Mathematical Analysis and its Contemporary Applications
Volume 7, Issue 3, 2025, 127–140
doi: 10.30495/maca.2025.2060635.1137
ISSN 2716-9898

A study of coincidence point theorems for multivalued mappings on extended m_b-metric spaces

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Abstract. In recent studies, the exploration of coincidence points represents a fresh development within the area of contractive-type single-valued and multivalued theory. This paper establishes new coincidence point theorems pertaining to both single-valued and multivalued mappings within the context of extended m_b-metric spaces. Employing methodologies derived from classical fixed point theorems, such as Banach's Contraction Principle and Kannan's fixed point results, we elucidate the requisite conditions under which these mappings exhibit coincidence points. To underscore the practical implications of our principal theorem, we present an illustrative example that validates the results. These contributions advance the understanding of fixed point theory in extended m_b-metric spaces and offer new avenues for further research in this area.

Keywords: Extended m_b-metric space, multivalued mapping, H_θ-contraction, coincidence points

2020 Mathematics Subject Classification: 54H25, 47H10

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1. Introduction

Multivalued functions arise within the domain of optimal control theory as a consequence of the existence of controls, leading to an intrinsic multivalence in the

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evolution of the system. This is particularly evident in differential inclusions and associated fields like game theory, where the utilization of the Kakutani fixed point theorem for multivalued mappings has been instrumental in establishing the existence of Nash equilibria and various concepts in physics. Notably, the renowned Banach and Schauder fixed-point theorems have been extended to encompass multivalued mappings. In addition to exploring the existence of solutions for nonlinear Volterra integral equations and nonlinear integro-differential equations within the framework of Banach spaces, this approach holds promise for numerous other potential practical implementations. Nonlinear Volterra integral equations find relevance in various scientific contexts, including the modelling of phenomena like the spread of epidemics, behaviour of semiconductor devices, and the dynamics of populations.

Using the Pompeiu-Hausdorff metric, Nadler [**12**] investigated the fixed points of multivalued mappings. Specifically, he derived an expansion of the Banach Contraction Principle pertaining to multivalued mappings. Subsequently, the realm of fixed points associated with multivalued mappings underwent substantial development, evolving into a comprehensive and productive theory. Following this, numerous scholars have attained noteworthy results concerning fixed points for multivalued mappings, with practical implications in domains such as control theory, differential equations, and convex optimization (see [**1**, **2**, **3**, **5**, **6**, **7**, **10**, **11**, **14**]). The broadening of fixed point theorems occurred not solely through the relaxation of contractive conditions but also by easing the restrictions imposed on the space.

In 1994, Matthews [8] presented the notion of partial metric spaces as a generalization of the classical idea of metric spaces, where the distance between two points may not necessarily be zero. The notion of partial metric spaces was recently expanded to m-metric spaces by Asadi *et al.* [4], who also produced Banach and Kannan fixed point theorems in this context. Later, to expand both m-metric and b-metric by combining both within a nonempty set, Ş*ahin et al.* [**13**] introduced the idea of m_b-metric. A generalization of an m_b-metric space and an extended b-metric space, the concept of extended m_b-metric spaces was introduced in [9].

Under certain restrictions, this work proves new coincidence point theorems for single-valued and multivalued mappings in extended m_b-metric spaces. The approach makes use of traditional fixed-point theorems, such as Banach's Contraction Principle and Kannan's fixed point theorem. Furthermore, examples are provided to illustrate the validity of the main results.

2. Some Basic Concepts

We commence by introducing fundamental notations, definitions, and essential outcomes in extended m_b-metric spaces.

Definition 2.1. [8] A partial metric on a nonempty set Z is a function $q : Z \times Z \to \mathbb{R}^+$ such that for all u, v, $w \in Z$,

(i) $q(u,u) = q(v,v) = q(u,v) \Longleftrightarrow u = v$,
(ii) $q(u,u) \leq q(u,v)$,
(iii) $q(u,v) = q(v,u)$,
(iv) $q(u,v) \leq q(u,w) + q(w,v) - q(w,w)$.

A pair (Z,q) where Z is a nonempty set and q represents a partial metric on Z is called a partial metric space.

Definition 2.2. [4] Let Z be a nonempty set. An m-metric is a function $\gamma : Z \times Z \to \mathbb{R}^+$ if the following conditions are satisfied: for all $u,\ v,\ w \in Z$,

(i) $\gamma(u,u) = \gamma(v,v) = \gamma(u,v) \Longleftrightarrow u = v$,
(ii) $m_{uv} \leq \gamma(u,v)$,
(iii) $\gamma(u,v) = \gamma(v,u)$,
(iv) $(\gamma(u,v) - m_{uv}) \leq (\gamma(u,w) - m_{uw}) + (\gamma(w,v) - m_{wv})$,

where $m_{uv} := min\,\{\gamma(u,u),\ \gamma(v,v)\}$. Then the pair (Z,γ) is called an m-metric space.

Definition 2.3. [13] Let Z be a nonempty set. An m_b-metric is a function $\gamma_b : Z \times Z \to \mathbb{R}^+$ if the following conditions are satisfied: for all $u,\ v,\ w \in Z$,

(i) $\gamma_b(u,u) = \gamma_b(v,v) = \gamma_b(u,v) \Longleftrightarrow u = v$,
(ii) $m_{buv} \leq \gamma_b(u,v)$,
(iii) $\gamma_b(u,v) = \gamma_b(v,u)$,
(iv) $(\gamma_b(u,v) - m_{buv}) \leq s[\,(\gamma_b(u,w) - m_{buw}) + (\gamma_b(w,v) - m_{bwv})\,]$,

where $m_{buv} := min\,\{\gamma_b(u,u),\ \gamma_b(v,v)\}$. Then the pair (Z,γ_b) is called an m_b-metric space.

Definition 2.4. [9] Let Z be a nonempty set and $\theta : Z \times Z \to [1,\infty)$ be a function. An extended m_b-metric is a function $\gamma_\theta : Z \times Z \to \mathbb{R}^+$ if the following conditions are satisfied: for all $u,\ v,\ w \in Z$,

$(m_\theta 1)$ $\gamma_\theta(u,u) = \gamma_\theta(v,v) = \gamma_\theta(u,v) \Longleftrightarrow u = v$,
$(m_\theta 2)$ $m_{\theta uv} \leq \gamma_\theta(u,v)$,
$(m_\theta 3)$ $\gamma_\theta(u,v) = \gamma_\theta(v,u)$,
$(m_\theta 4)$ $(\gamma_\theta(u,v) - m_{\theta uv}) \leq \theta(u,v)[\,(\gamma_\theta(u,w) - m_{\theta uw}) + (\gamma_\theta(w,v) - m_{\theta wv})\,]$,

where $m_{\theta uv} := min\,\{\gamma_\theta(u,u),\ \gamma_\theta(v,v)\}$. Then the pair (Z,γ_θ) is called an extended m_b-metric space. The following notation is useful in the sequel:

$$M_{\theta uv} := \max\{\gamma_\theta(u,u),\ \gamma_\theta(v,v)\}.$$

We observe that the definition of an m_b-metric space is obtained if $\theta(u,v) = s$ for $s \geq 1$.

Example 2.5. [9] Let $Z = C([a,b],\mathbb{R})$ be the set of all continuous real valued functions on $[a,b]$. We define the functions $\gamma_\theta : Z \times Z \to \mathbb{R}^+$ and $\theta : Z \times Z \to [1,\infty)$ by $\gamma_\theta(f(t),g(t)) = \sup\limits_{t\in[a,b]} |f(t) - g(t)|^2$ and $\theta(f(t),g(t)) = |f(t)| + |g(t)| + 2$.

Then (Z, γ_θ) is an extended m_b-metric space with the function θ.

Proposition 2.1. [9] *Let (Z, γ_θ) be an extended m_b-metric space and u, v, $w \in Z$. Then we have*

1. $0 \leq M_{\theta uv} + m_{\theta uv} = \gamma_\theta(u,u) + \gamma_\theta(v,v)$;
2. $0 \leq M_{\theta uv} - m_{\theta uv} =\mid \gamma_\theta(u,u) - \gamma_\theta(v,v) \mid$;
3. $M_{\theta uv} - m_{\theta uv} \leq \theta(u,v)[(M_{\theta uw} - m_{\theta uw}) + (M_{\theta wv} - m_{\theta wv})]$.

Let (Z, γ_θ) be an extended m_b-metric space, $u \in Z$ and $r > 0$. The open ball with centered at $u \in Z$ and radius $r > 0$ is denoted by

$$B_{\gamma_\theta}(u,r) = \{\, v \in Z : \gamma_\theta(u,v) < m_{\theta uv} + r \,\}.$$

Definition 2.6. [9] A subset U of an extended m_b-metric space (Z, γ_θ) is called open if and only if for all $u \in U$, there exists $r > 0$ such that $B_{\gamma_\theta}(u,r) \subseteq U$.

It can be demonstrated that a topology on Z, say τ_{m_θ}, is the family of all open subsets of Z. Closed sets are the complements of the elements of τ_{m_θ} in Z.

Remark 2.7. Let (Z, γ_θ) be an extended m_b-metric space, (u_n) be a sequence in Z and $u \in Z$. Then (u_n) converges to u with respect to(w.r.t.) τ_{m_θ} if $\lim\limits_{n\to\infty}(\gamma_\theta(u_n,u) - m_{\theta u_n u}) = 0$.

Suppose that $\lim\limits_{n\to\infty}(\gamma_\theta(u_n,u) - m_{\theta u_n u}) = 0$. We shall show that $u_n \to u$ w.r.t. τ_{m_θ}. Let $U \in \tau_{m_\theta}$ and $u \in U$. Then there exists $\epsilon > 0$ such that $u \in B_{\gamma_\theta}(u,\epsilon) \subseteq U$. Since $\lim\limits_{n\to\infty}(\gamma_\theta(u_n,u) - m_{\theta u_n u}) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\gamma_\theta(u_n,u) - m_{\theta u_n u} < \epsilon$ for all $n \geq n_0$. This ensures that $u_n \in B_{\gamma_\theta}(u,\epsilon)$ for all $n \geq n_0$ and hence $u_n \in U$ for all $n \geq n_0$. Therefore, (u_n) converges to u w.r.t. τ_{m_θ} on Z.

In light of the aforementioned observation, we put forth the subsequent definition of convergence in a sequence and m_θ-Cauchy sequence within extended m_b-metric spaces.

Definition 2.8. Let (Z, γ_θ) be an extended m_b-metric space. Then:

1. A sequence (u_n) in an extended m_b-metric space (Z, γ_θ) converges to a point $u \in Z$ if $\lim\limits_{n\to\infty}(\gamma_\theta(u_n,u) - m_{\theta u_n u}) = 0$.
2. A sequence (u_n) in an extended m_b-metric space (Z, γ_θ) is called an m_θ-Cauchy sequence if $\lim\limits_{n,m\to\infty}(\gamma_\theta(u_n,u_m) - m_{\theta u_n u_m}) = 0$ and $\lim\limits_{n,m\to\infty}(M_{\theta u_n u_m} - m_{\theta u_n u_m}) = 0$.
3. An extended m_b-metric space (Z, γ_θ) is said to be complete if every m_θ-Cauchy sequence (u_n) in Z converges to a point $u \in Z$ such that

$$\lim_{n\to\infty}(\gamma_\theta(u_n,u) - m_{\theta u_n u}) = 0 \textit{ and } \lim_{n\to\infty}(M_{\theta u_n u} - m_{\theta u_n u}) = 0.$$

3. Coincidence Point Results

In this section, we establish certain results regarding coincidence points for a pair of multivalued and single valued mappings within extended m_b-metric spaces.

Definition 3.1. A subset A of an extended m_b-metric space (Z, γ_θ) is called bounded if there exist $u \in Z$ and $r > 0$ such that $a \in B_{\gamma_\theta}(u, r)$, i.e., $\gamma_\theta(a, u) < m_{\theta a u} + r$, for all $a \in A$.

Let $CB_{\gamma_\theta}(Z)$ represents the collection of all nonempty, bounded, and closed sets within an extended m_b-metric space (Z, γ_θ). For every $C,\, D \in CB_{\gamma_\theta}(Z)$, define

$$H_\theta(C,\, D) = \max\{\, \delta_\theta(C,\, D),\, \delta_\theta(D,\, C)\},$$

where $\delta_\theta(C,\, D) = \sup\{\, \gamma_\theta(c,\, D) : c \in C\,\}$ and $\gamma_\theta(c,\, D) = \inf\{\, \gamma_\theta(c, d) : d \in D\,\}$.

The closure of a set C with respect to the extended m_b-metric γ_θ is denoted as $\overline{C}$. It is evident that C is closed in $(Z,\, \gamma_\theta)$ if and only if $\overline{C} = C$.

Lemma 3.1. *Let C be a nonempty subset of an extended m_b-metric space (Z, γ_θ), then $u \in \overline{C}$ if and only if $\gamma_\theta(u, C) = \sup_{c \in C} m_{\theta u c}$.*

Proof.

$$\begin{aligned} u \in \overline{C} \quad &\Longleftrightarrow \quad B_{\gamma_\theta}(u, r) \cap C \neq \emptyset,\ \text{for all } r > 0 \\ &\Longleftrightarrow \quad \gamma_\theta(u, c) < m_{\theta u c} + r,\ \text{for some } c \in C \\ &\Longleftrightarrow \quad \gamma_\theta(u, c) - m_{\theta u c} < r \\ &\Longleftrightarrow \quad \inf\{\gamma_\theta(u, c) - m_{\theta u c} : c \in C\} = 0 \\ &\Longleftrightarrow \quad \inf\{\gamma_\theta(u, c) : c \in C\} = \sup\{m_{\theta u c} : c \in C\} \\ &\Longleftrightarrow \quad \gamma_\theta(u, C) = \sup_{c \in C} m_{\theta u c}. \end{aligned}$$

□

Proposition 3.2. *For any $C,\ D \in CB_{\gamma_\theta}(Z)$, the following are true*

(i) $H_\theta(C, C) = \delta_\theta(C, C) = \sup_{u \in C}\{\sup_{v \in C} m_{\theta u v}\}$,

(ii) $H_\theta(C, D) = H_\theta(D, C)$.

Proof. (i) Since $C \in CB_{\gamma_\theta}(Z)$, we have $\overline{C} = C$. Then from Lemma (3.1), $\gamma_\theta(u, C) = \sup_{v \in C} m_{\theta u v}$. Therefore

$$H_\theta(C, C) = \delta_\theta(C, C) = \sup_{u \in C} \gamma_\theta(u, C) = \sup_{u \in C}\{\sup_{v \in C} m_{\theta u v}\}.$$

(ii) The definition of an extended m_b-metric space (Z, γ_θ) gives rise to the following implications.

□

Lemma 3.3. *Let (Z, γ_θ) be an extended m_b-metric space and $\theta : Z \times Z \to [1, \infty)$ be bounded above. Then for any subset A of Z and any u, $v \in Z$, we have*

$$\gamma_\theta(u, A) - \sup_{a\in A} m_{\theta ua} \leq \sup_{a\in A} \theta(u, a) \left[(\gamma_\theta(u, v) - m_{\theta uv}) + \left(\gamma_\theta(v, A) - \inf_{a\in A} m_{\theta va} \right) \right].$$

Proof. We have

$$\begin{aligned} \gamma_\theta(u,\, A) - \sup_{a\in A} m_{\theta ua} &= \inf_{a\in A} \gamma_\theta(u,\, a) + \inf_{a\in A}(-m_{\theta ua}) \\ &\leq \inf_{a\in A}[\gamma_\theta(u,\, a) - m_{\theta ua}] \\ &\leq \gamma_\theta(u,\, a) - m_{\theta ua}, \\ &\leq \theta(u,a) \left[(\gamma_\theta(u,\, v) - m_{\theta uv}) + (\gamma_\theta(v,\, a) - m_{\theta va}) \right], \\ &\leq \sup_{a\in A} \theta(u,\, a) \left[(\gamma_\theta(u,\, v) - m_{\theta uv}) + (\gamma_\theta(v,\, a) - \inf_{a\in A} m_{\theta va}) \right], \end{aligned}$$

