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Notes on fixed point results of C*-algebra-valued metric spaces

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Abstract. It is well-known that C^*-algebras have been applied in understanding the physics of several phenomena in quantum theory, statistical mechanics and more than a handful of other domains. In continuation of these roles of C^*-algebra, its notion has recently been incorporated in metric fixed point theory, and several results have been obtained thereof. However, in accordance with the existing literature, it is revealed that a collective and comparative analysis of the announced concepts of C^*-algebra-valued metric spaces and the associated invariant point theorems has not been considered. One of the uses of such analysis is that it gives researchers handy information to know what has been done and the available gaps in the corresponding notions. With this background orientation, the objective of this paper is twofold: first, important developments of the idea of C^*-algebra-valued metric spaces are surveyed. Thereafter, a comparative analysis of possible combinations and deductions based on the already obtained results is conducted. To achieve the latter objective, a few new results are formulated and non-trivial examples constructed, where necessary, to validate some of our observations.

Keywords: C^*-algebra, C^*-algebra-valued metric space, fixed point, contractive mapping

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1. Introduction

The theory of fixed points is an important area for various branches of mathematics and physics due to its broad applications. It plays a vital role in numerical analysis and approximation theory. The earliest fixed point result known as the Banach Contraction Principle was proved by Banach [5]. Thereafter, Many researchers studied the fixed point theory in different directions, for example, see [1, 9, 15, 58, 59] and the references therein.

A C^*-algebra (see [51]) is used to explain physical systems in quantum theory and statistical mechanics. Subsequently, it appeared as a significant area of research. In 2015, Ma et al. [25] introduced the concept of C^*-algebra-valued metric space and proved fixed point theorems for contractive and expansive mappings on a complete C^*-algebra-valued metric space. Since then, several authors (e.g. [3, 4, 6, 24, 43]) have extended and generalized the work of Ma et al. [25] in different directions with interesting theorems and applications.

It is interesting to know that the introduction of C^*-algebra-valued metric spaces and the corresponding fixed point theorems has widened the scope of fixed point theory and applications. However, the presentation of the available results in this space has divergent and sometimes overlapping techniques. Therefore, it is worthwhile to have comprehensive information in the form of a monograph regarding major developments in fixed point theory of C^*-algebra-valued metric spaces. Hence, this paper will concentrate on highlighting the distinct and important fixed point extensions in C^*-algebra-valued metric spaces to provide researchers in the area of fixed point theory with an insight into the advancements in fixed point theory in C^*-algebra-valued metric spaces.

2. Preliminaries

Throughout this article, $\mathbb{A}$ represents a unital C^*-algebra with a unit I. Take $\mathbb{A}_h = \{a \in \mathbb{A} : a = a^*\}$. We call $a \in \mathbb{A}$ a positive element, denoted by $a \succeq \theta$, if $a \in \mathbb{A}_h$ and $\sigma(a) \subset \mathbb{R}_+ = [0, \infty)$, where $\sigma(a)$ is the spectrum of a. Using positive elements, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $a \preceq b$ if and only if $b - a \succeq \theta$, where θ means the zero element in $\mathbb{A}$. Henceforth, $\mathbb{A}_+$ denotes the set $\{a \in \mathbb{A} : a \succeq \theta\}$ and $|a| = (aa^*)^{\frac{1}{2}}$.

Definition 2.1. [25] Let X be a nonempty set. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}$ satisfies:

1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in X$;
2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

The concept of C^*-algebra-valued metric space generalizes the concept of metric space by replacing the set of real numbers by $\mathbb{A}_+$.

Definition 2.2. [25] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. Suppose that $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $x \in X$. If for any $\varepsilon > 0$, there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$, then $\{x_n\}$ is said to be convergent with respect to $\mathbb{A}$ and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. $\{x_n\}$ is said to be a Cauchy sequence with respect to $\mathbb{A}$ If for any $\varepsilon > 0$, there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$. $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 2.3. [25] Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is a Lebesgue measurable set. By $B(H)$, we denote the set of bounded linear operators on Hilbert space H. Define $d : X \times X \longrightarrow B(H)$ by $d(f, g) = \pi_{|f-g|}$ $(\forall f, g \in X)$. Then, $(X, B(H), d)$ is a Complete C^*-algebra-valued metric space.

Definition 2.4. [25] Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space. We call a mapping $T : X \longrightarrow X$ a C^*-algebra-valued contractive mapping on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y)A, \quad \forall x, y \in X. \tag{1}$$

Definition 2.5. [25] Let X be a nonempty set. We call a mapping T a C^*-algebra-valued expansive mapping on X if $T : X \longrightarrow X$ satisfies:

i) $T(X) = X$;
ii) $d(Tx, Ty) \succeq A^* d(x, y)A, \quad \forall x, y \in X$,

where $A \in \mathbb{A}$ is an invertible element and $\|A^{-1}\| < 1$.

The first result on fixed point theory in C^*-algebra-valued metric space was established in 2014 by Ma et al. [25]. They showed that a self map, T on a complete C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ satisfying certain contractive conditions has a unique fixed point.

Theorem 2.1. *[25] If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X.*

Remark 2.6. Every contractive condition in the sense of [25] forces the mapping T to be continuous. We show this for completeness as follows:
let $\epsilon > 0$ be given. Then, for all $x, y \in X$ with $\|d(x, y)\| < \delta$, for some $\delta(\epsilon) > 0$,

$$\begin{aligned} \|d(Tx, Ty)\| &\leq \|A^* d(x, y)A\| \\ &< \|A^* A\delta\| \\ &= \|A\|^2\delta = \delta = \epsilon. \end{aligned}$$

Theorem 2.2. *[25] If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space, then for the expansive mapping T, there exists a unique fixed point in X.*

Lemma 2.3. *[11] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I*

i) *if $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$.*
ii) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.*
iii) *by $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I - a \in \mathbb{A}'_+$ is invertible operator, then $(I-a)^{-1}b \succeq (I-a)^{-1}c$.*

Using Lemma 2.3 above, Ma et al. [25] gave the following result.

Theorem 2.4. *[25] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose the mapping $T : X \longrightarrow X$ satisfies $\forall x, y \in X$,*

$$d(Tx, Ty) \preceq A\left(d(Tx, y) + d(Ty, x)\right)$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

In 1974, Ciric [9] introduced quasi contraction in metric space. We extend this notion to C^*-algebra-valued metric space.

Definition 2.7. Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a quasi contraction there exists $A \in \mathbb{A}'_+$ with $\|A\| \leq \frac{1}{2}$ such that for all $x, y \in X$,

$$d(Tx, Ty) \preceq A \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}.$$

We improve Theorems 2.1 and 2.4, using the following quasi contractive inequalities. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space and $T : X \longrightarrow X$ be a mapping satisfying

(Q1) $d(Tx, Ty) \preceq A \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}\ \forall\ x, y \in X,\ A \in \mathbb{A}'_+$.
(Q2) $d(Tx, Ty) \preceq A^* \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y,Tx)+d(x,Ty)}{2}\right\} A\ \forall\ x, y \in X,\ A \in \mathbb{A}$.

Theorem 2.5. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. $T : X \longrightarrow X$ be a mapping satisfying Q1 above with $\|A\| \leq \frac{1}{2}$, then T has a unique fixed point in X.*

Proof. For $A = \theta$, T maps X into a single point and the result is obvious. Assume $A \neq \theta$. Choose $x_0 \in X$ and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1 = Tx_0$, $x_2 = Tx_1 = TTx_0, \ldots, x_n = Tx_{n-1}$, $n = 1, 2, 3, \ldots$. Then,

$$\begin{aligned}
& d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\
\preceq\ & A \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\} \\
=\ & A \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_{n+1})\} \\
\preceq\ & A \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \\
=\ & A\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right). \\
=\ & Ad(x_{n-1}, x_n) + Ad(x_n, x_{n+1}).
\end{aligned}$$

Thus, $(I-A)d(x_n,x_{n+1}) \preceq Ad(x_{n-1},x_n)$. Since $A \in \mathbb{A}'_+$ with $\|A\| \leq \frac{1}{2}$, then $(I-A)^{-1} \in \mathbb{A}'_+$ and $A(I-A)^{-1} \in \mathbb{A}'_+$ by Lemma 2.3. Therefore,

$$\begin{aligned} d(x_n,x_{n+1}) &\preceq A(I-A)^{-1}d(x_{n-1},x_n) = \xi d(x_{n-1},x_n) \\ &\preceq \xi^2 d(x_{n-2},x_{n-1}) \preceq ... \preceq \xi^n d(x_0,x_1), \end{aligned}$$

where $\xi = A(I-A)^{-1}$, by D, we denote the element $d(x_0,x_1)$ in $\mathbb{A}$. Now, for $n,p>1$, we have

$$\begin{aligned} d(x_n,x_{n+p}) &\preceq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{n+p-1},x_{n+p}) \\ &\preceq \left(\xi^n + \xi^{n-1} + \xi^{n-2} + ... + \xi^{n+p-1}\right) D \\ &= \sum_{k=n}^{n+p-1} \xi^k D = \sum_{k=n}^{n+p-1} \xi^{\frac{k}{2}} \xi^{\frac{k}{2}} D^{\frac{1}{2}} D^{\frac{1}{2}} \\ &= \sum_{k=n}^{n+p-1} \left(\xi^{\frac{k}{2}} D^{\frac{1}{2}}\right)^* \left(\xi^{\frac{k}{2}} D^{\frac{1}{2}}\right) \\ &= \sum_{k=n}^{n+p-1} \left|\xi^{\frac{k}{2}} D^{\frac{1}{2}}\right|^2 \preceq \left\| \sum_{k=n}^{n+p-1} \left|\xi^{\frac{k}{2}} D^{\frac{1}{2}}\right|^2 \right\| \\ &\preceq \left\|D^{\frac{1}{2}}\right\|^2 \sum_{k=n}^{n+p-1} \left\|\xi^{\frac{k}{2}}\right\|^2 I \\ &\preceq \left\|D^{\frac{1}{2}}\right\|^2 \frac{\|\xi\|^n}{1-\|\xi\|} I \longrightarrow \theta \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,\mathbb{A},d)$ and by the completeness of this space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x$. Assume by contradiction $x \neq Tx$, $\forall x \in X$, i.e $\|d(Tx,x)\| > 0$ and by continuity of d, we have

$$\begin{aligned} & d(Tx,x) = d(Tx, \lim_{n\to\infty} Tx_{n-1}) = \lim_{n\to\infty} d(Tx,Tx_{n-1}) \\ \preceq\ & \lim_{n\to\infty} A\max\{d(x,x_{n-1}), d(x,Tx), d(x_{n-1},Tx_{n-1}), d(x,Tx_{n-1}), d(x_{n-1},Tx)\} \\ =\ & \lim_{n\to\infty} A\max\{d(x,x_{n-1}), d(x,Tx), d(x_{n-1},x_n), d(x,x_n), d(x_{n-1},Tx)\} \\ =\ & A\max\{\theta, d(x,Tx), \theta, \theta, d(x,Tx)\} = Ad(Tx,x). \end{aligned}$$

Then,

$$\|d(Tx,x)\| \leq \|Ad(Tx,x)\| \leq \frac{1}{2}\|d(Tx,x)\|.$$

A contradiction. Hence, $\|d(Tx,x)\|=0$, which implies $x=Tx$ i.e, x is a fixed point of T. For uniqueness, if $y(\neq x)$ is another fixed point, then

$$\begin{aligned}\theta \preceq d(x,y) &= d(Tx,Ty)\\ &\preceq A\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\\ &= A\max\{d(x,y),d(x,x),d(y,y),d(x,y),d(y,x)\}\\ &= Ad(x,y).\end{aligned}$$

Since $\|A\|\leq \frac{1}{2}$,

$$\begin{aligned}0 &\leq \|d(x,y)\|=\|d(Tx,Ty)\|\\ &\leq \|Ad(Tx,Ty)\|\leq \frac{1}{2}\|d(x,y)\|.\end{aligned}$$

This implies $d(x,y)=\theta \Leftrightarrow x=y$. □

Theorem 2.6. *Let $(X,\mathbb{A},d)$ be a complete C^*-algebra-valued metric space. $T: X\longrightarrow X$ be a mapping satisfying Q2 above with $\|A\|\leq 1$, then T has a unique fixed point in X.*

Proof. For $\mathbb{A}=\theta$, T maps X into a single point and the result is obvious. Assume $\mathbb{A}\neq\theta$. Choose $x_0\in X$ and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1=Tx_0$, $x_2=Tx_1=TTx_0,\ldots$, $x_n=Tx_{n-1}$, $n=1,2,3,\ldots$. Then

$$\begin{aligned}&d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n)\\ &\preceq A^*\max\left\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),\frac{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)}{2}\right\}A\\ &=A^*\max\left\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_n,x_n)+d(x_{n-1},x_{n+1})}{2}\right\}A\\ &\preceq A^*\max\left\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\}A.\end{aligned}$$

Assume that $\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\}=d(x_n,x_{n+1})$, then

$$\begin{aligned}d(x_n,x_{n+1}) &\preceq A^*d(x_n,x_{n+1})A\\ &= A^*\left(d(x_n,x_{n+1})\right)^{\frac{1}{2}}\left(d(x_n,x_{n+1})\right)^{\frac{1}{2}}A\\ &= \left((d(x_n,x_{n+1})A)^{\frac{1}{2}}\right)^*\left((d(x_n,x_{n+1})A)^{\frac{1}{2}}\right)\\ &= \left\|(d(x_n,x_{n+1})A)^{\frac{1}{2}}\right\|^2 I\\ &= \|A\|\|(d(x_n,x_{n+1})A)\|I.\end{aligned}$$

From the hypotheses $\|A\| < 1$, which is a contraction. Hence,

$$\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\} = d(x_{n-1},x_n).$$

Thus,

$$\begin{aligned} d(x_n,x_{n+1}) &\preceq A^* d(x_{n-1},x_n)A \\ &\preceq (A^*)^2 d(x_{n-2},x_{n-1})A^2 \preceq \cdots \\ &\preceq (A^*)^n d(x_0,x_1)A^n. \end{aligned}$$

For $n,p > 1$,

$$\begin{aligned} d(x_n,x_{n+p}) &\preceq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{n+p-1},x_{n+p}) \\ &\preceq (A^*)^n d(x_0,x_1)A^n + (A^*)^{n+1} d(x_0,x_1)A^{n+1} \\ &\quad + \cdots + (A^*)^{n+p-1} d(x_0,x_1)A^{n+p-1} \\ &= \sum_{k=n}^{n+p-1} (A^*)^k d(x_0,x_1)A^k \\ &= \sum_{k=n}^{n+p-1} (A^*)^k \left(d(x_0,x_1)\right)^{\frac{1}{2}} \left(d(x_0,x_1)\right)^{\frac{1}{2}} A^k \\ &= \sum_{k=n}^{n+p-1} \left(d(x_0,x_1)^{\frac{1}{2}} A^k\right)^* \left(d(x_0,x_1)^{\frac{1}{2}} A^k\right) \\ &\preceq \sum_{k=n}^{n+p-1} \left|d(x_0,x_1)^{\frac{1}{2}} A^k\right|^2 \preceq \left\|\sum_{k=n}^{n+p-1} d(x_0,x_1)^{\frac{1}{2}} A^k\right\|^2 I \\ &\preceq \left\|d(x_0,x_1)^{\frac{1}{2}}\right\|^2 \sum_{k=n}^{n+p-1} \|A\|^{2k} I \\ &\preceq \left\|d(x_0,x_1)^{\frac{1}{2}}\right\|^2 \frac{\|A\|^{2n}}{1-\|A\|} I \longrightarrow \theta \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,\mathbb{A},d)$ and by the completeness of this space, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} Tx_{n-1} = x$. Assume by contradiction $x \neq Tx$, for all $x \in X$, i.e $\|d(Tx,x)\| > 0$ and by continuity of d, we have

$$
\begin{aligned}
&d(Tx,x) = d(Tx, \lim_{n\to\infty} Tx_{n-1}) = \lim_{n\to\infty} d(Tx, Tx_{n-1})\\
&\preceq \lim_{n\to\infty} A^* \max\left\{d(x,x_{n-1}), d(x,Tx), d(x_{n-1},Tx_{n-1}), \frac{d(x,Tx_{n-1})+d(x_{n-1},Tx)}{2}\right\} A\\
&= \lim_{n\to\infty} A^* \max\left\{\theta, d(x,Tx), \theta, \frac{\theta + d(x,Tx)}{2}\right\} A\\
&= A^* d(x,Tx) A.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|d(Tx,x)\| &\leq \|A^* d(Tx,x) A\|\\
&= \|A^* A d(Tx,x)\|\\
&\leq \|A\|^2 \|d(Tx,x)\|.
\end{aligned}
$$

This implies that $\|d(Tx,x)\| < 0$. A contradiction hence $\|d(Tx,x)\| = 0$, which implies $x = Tx$ i.e, x is a fixed point of T. For uniqueness, if $y(\neq x)$ is another fixed point, then

$$
\begin{aligned}
\theta \preceq d(x,y) &= d(Tx,Ty)\\
&\preceq A^* \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} A\\
&= A^* \max\left\{d(x,y), d(x,x), d(y,y), \frac{d(x,y)+d(y,x)}{2}\right\} A\\
&= A^* d(x,y) A.
\end{aligned}
$$

Since $\|A\| < 1$,

$$
\begin{aligned}
0 &\leq \|d(x,y)\| = \|d(Tx,Ty)\|\\
&\leq \|A^* d(x,y) A\| = \|A^* A d(x,y)\|\\
&\leq \|A\|^2 \|d(x,y)\|.
\end{aligned}
$$

This implies $d(x,y) = \theta \Leftrightarrow x = y$. □

3. Sequent of Ma's Result

In this section, we give some important extensions of the work of Ma et al. [25]. One of the earliest extensions of Ma's result was given by Batul and Kamran [6]. Other results in related directions were given by [14, 21, 25, 43, 46, 57].

3.1. Batul and Kamran (2015). Batul and Kamran [6] established the new notion of C^*-algebra-valued contraction type mappings by weakening the contractive condition and proved the fixed point results for such mappings.

Definition 3.1. [16] Given a mapping $T : X \longrightarrow X$ and $x \in X$, the sequence of points $\mathcal{O}_T(x) = \{x, Tx, T^2x, ...\}$ is called the orbit of x with respect to T.

Definition 3.2. [6] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric. A mapping $T : X \longrightarrow X$ is said to be a C^*-algebra-valued contractive type mapping if there exists $x \in X$ and $A \in \mathbb{A}$ such that

$$d(Ty, T^2y) \preceq A^* d(y, Ty)A \quad \textit{with} \quad \|A\| < 1 \quad \forall y \in \mathcal{O}_T(x).$$

Remark 3.3. [6] A C^*-algebra-valued contraction mapping is a C^*-algebra-valued contractive type mapping, but the converse is not true as shown in the following example.

Example 3.4. [6] Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$ with $\|A\| = \max\{|a_1|, |a_2|, |a_3|, |a_4|\}$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space, where

$$d(x, y) = \begin{bmatrix} |x-y| & 0 \\ 0 & |x-y| \end{bmatrix}.$$

Define a mapping $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Then, for $y \in \mathcal{O}_T(x)$, $x \geq 0$, we see that T is a C^*-algebra-valued contractive type mapping. Since T is not continuous with respect to the C^*-algebra $\mathbb{A}$, then T is not a C^*-algebra-valued contraction mapping.

Lemma 3.1. *[6] Let $\mathbb{A}$ be a C^*-algebra with identity $1_{\mathbb{A}}$ and x be an element of $\mathbb{A}$. If $A \in \mathbb{A}$ is such that $\|A\| < 1$, then for $m < n$ we have*

$$\lim_{n\to\infty} \sum_{k=m}^{n} (A^*)^k x A^k = 1_{\mathbb{A}} \|(x)^{\frac{1}{2}}\|^2 \left(\frac{\|A\|^m}{1 - \|A\|} \right) \quad \textit{and} \quad \sum_{k=m}^{n} (A^*)^k x A^k \to \theta_{\mathbb{A}} \quad \textit{as} \quad m \to \infty.$$

The main result of Batul and Kamran [6] is the following.

Theorem 3.2. *[6] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ be a C^*-algebra-valued contractive type mapping. Then:*

A1) *there exist $x_0 \in X$ such that the sequence $T^n(x)$ converges to x_0;*

A2) $d(T^n(x), x_0) \leq \dfrac{\|a\|^{2n}}{1 - \|a\|} \|d(x, Tx)^{\frac{1}{2}}\|^2 1_{\mathbb{A}}$;

A3) *x_0 is a fixed point of T iff $G(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x_0 with respect to $\mathbb{A}$.*

3.2. Shehwar and Kamran (2015). Shehwar and Kamran [57] proved the fixed point theorem for self-mapping on C^*-algebra-valued metric space satisfying the contractive condition for pair of elements from the metric space which form edges of a graph in the metric space.

Property P1 [57] For any $\{f^n x\}$ in X such that $f^n x \to y \in X$ with $(f^{n+1}x, f^n x) \in E(G)$ there exists a subsequence $\{f^{n_k}x\}$ of $\{f^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, f^{n_k}x) \in E(G)$ for all $k \geq n_0$.

Definition 3.5. [57] Suppose $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space endowed with the graph $G = (V(G), E(G))$. A mapping $T : X \longrightarrow X$ is called a C^*-algebra-valued G-contraction on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \leq A^* d(x, y) A, \quad \forall (x, y) \in E(G). \tag{2}$$

Note that, a C^*-algebra-valued contractive mapping (1) satisfies C^*-algebra-valued G-contractive mapping (2) but the converse is not true.

Example 3.6. [57] Let $X = \mathbb{R}$, $\mathbb{A} = \mathbb{M}_{2\times 2}(\mathbb{R})$. Defining the norm on $\mathbb{A}$ by $\|\mathbb{A}\| = \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{\frac{1}{2}}$ and $* : \mathbb{A} = \mathbb{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{A} = \mathbb{M}_{2\times 2}(\mathbb{R})$ given by $A^* = A$. Then $\mathbb{A}$ is a C^*-algebra. Now, define $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{A}$ by

$$d(x, y) = \begin{pmatrix} |x - y| & 0 \\ 0 & |x - y| \end{pmatrix}.$$

Then $(\mathbb{R}, \mathbb{M}_{2\times 2}(\mathbb{R}), d)$ is a C^*-algebra-valued metric space. Defining a mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ by $Tx = \frac{x}{2}$ and consider the graph $G = (V(G), E(G))$ where $V(G) = \mathbb{R}$ and $E(G) = \left\{\left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) : n = 1, 2, 3, ...\right\} \cup \{(x, x) : x \in \mathbb{R}\}$. For each $n \in \mathbb{N}$, $\left(T\frac{1}{2^n}, T\frac{1}{2^{2n+1}}\right) = \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{4n+3}}\right) \in E(G)$. Also, for each $x \in \mathbb{R}$, $(Tx, Tx) = (\frac{x}{2}, \frac{x}{2})$, which is also an edge in the graph G. By taking

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

we have $\|A\| < 1$. It is easy to see that the contractive condition (2) holds for all edges that belong to the graph G but the contractive condition (1) is not satisfied. Hence T is C^*-algebra-valued G-contraction but not a C^*-algebra-valued contraction.

Lemma 3.3. *[57] Let $\mathbb{A}$ be a C^*-algebra and $a \in \mathbb{A}$ such that $\|a\| < 1$, then*

$$\lim_{m\to\infty} \sum_{k=m}^{n} \|a\|^k = 0.$$

The main result of Shehwar and Kamran [57] is the following.

Theorem 3.4. *[57] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space endowed with the graph $G = (V(G), E(G))$. Suppose $T : X \longrightarrow X$ is a C^*-algebra-valued G-contraction in X satisfying (P1) and the following conditions:*

i) *if $(x, y) \in E(G)$ then $(Tx, Ty) \in E(G)$;*
ii) *there exists an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$,*

then T has a fixed point. Moreover, if y, z are two fixed point of T and $(y, z) \in E(G)$ then $y = z$.

3.3. Xin, Jiang and Ma (2015). Xin et al. [46] obtained the coincidence fixed point and common fixed point theory for two mappings in a complete C^*-algebra-valued metric space which satisfy a new contractive condition.

Definition 3.7. [46] Two mappings T and S on a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is said to be compatible, if for arbitrary sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$, such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t \in X$, then $d(TSx_n, STx_n) \to \theta(n \to \infty)$.

Definition 3.8. [46] Let T and S be two mappings of the set X.

1) If $x = Tx = Sx$ for some $x \in X$, then x is called a common fixed point of T and S.
2) If $z = Tx = Sx$ for some $z \in X$, then x is called a coincidence point of T and S, and z is called a point of coincidence of T and S.
3) If T and S commute at all of their coincidence points, that is, $TSx = STx$ for all $x \in \{x \in X : Tx = Sx\}$, then T and S are called weakly compatible.

Note that for all $x \in X$, the mapping T and S commuting implies that they are weakly commuting, but the converse is not true.

Example 3.9. Let $X = \mathbb{R}$, $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$. Define $d(x, y) = \operatorname{diag}(|x - y|, \alpha |x - y|)$, where $x, y \in \mathbb{R}$ and $\alpha \geq 0$ is a constant. $(X, \mathbb{M}_2(\mathbb{R}), d)$ is a complete C^*-algebra-valued metric space by the completeness of $\mathbb{R}$. Defining two mapping $T, S : X \longrightarrow X$ by

$$Tx = \frac{x}{2}, \quad Sx = \frac{x}{x+2} \quad \forall x, y \in X.$$

Then, we have

$$\begin{aligned} d(TSx, STx) &= \operatorname{diag}\left(\left|\frac{x}{4+x} - \frac{x}{4+2x}\right|, \alpha\left|\frac{x}{4+x} - \frac{x}{4+2x}\right|\right) \\ &= \operatorname{diag}\left(\left|\frac{x^2}{(4+x)(4+2x)}\right|, \alpha\left|\frac{x^2}{(4+x)(4+2x)}\right|\right) \\ &\leq \operatorname{diag}\left(\left|\frac{x^2}{(4+2x)}\right|, \alpha\left|\frac{x^2}{(4+2x)}\right|\right) = \operatorname{diag}\left(\left|\frac{x}{2} - \frac{x}{x+2}\right|, \alpha\left|\frac{x}{2} - \frac{x}{x+2}\right|\right) \\ &= d(STx, TSx) \end{aligned}$$

for all $x \in X$. So T and S are weakly commuting, but they are not commuting since for all $x \in X$, $TSx \neq STx$.

Remark 3.10.

(i) The weak commutativity does not imply the existence of a sequence of points satisfying the condition of compatibility.
(ii) If S and T are compatible mappings, then $d(STx, TSx) = \theta$, whenever $d(Sx, Tx) = \theta$ for some $x \in X$.

The following example illustrates the above observation.

Example 3.11. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$. Define $d : X \times X \longrightarrow \mathbb{A}$ by $d(x, y) = (|x - y|, 0)$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space with norm $\|(x, y)\| = (x^2 + y^2)^{\frac{1}{2}}$.

Define two mappings $T, S : X \longrightarrow X$ by $Tx = x^2$, $Sx = 2 - x$ for all $x \in X$ respectively. Suppose $\{x_n\} \subseteq X$, then,

$$d(Tx_n, Sx_n) = (|x_n^2 + x_n - 2|, 0) = (|x_n + 2||x_n - 1|, 0) \to \theta \Longleftrightarrow x_n \to 1.$$

Therefore,

$$\lim_{n\to\infty} d(TSx_n, STx_n) = \lim_{n\to\infty} d\left((2 - x_n)^2, (2 - x_n^2)\right) = \lim_{n\to\infty} (2|x_n - 1|^2, 0) = \theta$$

as $x_n \to 1$. Hence T and S are compatible, but they are not weakly commuting since, if $x = 0$ in X,

$$d(TSx, STx) = (2, 0) > (-2, 0) = d(Tx, Sx).$$

Lemma 3.5. *[46] If the mapping T and S on the C^*-algbera-valued metric space $(X, \mathbb{A}, d)$ are compatible, then they are weakly compatible.*

Their main result is the following.

Theorem 3.6. *[46] Let $(X, \mathbb{A}, d)$ be a complete C^*-algbera-valued metric space. Suppose the mappings $T, S : X \longrightarrow X$ satifies:*

$$d(Tx, Sy) \preceq A^* d(x, y)A, \text{ for any } x, y \in X,$$

where $A \in \mathbb{A}$ with $\|A\| < 1$, then T and S have a unique fixed point in X.

Theorem 3.7. *[46] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that two mappings $T, S : X \longrightarrow X$ satisfies:*

$$d(Tx, Ty) \preceq A^* d(Sx, Sy)A, \text{ for any } x, y \in X,$$

where $A \in \mathbb{A}$ with $\|A\| < 1$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X, then T and S have a unique point of coincidence in X. Furthermore, if T and S are weakly compatible, T and S have a unique common fixed point in X.

Xin et al. [46] obtained the Kannan and Chatterjea common fixed point in C^*-algebra-valued metric space as follows:

Theorem 3.8. *[46] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that two mappings $T, S : X \longrightarrow X$ satisfies:*

$$d(Tx, Ty) \preceq Ad(Tx, Sx) + Ad(Ty, Sy), \text{ for any } x, y \in X,$$

where $A \in \mathbb{A}'_+$ with $\|A\| < \frac{1}{2}$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X, then T and S have a unique point of coincidence in X. Furthermore, if T and S are weakly compatible, T and S have a unique common fixed point in X.

Theorem 3.9. *[46] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that two mappings $T, S : X \longrightarrow X$ satisfies:*

$$d(Tx, Ty) \preceq Ad(Tx, Sy) + Ad(Sx, Ty), \text{ for any } x, y \in X,$$

where $A \in \mathbb{A}'_+$ with $\|A\| < \frac{1}{2}$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X, then T and S have a unique point of coincidence in X. Furthermore, if T and S are weakly compatible, T and S have a unique common fixed point in X.

Remark 3.12. If $S = I_X$, I_X is the identity mapping in X, then

i) Theorem 3.7 of Xin et al. [46] becomes Theorem 2.1 of Ma et al. [25].
ii) Theorem 3.9 of Xin et al. [46] becomes Theorem 2.4 of Ma et al. [25].

3.4. Gholamin and Khanehgir (2015). They established the structure of C^*-algebra-valued 2-metric space and gave some fixed point theorems for self-maps with contractive or expansive condition in such spaces

Definition 3.13. [14] Let X be a nonempty set, $\mathbb{A}$ be a C^*-algebra and $d : X \times X \times X \longrightarrow \mathbb{A}$ be a map satisfying the following conditions:

M1) for every pair of distinct element $x, y \in X$, there exists $z \in X$ such $d(x, y, z) \neq \theta$;
M2) if atleast two of three elements x, y, z are the same, then $d(x, y, z) = \theta$;
M3) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$;
M4) $d(x, y, z) \preceq d(t, x, y) + d(t, y, z) + d(t, x, z)$ for all $x, y, z, t \in X$.

Then d is called a C^*-algebra-valued 2-metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued 2-metric space.

Definition 3.14. [14] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued 2-metric space. A mapping $T : X \longrightarrow X$ is said to be

1. A C^*-algebra-valued 2-Contractive mapping on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that $d(Tx, Ty, w) \leq A^* d(x, y, w)A$ for all $x, y, w \in X$.
2. A C^*-algebra-valued 2-expansive mapping on X if it satisfies the following:
   E1) $T(X) = X$,
   E2) $d(Tx, Ty, w) \geq A^* d(x, y, w)A$ for each $x, y, w \in X$,
   where A is an invertible element in $\mathbb{A}$ such that $\|A^{-1}\| < 1$.

Their main result is the following.

Theorem 3.10. *[14] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued 2-metric space and $T : X \longrightarrow X$ is a C^*-algebra-valued 2-contractive mapping, then T has a unique fixed point in X.*

Theorem 3.11. *[14] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued 2-metric space and $T : X \longrightarrow X$ is a C^*-algebra-valued 2-expansive mapping, then T has a unique fixed point in X.*

Theorem 3.12. *[14] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued 2-metric space. Suppose the mapping $T : X \longrightarrow X$ satisfies the following conditions $\forall$ $x, y, w \in X$*

$$d(Tx, Ty, w) \leq A[d(Tx, x, w) + d(Ty, y, w)]$$
$$d(Tx, Ty, w) \leq A[d(Tx, y, w) + d(Ty, x, w)],$$

then T has a unique fixed point in X.

3.5. Ma and Jiang (2015). Ma and Jiang (2015) introduced the concept of C^*-algebra-valued b-metric space and gave some basic fixed point theorems for self-map with contractive conditions.

Definition 3.15. [25] Let X be a nonempty set, and $b \in \mathbb{A}'$ such that $b \geq 1$. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}$ satisfies

b1) $\theta \preceq d(x, y)$ for all $x, y \in X$;
b2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
b3) $d(x, y) \preceq b[d(x, z) + d(z, y)]$ for all $x, y, z \in X$;

Then d is called a C^*-algebra-valued b-metric on X and $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space.

Remark 3.16. [25] If $b = I$, then condition $(b3)$ becomes (3) of definition 2.1. Thus a C^*-algebra-valued b-metric space is an ordinary C^*-algebra-valued metric space. In particular, if $\mathbb{A} = \mathbb{C}$ and $\mathbb{A} = I$, the C^*-algebra-valued b-metric spaces are just the ordinary metric spaces.

Example 3.17. [25] Let $X = \mathbb{R}$ and $\mathbb{A} = \mathbb{M}_n(\mathbb{R})$. Define $d(x, y) = \text{diag}(c_1|x - y|^p, c_2|x - y|^p, ..., c_n|x - y|^p)$ and $x, y \in \mathbb{R}$, $c_i > 0$ $(i = 1, 2, ..., n)$ are constants and $p > 1$. It is easy to verify that $(X, \mathbb{M}_n(\mathbb{R}), d)$ is a complete C^*-algebra-valued b-metric space but not a C^*-algebra-valued metric space

The main result of Ma and Jiang [25] is the following.

Theorem 3.13. *[25] If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space and $T : X \longrightarrow X$ is a contractive mapping, there exists a unique fixed point in X.*

Also, they proved the Chatterjea and Kannan type fixed point theorem in C^*-algebra-valued b-metric space.

3.6. Ozer and Omran (2016). Ozer and Omran [43] established the existence and uniqueness of common fixed point theorem for self-maps in C^*-algebra-valued b-metric space. Their main result is the following.

Theorem 3.14. *[43] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Suppose the two mappings $T : X \longrightarrow X$ and $S : X \longrightarrow X$ satisfy $d(Tx, Sy) \leq A^* d(x, y)A$ for all $x, y \in X$ and $A \in \mathbb{A}$ with $\|A\| \leq 1$. Then T and S have a unique common fixed point in X.*

Remark 3.18. If $b = I$(the identity mapping) in C^*-algebra-valued b-metric space, theorem 3.20 of [43] becomes theorem 3.8 of [46].

3.7. Kamran et al. (2016). Kamran et al. [21] defined the concept of C^*-algebra-valued b-metric space using the concept of b-metric space introduced by Czerwick [10] and generalized the Banach contraction principle in this setting. Their main result is the following.

Theorem 3.15. *[21] Consider a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$ with coefficient b. Let $T : X \longrightarrow X$ be a contraction with contraction constant A such that $\|b\|\|A\|^2 < 1$. Then T has a unique fixed point in X.*

Remark 3.19. [21] If $\mathbb{A} = \mathbb{R}$ then the notion of C^*-algebra-valued b-metric space becomes b-metric space [10]. If $b = I$ in Definition 3.8, then d becomes the usual C^*-algebra-valued metric as defined in [25].

3.8. Bai (2016). Bai [4] established some couple fixed point theorems for mapping satisfying different contractive conditions on C^*-algebra-valued b-metric space.

Definition 3.20. [4] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $T : X \times X \longrightarrow X$ if $T(x, y) = x$ and $T(y, x) = x$.

Theorem 3.16. *[4] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Assume that the mapping $T : X \times X \longrightarrow X$ satisfies the following condition:*

$$d(T(x, y), T(u, v)) \leq A^* d(x, u)A + A^* d(y, v)A, \quad \forall x, y, u, v \in X,$$

where $A \in \mathbb{A}$ with $2\|A\|^2\|b\| \leq 1$. Then T has a unique coupled fixed point in X. Moreover, T has a unique fixed point in X.

Theorem 3.17. *[4] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Assume that the mapping $T : X \times X \longrightarrow X$ satisfies the following condition:*

$$d(T(x, y), T(u, v)) \leq A_1 d(T(x, y), u) + A_2 d(T(u, v), x), \quad \forall x, y, u, v \in X,$$

where $A_1, A_2 \in \mathbb{A}'_+$ with $\|A_1 + A_2\|\|b\|^2 < 1$. Then T has a unique coupled fixed point in X. Moreover, T has a unique fixed point in X.

Theorem 3.18. *[4] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Assume that the mapping $T : X \times X \longrightarrow X$ satisfies the following condition:*

$$d(T(x,y),T(u,v)) \leq A_1 d(T(x,y),x) + A_2 d(T(u,v),u), \quad \forall x,y,u,v \in X,$$

where $A_1, A_2 \in \mathbb{A}'_+$ with $\|A_1 + A_2\| \|b\|^2 < 1$. Then T has a unique coupled fixed point in X. Moreover, T has a unique fixed point in X.

3.9. Zada, Saifullahi and Ma (2016). Zada et al. [60] proved the common fixed point theorems for two mappings in complete C^*-algebra-valued metric space endowed with the graph $G = (V, E)$ which satisfies G-contractive condition.

Their main result is the following.

Theorem 3.19. *[60] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space endowed with the graph $G = (V, E)$. Suppose the mappings $T, S : X \longrightarrow X$ are C^*-algebra-valued G-contractive mappings on X satisfying:*

i) *for any $\{T^n x\} \in X$ such that $T^n x$ converges to $x \in X$ with $(T^{n+1}x, T^n x) \in E$, there exists a subsequence $\{T^{n_k}x\}$ and $n_0 \in \mathbb{N}$ such that $(x, T^{n_k}x) \in E$ for all $k \geq n_0$,*
ii) *if $(x, y) \in E$ then $(Tx, Sy) \in E$,*
iii) *there exists $z_0 \in X$ such that (z_0, Tz_0),$(z_0, Sz_0) \in E$.*

Then T and S have a unique common fixed point in X.

Theorem 3.20. *[60] Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space endowed with the graph G, and suppose that the mappings $T, S : X \longrightarrow X$ are G-contractive, satisfying $\|d(Tx, Sy)\| \leq \|A\| \|d(x, y)\|$, for all $(x, y) \in E$, where $A \in \mathbb{A}$ with $\|A\| < 1$. Then T and S have a unique common fixed point in X.*

Remark 3.21. [60] If $S = T$, Theorem 3.19 becomes [56, Theorem 3.5].

3.10. Shehwar et al. (2016). Shehwar et al. [56] extended the Caristi's fixed point theory to C^*-algebra-valued metric space using the concept of minimal element in C^*-algebra-valued metric space by introducing the notion of partial order on X.

Definition 3.22. [56] Let $(X, \mathbb{A}, d)$ be a C^*-algebra metric space. A mapping $\Phi : X \longrightarrow \mathbb{A}$ is said to be lower semi continuous at x_0 with respect to $\mathbb{A}$ if $\|\Phi(x_0)\| \leq \liminf_{x \to x_0} \|\Phi(x)\|$.

Lemma 3.21. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra metric space and let $\Phi : X \longrightarrow \mathbb{A}_+$ be a map. Define the order $\preccurlyeq_\Phi$ on X by $x \preccurlyeq_\Phi y \Leftrightarrow d(x, y) \leq \Phi(y) - \Phi(x)$ for any $x, y \in X$. Then $\preccurlyeq_\Phi$ is a partial order on X.*

The main result of Shehwar et al. [56] is the following.

Theorem 3.22. *[56] Let $(X, \mathbb{A}, d)$ be a C^*-algebra metric space and $\Phi : X \longrightarrow \mathbb{A}_+$ be a lower semi-continuous map. Then $(X, \preccurlyeq_\Phi)$ has a minimal element.*

Theorem 3.23. *[56] Let $(X, \mathbb{A}, d)$ be a C^*-algebra metric space and $\Phi : X \longrightarrow \mathbb{A}_+$ be a lower semi-continuous map. Let $T : X \longrightarrow X$ be such that for all $x \in X$*

$$d(x, Tx) \preccurlyeq \Phi(x) - \Phi(Tx),$$

then T has atleast one fixed point.

Remark 3.23. [56] Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be a C^*-algebra. Define $d : X \times X \longrightarrow \mathbb{A}$ by $d(x, y) = (|x - y|, 0)$. Let $\Phi : X \longrightarrow \mathbb{A}_+$, $\Phi(x) = (x, 0)$ be continuous map, and $T : X \longrightarrow X$ be define by $T(x) = x^2$. Then it is easy to see that all the conditions of Theorem 3.33 are satisfied and T has a fixed point. Note that contractive theorem stated in [25] is not applicable here, since contractive condition (1) does not hold.

3.11. Klin-eam and Kaskasem (2016). Klin-eam and Kaskasem [23] studied the fundamental properties of C^*-algebra-valued b-metric space which was introduced by Ma and Jiang [25] and gave some fixed point theorems for cyclic mapping with contractive and expansive condition on such space.

Theorem 3.24. *[23] Let P and Q be two nonempty closed subsets of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : P \cup Q \longrightarrow P \cup Q$ is cyclic mapping satisfying*

$$d(Tx, Ty) \preceq A^* d(x, y) A, \quad \forall x \in P, y \in Q,$$

where $A \in \mathbb{A}$ with $\|A\| < \dfrac{1}{\|b\|}$. Then T has a unique fixed point in $P \cap Q$.

Klin-eam and Kaskasem [23] proved the cyclic Kannan-type and the cyclic Chatterjea-type fixed point results as follows.

Theorem 3.25. *[23] Let P and Q be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : P \cup Q \longrightarrow P \cup Q$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq A[d(x, Tx) + d(y, Ty)], \quad \forall x \in P, \forall y \in Q,$$

where $A \in \mathbb{A}'_+$ with $\|A\| < \dfrac{1}{2\|b\|}$. Then, T has a unique fixed point in $P \cap Q$.

Theorem 3.26. *[23] Let P and Q be nonempty closed subsets of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : P \cup Q \longrightarrow P \cup Q$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq A[d(y, Tx) + d(x, Ty)], \quad \forall x \in P, \forall y \in Q,$$

where $A \in \mathbb{A}'_+$ with $\|A\| < \dfrac{1}{2\|b\|^2}$. Then, T has a unique fixed point in $P \cap Q$.

Note that Banach's contraction, Kannan's contraction and Chatterjea's contraction are independent (see [48]).

3.12. Kadelburg and Radenovic (2016). They showed that all the result in C^*-algebra-valued b-metric space can be directly obtained as consequences of their standard metric or b-metric counterparts.

Lemma 3.27. *[37] Let $\mathbb{A}$ denotes a unital C^*-algebra and $\mathbb{A}_h$ denotes the set of all self-adjoint elements of $\mathbb{A}$. Then,*

1) $\mathbb{A}_+ = \{a^*a : a \in \mathbb{A}\}$;
2) *if $a, b \in \mathbb{A}_h$, $a \preceq b$, and $c \in \mathbb{A}$, then $c^*ac \preceq c^*bc$;*
3) *for all $a, b \in \mathbb{A}_h$, if $\theta \preceq a \preceq b$ then $\|a\| \leq \|b\|$.*

The main result of Kadelburg and Radenovic [18] is the following.

Theorem 3.28. *[18] Theorem 2.1 of [25] is equivalent to the Banach Contraction principle (BCP).*

Theorem 3.29. *[18] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space and $T : X \longrightarrow X$ be a mapping. Suppose that there exists $A \in \mathbb{A}$ with $\|A\| < 1$ and that for all $x, y \in X$ there exists $u(x, y) \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ such that $d(Tx, Ty) \leq A^*u(x, y)A$. Then T has a unique fixed point in X.*

Theorem 3.30. *[12] Let (X, d, s) be a complete b-metric space and let $T : X \longrightarrow X$ be a map such that, for some $\lambda \in [0, 1)$ and for all $x, y \in X$,*

$$d(Tx, Ty) \leq \lambda d(x, y). \tag{3}$$

Then T has a unique fixed point in X.

Theorem 3.31. *[18] Theorem 3.30 above and Theorem 3.13 of [25] are equivalent.*

Remark 3.24. [18] We note some other results from [24, 22] that can be reduced in the same way to well-known results in b-metric spaces:

i) the Chatterjea-type fixed point result [25], with the contractive condition in the form $d(Tx, Ty) \preceq A^*(d(x, Ty) + d(y, Tx))A$, with $A \in \mathbb{A}$, $\|A\| < \frac{1}{\|b\|\sqrt{2}}$;
ii) the Kannan-type fixed point result [25], with the contractive condition in the form $d(Tx, Ty) \preceq A^*(d(x, Tx) + d(y, Ty))A$, with $A \in \mathbb{A}$, $\|A\| < \frac{1}{\sqrt{2\|b\|}}$;
iii) the Banach-type cyclic fixed point result [23], with the improved condition $\|\lambda\| < 1$ instead of $\|\lambda\| < \frac{1}{\|b\|}$;
iv) the Banach-type fixed point result for expansive mappings [23], with the improved condition $\|\lambda\| < 1$ instead of $\|\lambda\| < \frac{1}{\|b\|}$;
v) the cyclic Kannan-type, resp. cyclic Chatterjea-type fixed point results [23], with contractive conditions as in (ii), resp. (i).

3.13. Kadelburg et al. (2016). Kadelburg et al. [17] improved and generalized the result of Klin-eam and Kaskasem [23] about contractive and cyclic mappings established in the framework of C^*-algebra-valued b-metric spaces.

Theorem 3.32. *[17] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \longrightarrow X$ be a given mapping. Assume that there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that*

$$d(Tx, Ty) \preceq A^* d(x, y)A,$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Theorem 3.33. *[17] Let $(X, \mathbb{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \longrightarrow X$ be a given mapping. Assume that there exists $A \in \mathbb{A}$ with $\|A\| < \dfrac{1}{\sqrt{3\|b\|}}$ such that*

$$d(Tx, Ty) \preceq A^*[d(x, y) + d(x, Ty) + d(x, Ty)]A, \quad \forall x, y \in X.$$

Then T has a unique fixed point in X.

Theorem 3.34. *[17] Let $(X, \mathbb{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let P and Q be two nonempty closed subsets of X, and let $T : P \cup Q \longrightarrow P \cup Q$ be a cyclic mapping. Assume that there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that*

$$d(Tx, Ty) \preceq A^* d(x, y)A, \quad \forall x \in P, y \in Q.$$

Then T has a unique fixed point in $P \cap Q$.

Theorem 3.35. *[17]Let $(X, \mathbb{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let P and Q be two nonempty closed subsets of X, and let $T : P \cup Q \longrightarrow P \cup Q$ be a cyclic mapping. Assume that there exists $A, L \in \mathbb{A}_+$ with $\|A\| < \frac{1}{2\|b\|}$ and $AB = BA$ for all $B \in \mathbb{A}_+$ such that*

$$d(Tx, Ty) \preceq A(d(x, Tx) + d(y, Ty)), \text{ for any } x \in P,\ y \in Q.$$

Then T has a unique fixed point in $P \cap Q$.

Theorem 3.36. *[17] Let $(X, \mathbb{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let P and Q be two nonempty closed subsets of X, and let $T : P \cup Q \longrightarrow P \cup Q$ be a cyclic mapping. Assume that there exists $A, L \in \mathbb{A}_+$ with $\|A\| < \dfrac{1}{\|b\|(1 + \|b\|)}$ and $AB = BA$, $LB = BL$ for all $B \in \mathbb{A}_+$ such that*

$$d(Tx, Ty) \preceq Ad(x, Ty) + Ld(y, Tx), \text{ for any } x \in P,\ y \in Q.$$

Then T has a unique fixed point in $P \cap Q$.

Theorem 3.37. *[17] Let $(X, \mathbb{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let P and Q be two nonempty closed subsets of X, and let $T : P \cup Q \longrightarrow P \cup Q$ be a cyclic mapping. Assume that there exists $A \in \mathbb{A}$ with $\|A\| < \dfrac{1}{\sqrt{3\|b\|}}$ such that*

$$d(Tx, Ty) \preceq A^*[d(x, y) + d(x, Tx) + d(y, Ty)]A, \quad \forall x \in P, y \in Q.$$

Then T has a unique fixed point in $P \cap Q$.

Remark 3.25. [17] Putting $P = Q = X$ in Theorems 3.45, 3.46, 3.47 and 3.48, we obtain the non-cyclic version. This shows that each true cyclic type extension is in fact a generalization of usual non-cyclic type.

3.14. Gholamin et al. (2017). Gholamin et al. [15] introduced the notion of C^*-algebra-valued b_2-metric space and gave some fixed point theorems for self-maps with contractive or expansive condition on such space.

Definition 3.26. [15] Let X be a nonempty set, $s \geq 1$ be a real number, $\mathbb{A}$ be a C^*-algebra and $d : X \times X \times X \longrightarrow \mathbb{A}$ be a map satisfying the following conditions:

1) for every pair of distinct elements $x, y \in X$, $\exists\, z \in X$ such that $d(x, y, z) \neq \theta$;
2) if atleast two of the three elements x, y, z are the same, then $d(x, y, z) = \theta$;
3) $d(x, y, z) = d(x, z, y) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$;
4) $d(x, y, z) \leq b[d(t, x, y) + d(t, y, z) + d(t, x, z)]$ $\forall x, y, z, t \in X$.

Then d is called a C^*-algebra-valued b_2-metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued b_2-metric space with parameter b.

Definition 3.27. [15] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b_2-metric space, $T : X \longrightarrow X$ be a given mapping and $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a mapping. We say that

1. T is α-admissible if for all $x, y, z \in X$
$$\alpha(x, y, z) \geq 1 \quad \Rightarrow \alpha(Tx, Ty, z) \geq 1.$$
2. T is triangular α-admissible if it is α-admissible and for all $x, y, z, a \in X$
$$\alpha(x, y, a) \geq 1, \quad \alpha(y, z, a) \geq 1 \quad \Rightarrow \alpha(x, z, a) \geq 1.$$
3. T is α-contractive on X if T is triangular α-admissible and for all $x, y, a \in X$ there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that the following condition holds:
$$b\alpha(x, y, a)d(Tx, Ty, a) \leq A^* d(x, y, a)A.$$

The main result of Gholamin et al. [15] is the following.

Theorem 3.38. *[15] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b_2-metric space, $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a mapping and $T : X \longrightarrow X$ be an α-contractive mapping satisfying the following conditions*

i) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, a) \geq 1$ and $\alpha(x_0, Tx_0, a) \geq 1$;*
ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_{n+1}, x_n, a) \geq 1$ and $\alpha(x_n, x_{n+1}, a) \geq 1$ as $n \to \infty$ then $\alpha(x_n, x, a) \geq 1$.*

Then T has a fixed point in X.

Theorem 3.39. *[15] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b_2-metric space. Suppose $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ and $T : X \longrightarrow X$ are two mappings satisfying the following conditions*

i) *T is triangular α-admissible mapping and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, a) \geq 1$ and $\alpha(x_0, Tx_0, a) \geq 1$;*
ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_{n+1}, x_n, a) \geq 1$ and $\alpha(x_n, x_{n+1}, a) \geq 1$ as $n \to \infty$ then $\alpha(x_n, x, a) \geq 1$;*
iii) *for all $x, y, a \in X$, the following inequality holds:*
$$b^2\alpha(x, y, a)d(Tx, Ty, a) \leq A[d(Tx, y, a) + d(Ty, x, a)],$$
where $A \in \mathbb{A}'_+$ and $\|A\| \leq \frac{1}{2}$.

Then T has a fixed point in X.

Theorem 3.40. *[15] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b_2-metric space. Suppose $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ and $T : X \longrightarrow X$ are two mappings satisfying the following conditions*

i) *T is triangular α-admissible mapping and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, a) \geq 1$ and $\alpha(x_0, Tx_0, a) \geq 1$;*
ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_{n+1}, x_n, a) \geq 1$ and $\alpha(x_n, x_{n+1}, a) \geq 1$ as $n \to \infty$ then $\alpha(x_n, x, a) \geq 1$;*
iii) *for all $x, y, a \in X$, the following inequality holds:*
$$b\alpha(x, y, a)d(Tx, Ty, a) \leq A[d(Tx, x, a) + d(Ty, y, a)],$$
where $A \in \mathbb{A}'_+$ and $\|A\| \leq \frac{1}{2}$.

Then T has a fixed point in X.

Theorem 3.41. *[15] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b_2-metric space. $T : X \longrightarrow X$ be an expansive mapping X. