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Fixed point theorems for some mappings in 2-Banach spaces

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Abstract. In the present study, we introduce the notions of Meir-Keeler contraction mappings and Ćirić contraction mappings on a 2-Banach space. In particular, we discuss the existence and uniqueness of a fixed point of such mappings in a 2-Banach space. On the other hand, we define the concept of Hardy-Rogers contraction mappings on a 2-Banach space. In particular, we prove the existence and uniqueness of a fixed point of such a mapping in a 2-Banach space. However, many results are proved on fixed point theorems of some mappings in 2-Banach spaces.

Keywords: Fixed point theorems, 2-Banach spaces, closed and bounded sets

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1. Introduction

In a 2-Banach space framework, Gähler [2] initiated the study of 2-normed spaces. Recently, White [10] introduced and studied the concept of 2-Banach spaces. The fixed point theory played an important role in functional analysis. Moreover, it used in several branches of science such as biology, chemistry, economics, engineering and computer science. Recently, the authors [5] demonstrated that a contraction mapping has a one fixed point in closed and bounded subsets of a 2-normed space.

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However, Kir and Kiziltunc [7] demonstrated several results on fixed points in 2-Banach spaces. Moreover, Kannan [6] demonstrated the next result in a complete metric space:

Theorem 1.1. *Let* $(\mathcal{F}, d)$ *be a complete metric space and* $S : \mathcal{F} \longrightarrow \mathcal{F}$ *be a mapping such that*

$$d(Sz, Sf) \leq a[d(z, Sz) + d(f, Sf)] \tag{1}$$

where $a \in [0, \frac{1}{2})$ *and* $z, f \in \mathcal{F}$, *then* S *has a unique fixed point.*

In this paper, we demonstrate several fixed point theorems for some mappings in 2-Banach spaces.

2. Preliminaries

We begin with preliminaries:

Definition 2.1 ([2])**.** Let $\mathcal{X}$ be a real vector space with $\dim \mathcal{X} \geq 2$. A 2-norm on $\mathcal{X}$ is a function $\| \cdot \| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ such that

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) For all $x, y \in \mathcal{X}$, $\|x, y\| = \|y, x\|$.
(iii) For any $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$, $\|\lambda x, y\| = |\lambda| \|x, y\|$.
(iv) For each $x, y, z \in \mathcal{X}$, $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a 2-normed space.

Definition 2.2 ([1])**.** Let $\mathcal{X}$ be a 2-normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ is said to be a Cauchy sequence if $\lim\limits_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for any $y \in \mathcal{X}$.

Definition 2.3 ([10])**.** Let $\mathcal{X}$ be a 2-normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ converges in $\mathcal{X}$ if there exists an element $x \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $\lim\limits_{n\to\infty} \|x_n - x, y\| = 0$. If $(x_n)_{n\in\mathbb{N}}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 2.4 ([10])**.** A 2-normed space in which every Cauchy sequence converges will be called a 2-Banach space.

Definition 2.5 ([5])**.** Let $\mathcal{X}$ be a 2-normed space then the mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be a contraction if there exists some $k \in (0, 1)$ such that

$$\|Sx - Sy, z\| \leq k\|x - y, z\| \tag{2}$$

for all $x, y, z \in \mathcal{X}$.

Theorem 2.1 ([5])**.** *Let* $\mathcal{X}$ *be a 2-normed space and* $\mathcal{F}$ *be a nonempty closed and bounded subset of* $\mathcal{X}$. *Let* $S : \mathcal{F} \longrightarrow \mathcal{F}$ *be a contraction, then* S *has a unique fixed point.*

3. Main results

In this section, we start with the following definitions.

Definition 3.1. A mapping S on a 2-normed space $\mathcal{X}$ is called a Meir-Keeler contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,

$$\varepsilon \leq \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon. \tag{3}$$

Definition 3.2. A mapping S on a 2-normed space $\mathcal{X}$ is called a Ćirić contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,

$$\varepsilon < \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon. \tag{4}$$

Theorem 3.1. *Let $\mathcal{X}$ be a 2-Banach space and let S be a Meir-Keeler contraction on $\mathcal{X}$, then S has a unique fixed point in $\mathcal{X}$.*

Proof. From (3), we get

$$x \neq y \Rightarrow \|Sx - Sy, z\| < \|x - y, z\|.$$

Then S is continuous and has at most one fixed point. □

Theorem 3.2. *Let $\mathcal{X}$ be a 2-Banach space and let S be a Ćirić contraction on $\mathcal{X}$, then S has a unique fixed point in $\mathcal{X}$.*

Proof. From (4), we have

$$x \neq y \Rightarrow \|Sx - Sy, z\| < \|x - y, z\|.$$

Then S is continuous and has at most one fixed point. □

Definition 3.3. A mapping S on a 2-normed space $\mathcal{X}$ is called a Hardy-Rogers contraction if S is self-mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,

$$\|Sx - Sy, z\| \leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| + e\|y - Sx, z\| + f\|x - y, z\| \tag{5}$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

Consider the following condition

$$\begin{aligned} &x \neq y \Longrightarrow \\ \|Sx - Sy, z\| <& a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| + e\|y - Sx, z\| \\ &+ f\|x - y, z\|. \end{aligned} \tag{6}$$

Without loss of generality, we may suppose $a = b$ and $c = e$, from condition (5), we can derive

$$\begin{aligned} \|Sx - Sy, z\| \leq& \frac{a+b}{2}[\|x - Sx, z\| + \|y - Sy, z\|] + \frac{c+e}{2}[\|x - Sy, z\| + \|y - Sx, z\|] \\ &+ f\|x - y, z\|. \end{aligned} \tag{7}$$

Now, we state our main results.

Lemma 3.3. *Let $\mathcal{X}$ be a 2-Banach space. Suppose that (5) holds on $\mathcal{X}$. Then if $\beta < 1$, there exists $k < 1$ such that for each $x, z \in \mathcal{X}$,*

$$\|Sx - S^2x, z\| \leq k\|x - Sx, z\|.$$

Also if condition (6) holds with $\beta = 1$, then

$$x \neq Sx \Longrightarrow \|Sx - S^2x, z\| < \|x - Sx, z\|. \tag{8}$$

Proof. Assume that $\beta < 1$. Put $y = Sx$ in (5), we obtain

$$\|Sx - S^2x, z\| \leq \frac{a+f}{1-b}\|x - Sx, z\| + \frac{c}{1-b}\|x - S^2x, z\|. \tag{9}$$

Using the triangle inequality,

$$\|Sx - S^2x, z\| \geq \|S^2x - x, z\| - \|Sx - x, z\|.$$

By (9),

$$\|S^2x - x, z\| - \|Sx - x, z\| \leq \frac{a+f}{1-b}\|x - Sx, z\| + \frac{c}{1-b}\|x - S^2x, z\|, \tag{10}$$

then

$$\|S^2x - x, z\| \leq \frac{1+a+f-b}{1-b-c}\|x - Sx, z\|. \tag{11}$$

Substituting inequality (11) into inequality (9), we get

$$\|Sx - S^2x, z\| \leq \frac{a+c+f}{1-b-c}\|x - Sx, z\|. \tag{12}$$

By symmetry, we may exchange a with b and c with e in (12) to obtain

$$\|Sx - S^2x, z\| \leq \frac{b+e+f}{1-a-e}\|x - Sx, z\|. \tag{13}$$

Then

$$k = \min\{\frac{a+c+f}{1-b-c}, \frac{b+e+f}{1-a-e}\}$$

satisfies the first conclusion of the lemma. The proof for the remaining case is analogous, the main difference being that a side argument is needed to prove that without loss of generality we may suppose that the numbers $a + e$ and $b + c$ are less than 1. This follows essentially because of inequality (7) and the comment preceding it. □

Theorem 3.4. *Let $\mathcal{X}$ be a 2-normed space and S a self-mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,*

$$\|Sx - Sy, z\| \leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| + e\|y - Sx, z\| + f\|x - y, z\| \tag{14}$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

(i) *If $\mathcal{X}$ is complete and $\beta < 1$, then S has a unique fixed point.*

(ii) *If* (5) *is modified to the condition* (6) *and in this case* $\mathcal{X}$ *is compact,* S *is continuous and* $\beta = 1$*, then* S *has a unique fixed point.*