Taking infimum for all $a \in A$, we get

$$\gamma_\theta(u,\, A) - \sup_{a\in A} m_{\theta ua} \leq \sup_{a\in A} \theta(u,\, a) \left[(\gamma_\theta(u,\, v) - m_{\theta uv}) + \left(\gamma_\theta(v,\, A) - \inf_{a\in A} m_{\theta va} \right) \right].$$

□

Lemma 3.4. *Let C, $D \in CB_{\gamma_\theta}(Z)$ and $h > 1$. Then for every $u \in C$, there is atleast one $v \in D$ such that $\gamma_\theta(u,v) \leq h\, H_\theta(C,\, D)$.*

Proof. We assume that there exists an $u \in C$ such that $\gamma_\theta(u,v) > hH_\theta(C,D)$ for all $v \in D$. This implies that

$$\inf_{v\in D}\{\gamma_\theta(u,v)\} \geq h\, H_\theta(C,D).$$

Thus, $\gamma_\theta(u,D) \geq h\, H_\theta(C,D)$. Now, we observe that

$$H_\theta(C,D) \geq \delta_\theta(C,D) = \sup_{u\in C} \gamma_\theta(u,D) \geq \gamma_\theta(u,D) \geq h\, H_\theta(C,D).$$

Since $H_\theta(C,D) \neq 0$, $h \leq 1$, which is a contradiction. □

Definition 3.2. Let (Z, γ_θ) be an extended m_b-metric space and $G : Z \to CB_{\gamma_\theta}(Z)$ and $g : Z \to Z$ be a multivalued mapping and a single valued mapping respectively. If $v = gu \in Gu$ for some u in Z, then u is called a coincidence point of G and g and v is called a point of coincidence of G and g.

Definition 3.3. Let (Z, γ_θ) be an extended m_b-metric space with the function θ. Then a multivalued mapping $G : Z \to CB_{\gamma_\theta}(Z)$ and a single valued mapping $g : Z \to Z$ are called H_θ-Banach contraction if the following condition holds: there exists $k \in (0, 1)$ such that

$$H_\theta(Gu, Gv) \leq k\, \gamma_\theta(gu, gv), \tag{1}$$

for all u, $v \in Z$.

Definition 3.4. Let (Z, γ_θ) be an extended m_b-metric space with the function θ. Then a multivalued mapping $G : Z \to CB_{\gamma_\theta}(Z)$ and a single valued mapping $g : Z \to Z$ are called H_θ-Kannan contraction if the following condition holds: there exists $k \in (0, \frac{1}{2})$ such that

$$H_\theta(Gu, Gv) \leq k\,[\gamma_\theta(gu, Gu) + \gamma_\theta(gv, Gv)], \tag{2}$$

for all u, $v \in Z$.

Theorem 3.5. *Let (Z, γ_θ) be an extended m_b-metric space and $\theta : Z \times Z \to [1, \infty)$ be bounded above. Let $G : Z \to CB_{\gamma_\theta}(Z)$ and $g : Z \to Z$ be H_θ-Banach contraction with the constant $k \in (0, 1)$. Suppose that $G(Z) \subseteq g(Z)$ and $g(Z)$ is a complete subspace of Z with $\lim\limits_{n,m\to\infty} \theta(gu_n, gu_m) < \dfrac{1}{\sqrt{k}}$, where $gu_{n+1} \in Gu_n$, $n = 0, 1, 2, \cdots$, for every $u_0 \in Z$. Then g and G have a point of coincidence say a in Z with $\gamma_\theta(a, a) = 0$.*

Proof. Since Z is a nonempty set, we consider $u_0 \in Z$ and fix an element $gu_1 \in Gu_0$. Using Lemma 3.4, there exists $gu_2 \in Gu_1$ such that

$$\gamma_\theta(gu_1, gu_2) \leq \frac{1}{\sqrt{k}} H_\theta(Gu_0, Gu_1). \tag{3}$$

By using (1) we obtain,

$$H_\theta(Gu_0, Gu_1) \leq k\,\gamma_\theta(gu_0, gu_1).$$

Now from (3) we have,

$$\gamma_\theta(gu_1, gu_2) \leq \frac{1}{\sqrt{k}} k\gamma_\theta(gu_0, gu_1) = \sqrt{k}\,\gamma_\theta(gu_0, gu_1) = r\,\gamma_\theta(gu_0, gu_1)$$

where $r = \sqrt{k} < 1$. Again, from Lemma(3.4), there exists $gu_3 \in Gu_2$ such that

$$\gamma_\theta(gu_2, gu_3) \leq r\gamma_\theta(gu_1, gu_2).$$

In this manner, we obtain a sequence (gu_n) in $g(Z)$ such that $gu_{n+1} \in Gu_n$, for $n = 0, 1, 2, \ldots$ and

$$\gamma_\theta(gu_n, gu_{n+1}) \leq r\gamma_\theta(gu_{n-1}, gu_n),\ \textit{for all } n \geq 1. \tag{4}$$

Repeating the condition (4) n-times, we get

$$0 \leq \gamma_\theta(gu_n, gu_{n+1}) \leq r^n\gamma_\theta(gu_0, gu_1). \tag{5}$$

Taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \gamma_\theta(gu_n, gu_{n+1}) = 0. \tag{6}$$

