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Quadrature rule extended cubic spline approach for a class of nonlinear and linear Fredholm integral equations

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Abstract. In this research, we consider the linear and nonlinear Fredholm integral equations (FIEs). The main aim of research is to approximate the integral by Gauss-Turán quadrature rule and then using an extended cubic spline as the base function. The unknown coefficients are determined in combination by the collocation method. The arising system of linear and nonlinear equations can be solved. Error analysis is investigated theoretically. Numerical text problems are considered to justify the applicability and efficient nature of our approach; comparison of the results justifies the considerable accuracy and efficiency proposed methods. The extended parameter in valued in the spline can be chosen in such a way to improve the accuracy also.

Keywords: Linear and nonlinear FIEs, Extended cubic spline, Gauss-Turán quadrature rule, Error analysis

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1. Introduction

It is well known that FIEs are of the form

$$U(\xi) = g(\xi) + \int_{\alpha}^{\beta} K(\xi, y, U(y))dy, \ \xi \in [\alpha, \beta], \tag{1}$$

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where $K(\xi, y, U(y))$ is continuous on $[\alpha, \beta]$ and satisfies a uniform Lipschitz condition. FIE(1) arise in various fields of science and dynamics, such as the spread of epidemics, and semiconductor devices [**28, 5, 4**]. For the solution of the FIE (1), several numerical approaches have been proposed, such as the Homotopy-perturbation method [**7, 8, 20**], the wavelet basis [**21, 29**], and the collocation method basis [**23**], the superconvergent methods based on quasi-interpolating operators [**2**], the multistep collocation[**10**], the Taylor-series expansion methods [**31, 19, 3**], the Newton-Kantorovich-quadrature method [**26**], the Bernstein polynomial [**22**], the quadrature approach based on B-spline [**15**], hat functions operational matrix [**14**]. We develop a collocation by using an extended cubic spline to approximate in FIE(1).

2. Extended cubic spline collocation approach

We apply extended cubic spline collocation method to approximate solution of FIE(1). Let $\Delta_M : \{\alpha = y_0 < y_1 < \ldots < y_M = \beta\}$ be a uniform partition of the interval $[\alpha, \beta]$ with step size $h = \frac{\beta-\alpha}{M}$. The extended cubic spline $B_r(y, \theta)$ is defined as:

$$B_r(y,\theta) = \frac{1}{24h^4}\begin{cases} 4k(1-\theta)(y-y_{r-2})^3 + 3\theta(y-y_{r-2})^4, & y_{r-2} < y \leq y_{r-1} \\ (4-\theta)k^4 + 12k^3(y-y_{r-1}) + 6k^2(2+\theta)(y-y_{r-1})^2 & \\ -12k(y-y_{r-1})^3 - 3\theta(y-y_{r-1})^4, & y_{r-1} < y \leq y_r \\ (4-\theta)k^4 + 12k^3(y_{r+1}-y) + 6k^2(2+\theta)(y_{r+1}-y)^2 & \\ -12k(y_{r+1}-y)^3 - 3\theta(y_{r+1}-y)^4, & y_r < y \leq y_{r+1} \\ 4k(1-\theta)(y_{r+2}-y)^3 + 3\theta(y_{r+2}-y)^4, & y_{r+1} < y \leq y_{r+2} \\ 0, & otherwise. \end{cases} \tag{2}$$

The extended cubic spline function has one arbitrary parameter θ,when θ tends to zero the extended cubic spline reduced to convectional cubic spline function. For $\theta \geqslant -2$, spline and extended spline share the same properties: local support, non-negativity, partition of unity and C^2 continuity. The parameter θ control the tension of the solution curve [**30, 24**]. we consider a extended cubic spline $S(y)$ of the form [**27**]

$$\begin{aligned} S(y) &= \textstyle\sum_{r=-1}^{M+1} t_r B_r(y,\theta) \\ &= \tfrac{B_{-1}(y,\theta)}{B_{-1}(y_0,\theta)} W_0 + \tfrac{B_{M+1}(y,\theta)}{B_{M+1}(y_M,\theta)} W_1 + \textstyle\sum_{r=0}^{M} t_r \overline{B}_r(y,\theta), \end{aligned} \tag{3}$$

where $W_0 = U(\alpha)$, $W_1 = U(\beta)$ and the functions $\overline{B}_r(y, \theta)$ as follows:

$$\begin{aligned} \overline{B}_r(y,\theta) &= B_r(y,\theta) - \tfrac{B_r(y_0,\theta)}{B_{-1}(y_0,\theta)} B_{-1}(y,\theta),\ r = 0, 1, \\ \overline{B}_r(y,\theta) &= B_r(y,\theta),\ r = 2, \ldots, M-2, \\ \overline{B}_r(y,\theta) &= B_r(y,\theta) - \tfrac{B_r(y_M,\theta)}{B_{M+1}(y_M,\theta)} B_{M+1}(y,\theta),\ r = M-1, M. \end{aligned} \tag{4}$$

$\overline{B}_r(y, \theta),\ r = 0, \ldots, M$ is as the new set of redefined extended cubic spline functions which vanish on the Dirichlet's boundary conditions.

3. On quadrature rules of Gauss-Turán

Let P_n be the set of all algebraic polynomials of degree at most n. The Gauss-Turán quadrature rule in [13, 18] is

$$\int_{\alpha}^{\beta} g(y)d\chi(y) = \sum_{\tau=1}^{m}\sum_{r=0}^{2p} \varphi_{\tau,r} g^{(r)}(\nu_\tau) + E_{m,2p+1}(g), \tag{5}$$

where $m \in M$, $p \in M_0$ and $d\chi(y)$ is a nonnegative measure on the interval (α, β) which can be the real axis E, with compact or infinite support for which all moments:

$$\eta_\kappa = \int_{\alpha}^{\beta} y^\kappa d\chi(y),\ \kappa = 0, 1, \ldots, \tag{6}$$

exists, are finite, more over $\eta_0 > 0$, and $\varphi_{\tau,r} = \int_{\alpha}^{\beta} \gamma_{\tau,r}(y)d\chi(y), (r = 0, \ldots, 2p,$
$\tau = 1, \ldots, m)$ and $\gamma_{\tau,r}(y)$ are the fundamental polynomials of Hermite interpolation. The nodes ν_τ $(\tau = 1, \ldots, m)$ in Eq.(5) are the zeros of monic polynomial $\psi_m(y) = y^m + b_{m-1}y^{m-1} + \ldots + b_1x + b_0$ which minimizes the integral.

$$G(b_0, b_1, \ldots, b_{m-1}) = \int_{\alpha}^{\beta} [\psi_m(y)]^{2p+2} d\chi(y), \tag{7}$$

then the rule Eq.(5) is exact for all polynomials of degree at most $2(p+1)m-1$, that is, $E_{m,2p+1}(g) = 0, \forall g \in P_{2(p+1)m-1}$. The condition Eq.(7) is equivalent with the following conditions:

$$\int_{\alpha}^{\beta} [\psi_m(y)]^{2p+1} y^\kappa d\chi(x) = 0,\ \kappa = 0, \ldots, m-1, \tag{8}$$

$$\int_{-1}^{1} [\psi_4(y)]^5 y^\kappa dy = 0,\ \kappa = 0, 1, 2, 3, \tag{9}$$

where $\psi_4(y) = y^4 + b_3y^3 + b_2y^2 + b_1y + b_0$, by solving system Eq.(9) we can obtain $b_r, (r = 0, 1, 2, 3, 4)$ coefficients, on the other hand we have

$$\psi_{\tau+1}(y) = (y - \rho_\tau)\psi_\tau(yx) - \delta_\tau\psi_{\tau-1}(y),\ \tau = 0, 1, 2, 3,$$

$$\psi_{-1}(y) = 0,\ \psi_0(y) = 1,$$

where
$\rho_\tau = \rho_\tau(2, 4) = \frac{(y\psi_\tau,\psi_\tau)}{(\psi_\tau,\psi_\tau)} = \frac{\int_{-1}^{1} y\psi_\tau^2(y)\psi_m^{2p}(y)dy}{\int_{-1}^{1} \psi_\tau^2(y)\psi_m^{2p}(y)dy} = \frac{\int_{-1}^{1} y\psi_\tau^2(y)\psi_4^4(y)dy}{\int_{-1}^{1} \psi_\tau^2(y)\psi_4^4(y)dy}$
$\delta_\tau = \delta_\tau(2, 4) = \frac{(\psi_\tau,\psi_\tau)}{(\psi_{\tau-1},\psi_{\tau-1})} = \frac{\int_{-1}^{1} \psi_\tau^2(y)\psi_m^{2p}(y)dy}{\int_{-1}^{1} \psi_{\tau-1}^2(y)\psi_m^{2p}(y)dy} = \frac{\int_{-1}^{1} \psi_\tau^2(y)\psi_4^4(y)dy}{\int_{-1}^{1} \psi_{\tau-1}^2(y)\psi_4^4(y)dy}$
$\delta_0 = \int_{-1}^{1} \psi_4^4(y)dy,$
so that we can obtain the zeros of monic polynomial $\psi_4^{2,4}(y)$ of eigenvalue Jocobian

TABLE 1. Determined values of ν_τ, β_τ and δ_τ.

ν	ν_τ	β_τ	δ_τ
0	-0.899829212560986	0	0.132703088805391(-03)*
1	-0.365924354691640	0	0.424102581549750
2	0.365924354691679	0	0.263848849055045
3	0.899829212650986	0	0.255641814691793

*$0.132703088805391(-03) = 0.132703088805391 * 10^{-03}$.

matrix

$$J_4 = \begin{bmatrix} \rho_0 & \sqrt{\delta_1} & & \\ \sqrt{\delta_1} & \rho_1 & \sqrt{\delta_2} & \\ & \sqrt{\delta_2} & \rho_2 & \sqrt{\delta_3} \\ & & \sqrt{\delta_3} & \rho_3 \end{bmatrix},$$

and the values of ν_τ, ρ_τ and δ_τ which are tabulated in Table 1.

Finally, to determine $\varphi_{\tau,r}$, we use the following polynomial for approximation of function $g(y)$,

$$g_{\kappa,\tau}(y) = (y-\nu_\tau)^\kappa \Phi_\tau(y) = (y-\nu_\tau)^\kappa \prod_{r\neq\tau}(y-\nu_r)^{2p+1}, \tag{10}$$

where $0 \le \kappa \le 2w,\ 1 \le \tau \le m$ and

$$\Phi_\tau(y) = (\frac{\psi_m(y)}{y-\nu_\tau})^{2p+1} = \prod_{r\neq\tau}(y-\nu_r)^{2p+1}, \tau = 1,\dots,m,$$

since Eq.(5) is exact for all polynomials of degree at most $2(p+1)m-1$ then accuracy degree $g_{\kappa,\tau}$ is

$$deg g_{\kappa,\tau} = (m-1)(2p+1) + \kappa \le (2p+1)m - 1.$$

Then Eq.(5) is exact for polynomials Eq.(10), that is, $E(g_{\kappa,\tau}) = 0,\ (0 \le \kappa \le 2p, 1 \le \tau \le m)$ then by replacing $g_{\kappa,\tau}(y)$ instead of $g(y)$ in Eq.(5) we have

$$\begin{aligned} \textstyle\sum_{l=1}^{m}\sum_{r=0}^{2p} \varphi_{l,r} g_{\kappa,\tau}^{(r)}(\nu_l) &= \textstyle\int_\alpha^\beta g_{\kappa,\tau}(y)d\chi(y) \\ &= \eta_{\kappa,\tau}, \end{aligned}$$

therefore for each $\tau = l$, we get the following linear system $(2p+1)\times(2p+1)$, where $\varphi_{\tau,r}$ are unknowns $r = 0,\dots,2w,\ \tau = 1,\dots,m$,

$$\begin{bmatrix} g_{0,\tau}(\nu_\tau) & g'_{0,\tau}(\nu_\tau) & \dots & g_{0,\tau}^{(2p)}(\nu_\tau) \\ & g'_{1,\tau}(\nu_\tau) & \dots & g_{1,\tau}^{(2p)}(\nu_\tau) \\ & & \ddots & \\ & & & g_{2p,\tau}^{(2p)}(\nu_\tau) \end{bmatrix} \begin{bmatrix} \varphi_{\tau,0} \\ \varphi_{\tau,1} \\ \vdots \\ \varphi_{\tau,2p} \end{bmatrix} = \begin{bmatrix} \eta_{0,\tau} \\ \eta_{1,\tau} \\ \vdots \\ \eta_{2p,\tau} \end{bmatrix},$$

solving the above system for $p = 2$ and $\tau = 1,2,3,4$, we obtain the values of $\varphi_{\tau,r},\ \tau = 1,\dots,4,\ r = 0,\dots,4$, which are tabulated in Table 2.

TABLE 2. Determined values of $\varphi_{\tau,r},\ \tau = 1,\dots,4,\ r = 0,\dots,4$.

$\varphi_{1,0} = 0.315604206062624$	$\varphi_{1,2} = 0.001213976533015$
$\varphi_{2,0} = 0.684395793937405$	$\varphi_{2,2} = 0.0104801638359508$
$\varphi_{3,0} = 0.684395793937377$	$\varphi_{3,2} = 0.010480163835949$
$\varphi_{4,0} = 0.315604206062603$	$\varphi_{4,2} = 0.00121397653301490$
$\varphi_{1,1} = 0.0151791927277847$	$\varphi_{1,3} = 2.67403743470878 * 10^{-5}$
$\varphi_{2,1} = 0.013556093515529$	$\varphi_{2,3} = 0.0001128025099388$
$\varphi_{3,1} = -0.135560935155336 * 10^{-1}$	$\varphi_{3,3} = -0.11280250993880 * 10^{-3}$
$\varphi_{4,1} = -0.151791927277821 * 10^{-1}$	$\varphi_{4,3} = -0.267403743470821 * 10^{-4}$
$\varphi_{1,4} = 5.42643518348675 * 10^{-7}$	$\varphi_{2,4} = 0.00002636423549605$
$\varphi_{3,4} = 0.000026364235496$	$\varphi_{4,4} = 5.42643518348595 * 10^{-7}$

4. Nonlinear Fredholm integral equation

In the given nonlinear FIE(1), we can approximate the unknown function by extended cubic spline Eq.(3), we have:

$$S(\xi) = g(\xi) + \int_{\alpha}^{\beta} K(\xi, y, S(y))dy, \tag{11}$$

Now collocated Eq.(11) for a fixted t in $\alpha \le \xi \le \beta$ at the points $\xi_r = \alpha + rh$, $h = \frac{\beta-\alpha}{M},\ r = 0,1,\dots,M$, we obtain

$$\begin{aligned}&\textstyle\int_{\alpha}^{\beta} K(\xi_r, y, (\frac{B_{-1}(y,\theta)}{B_{-1}(y_0,\theta)}W_0 + \frac{B_{M+1}(y,\theta)}{B_{M+1}(y_M,\theta)}W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(y,\theta)))dy + g(\xi_r)\\ &= \textstyle\frac{B_{-1}(\xi_r,\theta)}{B_{-1}(\xi_0,\theta)}W_0 + \frac{B_{M+1}(\xi_r,\theta)}{B_{M+1}(\xi_M,\theta)}W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(\xi_r,\theta),\ r = 0,1,\dots,M.\end{aligned} \tag{12}$$

