 the equation (10) has a solution in $C(\Omega, R)$. □

Acknowledgment

All the authors are grateful to the anonymous referees for their excellent suggestions, which greatly improved the presentation of the paper.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **2** (1922), 133–181.

[2] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum **9** (2004), 43–53.

[3] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst., **64** (1994), 395–399.

[4] V. Gregori and A. Sapena, *On fixed-point theorems in fuzzy metric spaces*, Fuzzy Sets Syst., **125** (2002), 245–252.

[5] V. Gregori and J. J. Minnana, *Some remarks on fuzzy contractive mappings*, Fuzzy Sets Syst., **251** (2014), 101–103.

[6] H. Huang, B. Carić, T. Došenović, D. Rakić, and M. Brdar, *Fixed-point theorems in fuzzy metric spaces via fuzzy F-contractions*, Mathematics, **9**(6) (2021), 641.

[7] S. Jain and S. Jain, *Fuzzy generalized weak contraction and its application to Fredholm non-linear integral equation in fuzzy metric space*, J. Anal., **29** (2021), 619–632.

[8] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces,* Kybernetica, **15** (1975), 326–334.

[9] E. Karapinar, *Revisiting the Kannan type contractions via interpolation*, Adv. Theory Nonlinear Anal. Appl., **2**(2) (2018), 85–87.

[10] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Courier Corporation, 2011.

[11] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **94** (2012).

[12] L. A. Zadeh, *Fuzzy Sets*, Inf. Control **89** (1965), 338–353.

Received: May 2025
Accepted: June 2025

(Jain) Shri Vaishnav Vidyapeeth Vishwavidyalaya, Indore (M.P.), India
Email address: `shobajain1@yahoo.com`

(Jain) Shri Vaishnav Vidyapeeth Vishwavidyalaya, Indore (M.P.), India
Email address: `jainshishir11@rediffmail.com`

(Radenović) Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11 120 Beograd 35, Serbia
Email address: `radens@beotel.rs`