Now, for $m, n \in \mathbb{N}$ with $m > n$, we have by using the triangle inequality of the extended m_b-metric space and condition (5) that

$$\begin{aligned} & \gamma_\theta(gu_n, gu_m) - m_{\theta gu_n gu_m} \\ \leq\ & \theta(gu_n, gu_m) r^n \gamma_\theta(gu_0, gu_1) + \theta(gu_n, gu_m)\theta(gu_{n+1}, gu_m) r^{n+1}\gamma_\theta(gu_0, gu_1) \\ & + \cdots + \theta(gu_n, gu_m)\cdots\theta(gu_{m-1}, gu_m) r^{m-1}\gamma_\theta(gu_0, gu_1) \\ \leq\ & \gamma_\theta(gu_0, gu_1)[\theta(gu_1, gu_m)\theta(gu_2, gu_m)\cdots\theta(gu_{n-1}, gu_m)\theta(gu_n, gu_m) r^n \\ & +\theta(gu_1, gu_m)\theta(gu_2, gu_m)\cdots\theta(gu_n, gu_m)\theta(gu_{n+1}, gu_m) r^{n+1} \\ & + \cdots + \theta(gu_1, gu_m)\theta(gu_2, gu_m)\cdots\theta(gu_{m-2}, gu_m)\theta(gu_{m-1}, gu_m) r^{m-1}]. \end{aligned} \tag{7}$$

Since $\lim\limits_{n,m\to\infty} \theta(gu_n, gu_m) r < 1$, the series $\sum\limits_{n=1}^{\infty} r^n \prod\limits_{i=1}^{n} \theta(gu_i, gu_m)$ converges by ratio test for large $m \in \mathbb{N}$. Let $S = \sum\limits_{n=1}^{\infty} r^n \prod\limits_{i=1}^{n} \theta(gu_i, gu_m)$, where $S_n = \sum\limits_{j=1}^{n} r^j \prod\limits_{i=1}^{j} \theta(gu_i, gu_m)$. Thus, for $m > n$, we deduce that,

$$\gamma_\theta(gu_n, gu_m) - m_{\theta gu_n gu_m} \leq \gamma_\theta(gu_0, gu_1)[S_{m-1} - S_n].$$

Taking the limit as $n \to \infty$, we conclude that

$$\lim_{n,m\to\infty} (\gamma_\theta(gu_n, gu_m) - m_{\theta gu_n gu_m}) = 0.$$

By using the proposition 3.2 and condition (1), we have for all $n \geq 1$

$$\begin{aligned} 0 \leq m_{\theta gu_{n+1} gu_{n+1}} \leq \sup_{gu_{n+1}\in Gu_n} \{ \sup_{gu_{n+1}\in Gu_n} m_{\theta gu_{n+1} gu_{n+1}} \} = H_\theta(Gu_n, Gu_n) \\ \implies 0 \leq \gamma_\theta(gu_{n+1}, gu_{n+1}) \leq k\gamma_\theta(gu_n, gu_n) < \gamma_\theta(gu_n, gu_n). \end{aligned}$$

This shows that the sequence $(\gamma_\theta(gu_n, gu_n))$ is decreasing. So $\lim\limits_{n\to\infty} \gamma_\theta(gu_n, gu_n)$ exists. By using $(m_\theta 2)$, we get

$$0 \leq m_{\theta gu_n gu_{n+1}} = \gamma_\theta(gu_{n+1}, gu_{n+1}) \leq \gamma_\theta(gu_n, gu_{n+1}).$$

Taking the limit of the above inequality as $n \to \infty$ and using the condition (6), we obtain

$$\lim_{n\to\infty} \gamma_\theta(gu_n, gu_n) = 0 \tag{8}$$

As $0 \leq M_{\theta gu_n gu_m} - m_{\theta gu_n gu_m} = |\gamma_\theta(gu_n, gu_n) - \gamma_\theta(gu_m, gu_m)|$, it follows that

$$\lim_{n,m\to\infty} (M_{\theta gu_n gu_m} - m_{\theta gu_n gu_m}) = 0.$$

Therefore, (gu_n) is an m_θ-Cauchy sequence in $g(Z)$. Since $g(Z)$ is complete, there exists $a \in g(Z)$ such that $gu_n \to a = gt$ for some $t \in Z$. So, it must be the case that

$$\begin{aligned} \lim_{n\to\infty} (\gamma_\theta(gu_n, gt) - m_{\theta gu_n gt}) = 0 \\ \textit{and} \lim_{n\to\infty} (M_{\theta gu_n gt} - m_{\theta gu_n gt}) = 0. \end{aligned} \tag{9}$$

As $\lim_{n\to\infty} m_{\theta g u_n g t} = 0$, it follows that

$$\lim_{n\to\infty} \gamma_\theta(gu_n, gt) = 0 \tag{10}$$

$$\text{and} \quad \lim_{n\to\infty} M_{\theta g u_n g t} = 0.$$

From 1, we have

$$0 \le H_\theta(Gu_n, Gt) \le k\,\gamma_\theta(gu_n, gt).$$

Taking the limit as $n \to \infty$ and using the condition (10), we obtain

$$\lim_{n\to\infty} H_\theta(Gu_n, Gt) = 0. \tag{11}$$

Now, since $gu_{n+1} \in Gu_n$, we have

$$0 \le \gamma_\theta(gu_{n+1}, Gt) \le \delta_\theta(Gu_n, Gt) \le H_\theta(Gu_n, Gt).$$

Taking the limit as $n \to \infty$ and using (11), we get

$$\lim_{n\to\infty} \gamma_\theta(gu_{n+1}, Gt) = 0. \tag{12}$$

Now, using the proposition 3.3

$$\begin{aligned}
\gamma_\theta(gt, Gt) - \sup_{x\in Gt} m_{\theta g t x} &\le \sup_{x\in Gt} \theta(gt, x)[(\gamma_\theta(gt, gu_{n+1}) - m_{\theta g t g u_{n+1}}) \\
&+ (\gamma_\theta(gu_{n+1}, Gt) - \inf_{x\in Gt} m_{\theta g u_{n+1} x})] \\
&\le \sup_{x\in Gt} \theta(gt, x)[(\gamma_\theta(gt, gu_{n+1}) - m_{\theta g t g u_{n+1}}) + \gamma_\theta(gu_{n+1}, Gt)]
\end{aligned}$$

Taking limit as $n \to \infty$ and using (9) and (12), we get

$$\gamma_\theta(gt, Gt) \le \sup_{x\in Gt} m_{\theta g t x}. \tag{13}$$

Again, we have $\lim_{n\to\infty}(M_{\theta g u_n g t} + m_{\theta g u_n g t}) = 0$. This implies that $\lim_{n\to\infty}(\gamma_\theta(gu_n, gu_n) + \gamma_\theta(gt, gt)) = 0$, Then

$$\lim_{n\to\infty} \gamma_\theta(gu_n, gu_n) + \gamma_\theta(gt, gt) = 0$$

Now, by using (9), we have $\gamma_\theta(gt, gt) = 0$. So,

$$\sup_{x\in Gt} m_{\theta g t x} = \sup_{x\in Gt} \min\{\gamma_\theta(gt, gt), \gamma_\theta(x, x)\} = 0.$$

So, by using (13), we have $\gamma_\theta(gt, Gt) = 0$. Thus, $\gamma_\theta(gt, Gt) = \sup_{x\in Gt} m_{\theta g t x}$. By using Lemma 3.1, we get $gt \in \overline{Gt} = Gt$, as Gt is closed. So, a is a point of coincidence of g and G in Z with $\gamma_\theta(a, a) = 0$. □

Corollary 3.6. *Let (Z, γ_θ) be a complete extended m_b-metric space and $\theta : Z \times Z \to [1, \infty)$ be bounded above. Suppose that $G : Z \to CB_{\gamma_\theta}(Z)$ satisfies the following condition: there exists $k \in (0, 1)$ such that*

$$H_\theta(Gu, Gv) \le k\,\gamma_\theta(u, v),$$

for all $u, v \in Z$. *Also assume that* $\lim_{n,m\to\infty} \theta(u_n, v_m) < \dfrac{1}{\sqrt{k}}$, *where* $u_{n+1} \in Gu_n$, $n = 0, 1, 2, \ldots$, *for every* $u_0 \in Z$. *Then* G *has a fixed point* a*(say) in* Z *with* $\gamma_\theta(a, a) = 0$.

Proof. By taking $g = I$, the identity map on Z, the proof is derived from Theorem 3.5. □

Theorem 3.7. *Let* (Z, γ_θ) *be an extended* m_b*-metric space and* $\theta : Z \times Z \to [1, \infty)$ *be bounded above by 2. Let* $G : Z \to CB_{\gamma_\theta}(Z)$ *and* $g : Z \to Z$ *be* H_θ*-Kannan contraction with the constant* $k \in (0, \frac{1}{2})$. *Suppose that* $G(Z) \subseteq g(Z)$ *and* $g(Z)$ *is a complete subspace of* Z *with* $\lim_{n,m\to\infty} \theta(gu_n, gu_m) < \dfrac{1}{r}$, *where* $r = \frac{k}{1-k}$ *and* $gu_{n+1} \in Gu_n$, $n = 0, 1, 2, \cdots$, *for every* $u_0 \in Z$. *Then* g *and* G *have a point of coincidence say* a *in* Z *with* $\gamma_\theta(a, a) = 0$.

Proof. Let $u_0 \in Z$ be arbitrary. As $Gu_0 \subseteq g(Z)$, we can choose an element $u_1 \in Z$ such that $gu_1 \in Gu_0$. As Gu_1 is a closed subset of Z, we can choose $gu_2 \in Gu_1$ such that

$$\gamma_\theta(gu_1, gu_2) = \gamma_\theta(gu_1, Gu_1) \leq \delta_\theta(Gu_0, Gu_1) \leq H_\theta(Gu_0, Gu_1).$$

Again, we can choose $gu_3 \in Gu_2$ such that

$$\gamma_\theta(gu_2, gu_3) \leq H_\theta(Gu_1, Gu_2).$$

Continuing in this way, we get a sequence (gu_n) such that $gu_{n+1} \in Gu_n$ with

$$\gamma_\theta(gu_n, gu_{n+1}) \leq H_\theta(Gu_{n-1}, Gu_n). \tag{14}$$

Using (2) in (14), we get

$$\begin{aligned} \gamma_\theta(gu_n, gu_{n+1}) &\leq k[\gamma_\theta(gu_{n-1}, Gu_{n-1}) + \gamma_\theta(gu_n, Gu_n)] \\ &\leq k[\gamma_\theta(gu_{n-1}, gu_n) + \gamma_\theta(gu_n, gu_{n+1})] \end{aligned}$$

$$\textit{Thus},\ \gamma_\theta(gu_n, gu_{n+1}) \leq \frac{k}{1-k}\, \gamma_\theta(gu_{n-1}, gu_n).$$

Let $r = \frac{k}{1-k}$. Since $k < \frac{1}{2}$, we have $r < 1$. So

$$\gamma_\theta(gu_n, gu_{n+1}) \leq r\, \gamma_\theta(gu_{n-1}, gu_n). \tag{15}$$

Repeating the condition (15) n-times, we get

$$0 \leq \gamma_\theta(gu_n, gu_{n+1}) \leq r^n\, \gamma_\theta(gu_0, gu_1). \tag{16}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \gamma_\theta(gu_n, gu_{n+1}) = 0. \tag{17}$$

Proceed similarly as in Theorem (3.5), we get

$$\lim_{n,m\to\infty} (\gamma_\theta(gu_n, gu_m) - m_{\theta gu_n gu_m}) = 0$$

By using the Proposition 3.2 and condition (2), we have for all $n \geq 1$

$$0 \leq m_{\theta g u_{n+1} g u_{n+1}} \leq \sup_{g u_{n+1} \in G u_n} \{ \sup_{g u_{n+1} \in G u_n} m_{\theta g u_{n+1} g u_{n+1}} \} = H_\theta(G u_n, G u_n)$$
$$\implies 0 \leq \gamma_\theta(g u_{n+1}, g u_{n+1}) \leq 2k \gamma_\theta(g u_n, G u_n) \leq 2k\, \gamma_\theta(g u_n, g u_{n+1}).$$

Taking the limit of the above inequality as $n \to \infty$ and using the condition (17), we obtain

$$\lim_{n\to\infty} \gamma_\theta(g u_{n+1}, g u_{n+1}) = 0. \tag{18}$$

As $0 \leq M_{\theta g u_n g u_m} - m_{\theta g u_n g u_m} = |\gamma_\theta(g u_n, g u_n) - \gamma_\theta(g u_m, g u_m)|$, it follows that

$$\lim_{n,m\to\infty} (M_{\theta g u_n g u_m} - m_{\theta g u_n g u_m}) = 0.$$

Therefore, $(g u_n)$ is an m_θ-Cauchy sequence in $g(Z)$. Since $g(Z)$ is complete, there exists $a \in g(Z)$ such that $g u_n \to a = gt$ for some $t \in Z$. So, it must be the case that

$$\lim_{n\to\infty} (\gamma_\theta(g u_n, gt) - m_{\theta g u_n gt}) = 0 \tag{19}$$

and

$$\lim_{n\to\infty} (M_{\theta g u_n gt} - m_{\theta g u_n gt}) = 0. \tag{20}$$

As $0 \leq m_{\theta g u_n gt} \leq \gamma_\theta(g u_n, g u_n)$. Taking limit as $n \to \infty$ and using (18), we get

$$\lim_{n\to\infty} m_{\theta g u_n gt} = 0. \tag{21}$$

Using (21) in (19) and (20), we get

$$\lim_{n\to\infty} \gamma_\theta(g u_n, gt) = 0 \tag{22}$$

and

$$\lim_{n\to\infty} M_{\theta g u_n gt} = 0. \tag{23}$$

Adding (21) and (23), we get $\lim_{n\to\infty} M_{\theta g u_n gt} + \lim_{n\to\infty} m_{\theta g u_n gt} = 0$. This implies that $\lim_{n\to\infty}[M_{\theta g u_n gt} + m_{\theta g u_n gt}] = 0$ and so $\lim_{n\to\infty}[\gamma_\theta(g u_n, g u_n) + \gamma_\theta(gt, gt)] = 0$. Thus, $\lim_{n\to\infty} \gamma_\theta(g u_n, g u_n) + \gamma_\theta(gt, gt) = 0$. Hence,

$$\gamma_\theta(gt, gt) = 0. \tag{24}$$

Now,

$$\begin{aligned} \gamma_\theta(g u_{n+1}, Gt) &\leq \delta_\theta(G u_n, Gt) \leq H_\theta(G u_n, Gt) \\ &\leq k\,[\gamma_\theta(g u_n, G u_n) + \gamma_\theta(gt, Gt)]. \end{aligned}$$

Taking limit as $n \to \infty$ and using (17), we get

$$\lim_{n\to\infty} \gamma_\theta(g u_{n+1}, Gt) \leq k\, \gamma_\theta(gt, Gt). \tag{25}$$

Using the proposition 3.3, we have

$$\begin{aligned}\gamma_\theta(gt,Gt)-\sup_{x\in Gt} m_{\theta gtx} &\leq \sup_{x\in Gt}\theta(gt,x)\big[(\gamma_\theta(gt,gu_{n+1})-m_{\theta gtgu_{n+1}})\\ &\quad +(\gamma_\theta(gu_{n+1},Gt)-\inf_{x\in Gt} m_{\theta gu_{n+1}x})\big].\end{aligned}$$

Taking limit as $n\to\infty$ and using (19), we get

$$\begin{aligned}\gamma_\theta(gt,Gt)-\sup_{x\in Gt} m_{\theta gtx} &\leq \sup_{x\in Gt}\theta(gt,x)\lim_{n\to\infty}\gamma_\theta(gu_{n+1},Gt)\\ &\leq \sup_{x\in Gt}\theta(gt,x)\,k\,\gamma_\theta(gt,Gt).\end{aligned}$$

As $\gamma_\theta(gt,gt)=0$, so $\sup\limits_{x\in Gt} m_{\theta gtx}=0$.

$$\gamma_\theta(gt,Gt)\leq \sup_{x\in Gt}\theta(gt,x)\,k\,\gamma_\theta(gt,Gt).$$

If $\gamma_\theta(gt,Gt)>0$, then $1\leq \sup\limits_{x\in Gt}\theta(gt,x)\,k\leq 2k$ i.e, $\frac{1}{2}\leq k$. This is a contradiction. So, we have $\gamma_\theta(gt,Gt)=0$. Thus, $\gamma_\theta(gt,Gt)=\sup\limits_{x\in Gt} m_{\theta gtx}$. By using Lemma 3.1, we get $gt\in\overline{Gt}=Gt$, as Gt is closed. So, a is a point of coincidence of g and G in Z with $\gamma_\theta(a,a)=0$. □

Corollary 3.8. *Let (Z,γ_θ) be a complete extended m_b-metric space and $\theta: Z\times Z\to[1,\infty)$ be bounded above by 2. Suppose that $G: Z\to CB_{\gamma_\theta}(Z)$ satisfies the following condition: there exists $k\in(0,\frac{1}{2})$ such that*

$$H_\theta(Gu,Gv)\leq k\,[\gamma_\theta(u,Gu)+\gamma_\theta(v,Gv)],$$

for all u, $v\in Z$. Also assume that $\lim\limits_{n,m\to\infty}\theta(u_n,u_m)<\dfrac{1}{r}$, where $r=\frac{k}{1-k}$ and $u_{n+1}\in Gu_n$, $n=0,1,2,\cdots$, for every $u_0\in Z$. Then G has a fixed point a(say) in Z with $\gamma_\theta(a,a)=0$.

Proof. By using $g=I$, the proof is derived from Theorem 3.7. □

Example 3.5. Let $Z=\{1,\,2,\,3\}$ and the function $\theta: Z\times Z\to[1,\infty)$ be defined by $\theta(u,v)=uv$, for all u, $v\in Z$. We take the function $\gamma_\theta: Z\times Z\to[0,\infty)$ as

$$\begin{aligned}\gamma_\theta(1,1)=\gamma_\theta(2,2)=\gamma_\theta(3,3)&=0,\\ \gamma_\theta(1,2)=\gamma_\theta(2,1)&=6,\\ \gamma_\theta(1,3)=\gamma_\theta(3,1)=294=\gamma_\theta(2,3)&=\gamma_\theta(3,2).\end{aligned}$$

Then (Z,γ_θ) is a complete extended m_b-metric space. Let $G: Z\to CB_{\gamma_\theta}(Z)$ be defined by $G1=G2=\{1\}$, $G3=\{1,2\}$ and $g: Z\to Z$ be defined by $g1=2$, $g2=1$, $g3=3$. Then $g(Z)(=Z)$ is a complete extended m_θ-metric space. For every $u\in Z$, it is simple to verify that each Gu is a closed and bounded subset of Z. We now examine the following possible cases:

Case-I: $u,\, v \in \{1,\, 2\}$. In this case, $H_\theta(Gu, Gv) = H_\theta(\{1\},\, \{1\}) = \gamma_\theta(1,1) = 0 \leq \frac{1}{49}\,\gamma_\theta(gu, gv)$.

Case-II: $u \in \{1,\, 2\},\ v = 3$. Then,

$$\begin{aligned} H_\theta(Gu, Gv) &= H_\theta(\{1\},\, \{1,\, 2\}) \\ &= \max\{\delta_\theta(\{1\},\, \{1,\, 2\}), \delta_\theta(\{1,\, 2\}, \{1\})\} \\ &= \max\{0,\, 6\} \\ &= 6. \end{aligned}$$

If $u \in \{1,\, 2\},\ v = 3$, then $\gamma_\theta(gu, gv) = 294$ and so

$$H_\theta(Gu, Gv) = 6 = \frac{1}{49}\gamma_\theta(gu, gv).$$

Case-III: $u = v = 3$. Then,

$$\begin{aligned} H_\theta(Gu, Gv) = \delta_\theta(G3,\, G3) &= \sup\{\gamma_\theta(w, G3) : w \in G3\} \\ &= \max\{\gamma_\theta(1, G3),\, \gamma_\theta(2, G3)\} \\ &= 0 \\ &\leq \frac{1}{49}\,\gamma_\theta(gu, gv). \end{aligned}$$

Thus, we have

$$H_\theta(Gu, Gv) \leq k\,\gamma_\theta(gu, gv)$$

for all $u,\, v \in Z$ with $k = \frac{1}{49}$. Also, $\lim\limits_{n,m\to\infty} \theta(gu_n, gu_m) < \dfrac{1}{\sqrt{k}}$, where $gu_{n+1} \in Gu_n$, $n = 0, 1, 2, \ldots$, for every $u_0 \in Z$. We see that 1 is a point of coincidence of g and G in $g(Z)$ with $\gamma_\theta(1, 1) = 0$, and all the requirements of Theorem 3.5 are satisfied.

4. Conclusions

This paper presents new coincidence point theorems for single-valued and multi-valued mappings in extended m_b-metric spaces. Using classical fixed point methods, we identify the necessary conditions for the existence of coincidence points for these mappings and validate the results with examples, contributing to both theory and future research in this field.

Acknowledgment

The author would like to thank the referees for their valuable comments.

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Received: May 2025
Accepted: August 2025

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