By partitioning the interval $[\alpha,\beta]$ to M equal subintervals we obtain

$$\begin{aligned}&\textstyle\sum_{j=0}^{M-1}\int_{\xi_j}^{\xi_{j+1}} K(\xi_r, y, (\frac{B_{-1}(y,\theta)}{B_{-1}(y_0,\theta)}W_0 + \frac{B_{M+1}(y,\theta)}{B_{M+1}(y_M,\theta)}W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(y,\theta)))dy\\ &\textstyle+g(\xi_r) = \frac{B_{-1}(\xi_r,\theta)}{B_{-1}(\xi_0,\theta)}W_0 + \frac{B_{M+1}(\xi_r,\theta)}{B_{M+1}(\xi_M,\theta)}W_1 + \sum_{r=0}^{M} t_r \overline{B}_r(\xi_r,\theta),\ r = 0,1,\dots,M.\end{aligned} \tag{13}$$

For using the Gauss-Turán rule we need to change each subinterval $[\xi_j, \xi_{j+1}]$ to the interval $[-1,1]$. Then by the following change of variable, we have

$$y = \frac{1}{2}[(\xi_{j+1} - \xi_j)u + (\xi_{j+1} - \xi_j)],\ dy = \frac{\xi_{j+1} - \xi_j}{2}du = \frac{h}{2}du.$$

To approximate the integral Eq.(13), we can use the Gauss-Turàn quadrature rule in the case $m = 4$ and $w = 2$, then we get the following $(M+1) \times (M+1)$, nonlinear system

$$\frac{B_{-1}(\xi_r,\theta)}{B_{-1}(\xi_0,\theta)}W_0 + \frac{B_{M+1}(\xi_r,\theta)}{B_{M+1}(\xi_M,\theta)}W_1 + \sum_{r=0}^{M} t_r\overline{B}_r(\xi_r,\theta) = \frac{h}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}$$
$$\times(K(\xi_r,\zeta_{j\tau},(\frac{B_{-1}(\zeta_{j\tau},\theta)}{B_{-1}(\xi_0,\theta)}W_0 + \frac{B_{M+1}(\zeta_{j\tau},\theta)}{B_{M+1}(\xi_M,\theta)}W_1 + \sum_{r=0}^{M} t_r\overline{B}_r(\zeta_{j\tau},\theta))))^{(\gamma)} + g(\xi_r),$$
$$r = 0,1,\ldots,M, \tag{14}$$

where $\zeta_{j\tau} = \frac{(\xi_{j+1}-\xi_j)\nu_\tau + (\xi_{j+1}+\xi_j)}{2}$ and ν_τ we have the nodes and coefficients $\varphi_{\tau,\gamma}$ of previous section. By solving the above nonlinear system via iterative method we determine the coefficients $t_r, r = 0,\ldots,M$ by setting t_r in Eq.(3), we obtain the approximate solution for FIE(1).

5. Error analysis

To obtain the error estimation of our approach, the first of all we recall the following definition and Theorem in [**13**, **18**, **9**].

Definition 5.1. The Gauss-Tur*á*n quadrature rule with multiple nodes,

$$\int_\alpha^\beta g(y)\chi(y)dy = \sum_{\tau=1}^{m}\sum_{r=0}^{2p}\varphi_{\tau,r}g^{(r)}(\nu_\tau) + E_{m,2p+1}(g), \tag{15}$$

is exact for all polynomials of degree at most $2(p+1)m-1$, that is,

$$E_{m,2p+1}(g) = 0,\ \forall g \in P_{2(p+1)m-1}.$$

Theorem 5.1. *Let $U(\xi) \in C^4[\alpha,\beta], \triangle$ be the partition of $[\alpha,\beta]$ and $S(\xi)$ be the spline interpolation function $U(\xi)$, we have*

$$\|D^r(S-U)\|_\infty \leq \phi_r h^{4-r}, r = 0,\ldots,3. \tag{16}$$

For the proof see [**9**].

Next we will prove the following theorem for convergence of our method in Eq.(14).

Theorem 5.2. *The approximate method Eq.(14)*

$$S(\xi_r) = \frac{h}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}(K(\xi_r,\zeta_{j\tau},S(\zeta_{j\tau})))^{(\gamma)} + g(\xi_r), r = 0,1,\ldots,M, \tag{17}$$

for solution of the nonlinear FIE(1) is converge and the error bounded is

$$|E_r| \leq \frac{hL}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}|\varphi_{\tau,\gamma}||E_{j\tau}|, \tag{18}$$

where $E_{j\tau} = S_{j\tau} - U_{j\tau}, E_r = S_r - U_r, r = 0,\ldots,M$ and kernel K satisfy Lipschitz condition in their third argument with L Lipschitz constant.

PROOF. We suppose that at the points $\xi_r = \alpha + rh,\ h = \frac{\beta-\alpha}{M},\ r = 0, 1, \ldots, M,$ the corresponding approximation method for nonlinear FIE(1) is

$$S(\xi_r) = \frac{h}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}(K(\xi_r,\zeta_{j\tau},S(\zeta_{j\tau})))^{(\gamma)} + g(\xi_r),\ r = 0, 1, \ldots, M. \tag{19}$$

By discrediting FIE(1) and approximate the integral by the Gauss-Tur$\acute{a}$n rule, we can obtain

$$U(\xi_r) = \frac{h}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}(K(\xi_r,\zeta_{j\tau},U(\zeta_{j\tau})))^{(\gamma)} + g(\xi_r),\ r = 0, 1, \ldots, M. \tag{20}$$

By subtracting Eq.(20) from Eq.(19) and using interpolatory condition of cubic spline, we get

$$S(\xi_r) - U(\xi_r) = \frac{h}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}\varphi_{\tau,\gamma}[K^{(\gamma)}(\xi_r,\zeta_{j\tau},S(\zeta_{j\tau})) - K^{(\gamma)}(\xi_r,\zeta_{j\tau},U(\zeta_{j\tau}))],$$

we suppose that, $S(y_r) = S_r,\ U(y_r) = U_r,\ r = 0, \ldots, M,$ and kernel $K^{(\gamma)}$ satisfies Lipschitz condition in their third argument for all $\mu_1, \mu_2 \in R$ is of the form

$$|K^{(\gamma)}(y,\zeta,\mu_1) - K^{(\gamma)}(y,\zeta,\mu_2)| \leq L|\mu_1 - \mu_2|,$$

where L is independent of y, ζ, μ_1 and μ_2. We get

$$|S_r - U_r| \leq \frac{h}{2}L\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}|\varphi_{\tau,\gamma}||S_{j\tau} - U_{j\tau}|,$$

$$|E_r| \leq \frac{hL}{2}\sum_{j=0}^{M-1}\sum_{\tau=1}^{4}\sum_{\gamma=0}^{4}|\varphi_{\tau,\gamma}||E_{j\tau}|,$$

where $E_r = S_r - U_r,\ r = 0, \ldots, M$. When $h \to 0$ then the above terms are zero and also these terms are due to interpolating of $U(\xi)$ by cubic spline (Theorem 5.1). We get for a fixed r,

$$|E_r| \to 0 \ as\ h \to 0.$$

□

6. Numerical examples

We consider text problems of nonlinear and linear FIEs. Our numerical results are compared with methods in [**6**, **1**, **12**, **11**, **16**, **25**, **17**], program preformed by Mathematica for examples , the running time is also reported in seconds(CPU time(s)).

Example 6.1. Consider the following nonlinear FIE with exact solution $U(\xi) = \xi$,

$$U(\xi) = (\xi - \frac{\pi}{8}) + \frac{1}{2}\int_0^1 \frac{1}{(1+U^2(y))}dy,\ \xi \in [0,1]. \tag{21}$$

We apply the presented method Eq.(14), the maximum absolute errors(MAEs) in the solutions for $\theta = 0$ with different values of $M = 8, 10, 15, 16, 20$ are tabulated in Table 3, and compared with the results [1, 6]. The MAEs in the solution for the different values of $\theta = -1.99, -1, 1, 3$ with $M = 20$ are tabulated in Table 4.

TABLE 3. The MAEs for different values of M.

M	our Method $\theta = 0$,	CPU time (s)	Method in [6]	Method in [1]
8	5.55112(-17)	1.954	7.5(-10)	—
10	7.70186(-17)	2.89	1.9(-10)	1.5(-09)
15	4.11682(-17)	9.499	—	2.9(-11)
16	1.38778(-17)	9.969	1.1(-11)	—
20	1.14186(-16)	20.344	3.1(-13)	7.3(-14)

TABLE 4. The MAEs at particular points for $M = 20$.

ξ_r	$\theta = -1.99$	$\theta = -1$	$\theta = 1$	$\theta = 3$
0.1	1.39(-17)	0	0	1.39(-17)
0.2	2.78(-17)	2.78(-17)	0	2.78(-17)
0.3	0	0	0	0
0.4	5.55(-17)	5.55(-17)	5.55(-17)	5.55(-17)
0.5	1.11(-16)	0	1.11(-16)	1.11(-16)
0.6	0	0	1.11(-16)	0
0.7	0	0	0	0
0.8	2.22(-16)	1.11(-16)	0	2.22(-16)
0.9	2.22(-16)	0	0	2.22(-16)

Example 6.2. Consider the following linear FIE with exact solution $U(\xi) = e^{2\xi}$,

$$U(\xi) = e^{2\xi + \frac{1}{3}} - \int_0^1 \frac{1}{3} e^{2\xi - \frac{5}{3}y} U(y)dy,\ \xi \in [0,1]. \tag{22}$$

We apply the presented method Eq.(14), the MAEs in the solutions for $\theta = 0$ with $M = 16$ are tabulated in Table 5, and compared with the results [11, 12]. The MAEs in the solution for the different values of $\theta = -1.99, -1, 1, 3$ with $M = 30$ are tabulated in Table 6.

Example 6.3. Consider the following linear FIE with exact solution $U(\xi) = e^{\xi}$,

$$U(\xi) = e^{\xi} + \frac{1}{2}(e^2 - 1)sin(\xi) - \int_0^1 sin(\xi)e^y U(y)dy,\ \xi \in [0,1]. \tag{23}$$

We apply the presented method Eq.(14), the MAEs in the solutions for $\theta = 0$ for the different values of M are tabulated in Table 7, and compared with the results

TABLE 5. The MAEs at particular points for $M = 16$.

ξ_r	our Method $\theta = 0$ CPU time $= 5.625s$	Hybrid function method in [12]	Haar function method in [11]
0	0	0	6.26(-04)
0.0625	2.22045(-16)	2.44027(-10)	-
0.125	2.22045(-16)	3.24913(-11)	1.51(-03)
0.1875	2.22045(-16)	2.80845(-10)	9.78(-04)
0.25	6.66134(-16)	7.42122(-11)	-
0.3125	0	3.28122(-10)	1.08(-03)
0.375	8.88178(-16)	1.27778(-10)	-
0.4375	1.33227(-15)	3.88825(-10)	2.81(-03)
0.5	4.44089(-16)	1.96564(-10)	1.70(-03)
0.5625	1.33227(-15)	4.66775(-10)	-
0.625	1.33227(-15)	2.84886(-10)	4.09(-03)
0.6875	1.77636(-15)	5.66856(-10)	2.66(-03)
0.75	8.88178(-16)	3.98295(-10)	-
0.8125	2.66454(-15)	6.95369(-10)	2.94(-03)
0.875	1.77636(-15)	5.43908(-10)	-
0.9375	8.88178(-16)	8.60371(-10)	7.63(-03)
1	8.88178(-16)	7.30885(-10)	-

TABLE 6. The MAEs at particular points for $M = 30$.