Proof. Existence of (ii) follows easily from the second part of Lemma 3.3. For $\inf\{\|x - Sx, z\| : x \in \mathcal{X}\} = \|y - Sy, z\|$ for some $y \in \mathcal{X}$ because S is continuous and $\mathcal{X}$ is compact. Condition (8) implies now that y must be fixed under S. Also, it is easy to verify that the conditions in both (i) and (ii) imply uniqueness. We proceed to demonstrate the existence of (i). Using the first part of Lemma 3.3, there is $k < 1$ such that $\|Sx - S^2x, z\| \leq k\|x - Sx, z\|$. Let $m > n$, then

$$\begin{aligned}\|S^m x - S^n x, z\| &\leq \|S^m x - S^{m-1}x, z\| + \cdots + \|S^{n+1}x - S^n x, z\| \\ &= k^n(1 + k + \cdots + k^{m-n})\|x - Sx, z\| \\ &\leq \frac{k^n}{1-k}\|x - Sx, z\|.\end{aligned}$$

Hence $(S^n x)_n$ is a Cauchy sequence and so converges to some $x_0 \in \mathcal{X}$. Now, we claim that $x_0 = Sx_0$. From (5), we obtain

$$\begin{aligned}\|x_0 - Sx_0, z\| &\leq \|S^{n+1}x - Sx_0, z\| + \|S^{n+1}x - x_0, z\| \\ &\leq a\|S^n x - S^{n+1}x, z\| + b\|x_0 - Sx_0, z\| + c\|S^n x - Sx_0, z\| \\ &\quad +(e+1)\|S^{n+1}x - x_0, z\| + f\|x_0 - S^n x, z\|.\end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\|x_0 - Sx_0, z\| \leq (b + c)\|x_0 - Sx_0, z\|,$$

which contradicts $b + c < 1$, unless $Sx_0 = x_0$. □

Now, we proceed to generalize (i) of Theorem 3.4.

Theorem 3.5. *Let* $\mathcal{X}$ *be a* 2*-Banach space,* a, b, c, e, f *be monotonically decreasing functions from* $[0, \infty)$ *to* $[0, 1)$ *and let the sum of these five functions be less than* 1*. Assume that* $S : \mathcal{X} \longrightarrow \mathcal{X}$ *satisfies condition* (5) *with* $a = a(\|x - y, z\|), \cdots, f = f(\|x - y, z\|)$ *for each* $x, y, z \in \mathcal{X}$*. Then* S *has a unique fixed point.*

Proof. Such as in the proof of Lemma 3.3, there exists a monotone decreasing function $k(t) = k(a(t), b(t), c(t), e(t), f(t))$ such that $0 \leq k(t) < 1$ and $\|S^2x - Sx, z\| \leq k(\|x - Sx, z\|)\|x - Sx, z\|$. Then

$$\begin{aligned}\|S^n x - S^{n+1}x, z\| &\leq k(\|S^{n-1}x - S^n x, z\|)\|S^{n-1}x - S^n x, z\| \\ &< \|S^{n-1}x - S^n x, z\|\end{aligned}$$

therefore $\|S^{n+1}x - S^n x, z\|$ is a decreasing sequence. Let $h = \lim_{n\longrightarrow\infty} \|S^{n+1}x - S^n x, z\|$. If $h > 0$, then $\|S^{n+1}x - S^n x, z\| \geq h$. Since k is monotone decreasing, then for all $x, z \in \mathcal{X}$,

$$\begin{aligned}\|S^{n+1}x - S^n x, z\| &< k(h)\|S^{n-1}x - S^n x, z\| \\ &\leq \cdots \leq k^n(h)\|x - Sx, z\| \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$ a contradiction. Therefore $h = 0$. Set for all $z \in \mathcal{X}, a = a(\|S^{n-1}x - S^{m-1}x, z\|)$ and analogously for b, c, e, f. Without loss of generality, we suppose that $S^{n-1}x \neq S^{m-1}x$. Applying condition (5) and by the triangle inequality, we obtain

$$\begin{aligned}
\|S^n x - S^m x, z\| &\leq a\|S^{n-1}x - S^n x, z\| + b\|S^{m-1}x - S^m x, z\| + c\|S^{n-1}x - S^m x, z\| \\
&\quad + e\|S^{m-1}x - S^n x, z\| + f\|S^{n-1}x - S^{m-1}x, z\| \\
&\leq a\|S^{n-1}x - S^n x, z\| + b\|S^{m-1}x - S^m x, z\| \\
&\quad + c(\|S^{n-1}x - S^n x, z\| + \|S^n x - S^m x, z\|) \\
&\quad + e(\|S^{m-1}x - S^m x, z\| + \|S^m x - S^n x, z\|) \\
&\quad + f(\|S^{m-1}x - S^m x, z\| + \|S^m x - S^n x, z\| + \|S^{n-1}x - S^n x, z\|).
\end{aligned}$$