ξ_r	$\theta = -1.99$	$\theta = -1$	$\theta = 1$	$\theta = 3$
0.1	2.22045(-16)	2.11738(-16)	2.22045(-16)	4.44089(-16)
0.2	4.44089(-16)	0	6.66134(-16)	4.44089(-16)
0.3	0	0	4.44089(-16)	4.44089(-16)
0.4	8.88178(-16)	8.88178(-16)	8.88178(-16)	8.88178(-16)
0.5	8.88178(-16)	1.77636(-15)	8.88178(-16)	1.33227(-16)
0.6	0	0	4.44089(-16)	4.44089(-16)
0.7	8.88178(-16)	8.88178(-16)	8.88178(-16)	8.88178(-16)
0.8	1.77636(-15)	1.77636(-15)	1.77636(-15)	2.66454(-16)
0.9	2.66454(-15)	1.77636(-15)	2.66454(-15)	3.55271(-16)
1	0	0	0	0

[16] . The MAEs in the solution for the different values of $\theta = -1.99, -1, 1, 3$ with $M = 5, 10, 15, 20$ are tabulated in Table 8.

TABLE 7. The MAEs for the different values of M.

M	CPU time(s)	our Method $\theta = 0$	Method in [16]	Method in [16]
6	89.671	2.40(-16)	5.54(-06)	1.84(-09)
11	93.139	3.86(-16)	3.53(-07)	2.91(-11)
21	115.044	2.37(-16)	2.22(-08)	4.56(-13)
41	375.154	3.92(-16)	1.39(-09)	7.99(-15)

TABLE 8. The MAEs for the different values of M.

M	$\theta=-1.99$	$\theta=-1$	$\theta=1$	$\theta=3$
5	4.965(-16)	2.22(-16)	1.40(-16)	2.63(-16)
10	6.397(-16)	2.53(-16)	2.98(-16)	5.06(-16)
15	1.404(-16)	3.85(-16)	1.28(-16)	2.00(-16)
20	4.684(-16)	1.65(-16)	2.16(-16)	3.29(-16)

Example 6.4. Consider the following nonlinear FIE with exact solution $U(\xi)=\xi$,

$$U(\xi)=e\ \xi-\int_0^1 \xi e^{U(y)}dy,\ \ \xi\in[0,1]. \tag{24}$$

We apply the presented method Eq.(14), the (MAEs) at particular points are tabulated in Table 9, and compared with the results [**17, 25**]. The MAEs in the solution for the different values of $\theta=-1.99,-1,1,3$ with $M=5,10,15,20$ are tabulated in Table 10.

TABLE 9. The MAEs at particular points.

	our Method CPU time=14.59s	Method in [**25**]	Method in [**17**]
ξ	$\theta=0, M=10$	$M=10$	$M=32$
0.2	8.33(-16)	1.88(-05)	3.43(-11)
0.4	1.67(-16)	3.76(-05)	6.00(-11)
0.6	0	5.65(-05)	6.91(-11)
0.8	3.33(-16)	7.52(-05)	5.78(-11)

TABLE 10. The MAEs for the different values of M.

M	$\theta=-1.99$	$\theta=-1$	$\theta=1$	$\theta=3$
5	2.04(-16)	1.43(-16)	8.95(-17)	1.78(-16)
10	2.39(-16)	1.34(-16)	1.14(-16)	1.53(-16)
15	1.44(-16)	9.73(-16)	1.62(-16)	1.32(-16)
20	1.90(-16)	1.06(-16)	1.22(-16)	1.46(-16)

7. Conclusions

We developed a method to find the solution of linear and nonlinear FIEs. The over approach is based on the Gauss-Turán quadrature rule and then using an extended cubic spline as the base function. The unknown coefficients in combination are determined by the collocation method. The arising system of linear and nonlinear equations can be solved. Numerical text problems are considered to justify the applicability and efficient nature of our approach; comparison of the results justifies the considerable accuracy and efficiency proposed methods.

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