Simplifying this expression, we get

$$\begin{aligned}
\|S^n x - S^m x, z\| &\leq \frac{a + c + f}{1 - (c + e + f)}\|S^{n-1}x - S^n x, z\| \\
&\quad + \frac{b + e + f}{1 - (c + e + f)}\|S^m x - S^{m-1}x, z\| \\
&= \beta_1\|S^{n-1}x - S^n x, z\| + \beta_2\|S^m x - S^{m-1}x, z\|,
\end{aligned}$$

where $\beta_1 = \frac{a+c+f}{1-(c+e+f)}$ and $\beta_2 = \frac{b+e+f}{1-(c+e+f)}$. Then

$$\|S^n x - S^m x, z\| \leq \beta_1\|S^{n-1}x - S^n x, z\| + \beta_2\|S^m x - S^{m-1}x, z\|. \tag{15}$$

Choose $\varepsilon > 0$. If $\|S^{n-1}x - S^{m-1}x, z\| \geq \varepsilon$, then from (15), we obtain

$$\|S^n x - S^m x, z\| \leq \beta_1(\varepsilon)\|S^{n-1}x - S^n x, z\| + \beta_2(\varepsilon)\|S^m x - S^{m-1}x, z\|. \tag{16}$$

If $\|S^{n-1}x - S^{m-1}x, z\| \leq \varepsilon$, we obtain

$$\begin{aligned}
\|S^n x - S^m x, z\| &\leq \|S^n x - S^{n-1}x, z\| + \|S^{n-1}x - S^{m-1}x, z\| + \|S^{m-1}x - S^m x, z\| \\
&\leq \|S^n x - S^{n-1}x, z\| + \|S^{m-1}x - S^m x, z\| + \varepsilon.
\end{aligned}$$

Then

$$\|S^n x - S^m x, z\| \leq \|S^n x - S^{n-1}x, z\| + \|S^{m-1}x - S^m x, z\| + \varepsilon. \tag{17}$$

Since $\|S^n x - S^{n-1}x, z\| \longrightarrow 0$, it is clear from (16) and (17) that $(S^n x)_n$ is a Cauchy sequence. Let $S^n x \longrightarrow w$. We may suppose that $S^{n-1}x \neq w$ and $b + c < \frac{1}{2}$.

Set $a = a(\|S^{n-1}x - x, z\|)$, etc. We get

$$\begin{aligned}
\|Sw - w, z\| &\leq \|w - S^n x, z\| + \|S^n x - Sw, z\| \\
&\leq \|w - S^n x, z\| + a\|S^{n-1}x - S^n x, z\| + b\|w - Sw, z\| \\
&\quad + c\|S^{n-1}x - w, z\| + c\|w - Sw, z\| + e\|w - S^n x, z\| \\
&\quad + f\|S^{n-1}x - w, z\| \\
&\leq 2\|w - S^n x, z\| + 2\|w - S^{n-1}x, z\| + \|S^{n-1}x - S^n x, z\| \\
&\quad + \frac{1}{2}\|w - Sw, z\| \longrightarrow \frac{1}{2}\|w - Sw, z\|.
\end{aligned}$$

Consequently, w must be a fixed point of S. Uniqueness follows easily from condition (5). □

Theorem 3.6. *Let $\mathcal{X}$ be a 2-Banach space and $S_n : \mathcal{X} \longrightarrow \mathcal{X}$, $n = 1, 2, \cdots$ satisfy the conditions of Theorem 3.5 with the coefficients a, b, c, e, f. Let $S_n x_n = x_n$ and assume that $S_n \longrightarrow S$ pointwise on $\mathcal{X}$. Then $x = \lim_{n \longrightarrow \infty} x_n$ is the unique fixed point of S.*

PROOF. Using the continuity of the 2-norm and condition (5), the limit S also satisfies condition (5) and so has a unique fixed point, call it x. Putting $a = a(\|x_n - x, z\|)$ and similarly with b, c, e, f, we get

$$\begin{aligned}
\|x_n - x, z\| &= \|S_n x_n - Sx, z\| \\
&\leq \|S_n x_n - S_n x, z\| + \|S_n x - Sx, z\| \\
&\leq a\|x_n - S_n x_n, z\| + b\|x - S_n x, z\| + c\|x_n - S_n x, z\| + e\|S_n x_n - x, z\| \\
&\quad + f\|x_n - x, z\| + \|S_n x - Sx, z\| \\
&\leq b\|x - S_n x, z\| + c(\|x_n - x, z\| + \|S_n x - x, z\|) + e\|S_n x_n - x, z\| \\
&\quad + f\|x_n - x, z\| + \|S_n x - Sx, z\|.
\end{aligned}$$

Hence

$$\|x_n - x, z\| \leq \frac{1 + b + c}{1 - (c + e + f)}\|S_n x - x, z\| \leq \frac{2}{1 - (c + e + f)}\|S_n x - x, z\|.$$

Choose $\varepsilon > 0$. There exists N such that $n \geq N$ implies

$$\|S_n x - Sx, z\| < \frac{\varepsilon}{2}(1 - (c(\varepsilon) + e(\varepsilon) + f(\varepsilon))).$$

Take $n \geq N$, hence if $\|x_n - x, z\| \geq \varepsilon$, it follows that

$$\|x_n - x, z\| \leq \frac{2}{1 - (c(\varepsilon) + e(\varepsilon) + f(\varepsilon))}\|S_n x - x, z\| < \varepsilon.$$

This contradiction implies $\|x_n - x, z\| < \varepsilon$. Then $x_n \longrightarrow x$ and the proof is complete. □

Theorem 3.7. *Let $\mathcal{X}$ be a 2-Banach space and $S_n : \mathcal{X} \longrightarrow \mathcal{X}$, $n = 1, 2, \cdots$ be functions with at least one fixed point $x_n, n = 1, 2, \cdots$. Let S satisfy the hypothesis of Theorem 3.5 and $S_n \longrightarrow S$ uniformly on $\mathcal{X}$. Then $x = \lim_{n\longrightarrow\infty} x_n$ is the unique fixed point of S.*

Proof. Putting $a = a(\|x_n - x, z\|)$ and similarly for the other coefficients, we get

$$\begin{aligned}
\|x_n - x, z\| &= \|S_n x_n - Sx, z\| \\
&\leq \|S_n x_n - Sx_n, z\| + \|Sx_n - Sx, z\| \\
&\leq \|S_n x_n - Sx_n, z\| + a\|x_n - Sx_n, z\| + b\|x - Sx, z\| + c\|x_n - Sx, z\| \\
&\quad + e\|Sx_n - x, z\| + f\|x_n - x, z\| \\
&\leq \|S_n x_n - Sx_n, z\| + a\|S_n x_n - Sx_n, z\| + c\|x_n - x, z\| \\
&\quad + e(\|Sx_n - S_n x_n, z\| + \|x_n - x, z\|) + f\|x_n - x, z\|.
\end{aligned}$$

Then

$$\|x_n - x, z\| \leq \frac{1 + a + e}{1 - (c + e + f)} \|S_n x_n - Sx_n, z\| \leq \frac{2}{1 - (c + e + f)} \|S_n x_n - Sx_n, z\|.$$

Choose $\varepsilon > 0$ and N such that $n \geq N$ implies

$$\|S_n x_n - Sx_n, z\| < \frac{\varepsilon}{2}(1 - (c(\varepsilon) + e(\varepsilon) + f(\varepsilon))).$$

Then

$$\|x_n - x, z\| < \varepsilon$$

and the proof is complete. □

Now, we establish a generalization of Banach's principle of contraction mappings in 2-Banach spaces.

Theorem 3.8. *Let $\mathcal{X}$ be a 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a continuous mapping such that S^k is a contraction for some $k \in \mathbb{N}^*$, then S has a unique fixed point.*

Proof. Since for some $k \in \mathbb{N}^*$, S^k is a contraction, from Theorem 2.1, we have S^k has a unique fixed point, denoted x_0. Then $S^k x_0 = x_0$. Now, we will prove that $Sx_0 = x_0$. Since for all $z \in \mathcal{F}$,

$$\begin{aligned}
\|Sx_0 - x_0, z\| &= \|Sx_0 - x_0, z\| \\
&= \|SS^k x_0 - S^k x_0, z\| \\
&= \|S^k Sx_0 - S^k x_0, z\| \\
&\leq \alpha\|Sx_0 - x_0, z\|
\end{aligned}$$

where $\alpha \in (0, 1)$, then $\|Sx_0 - x_0, z\| = 0$ for each $z \in \mathcal{F}$. Consequently, $Sx_0 = x_0$. □

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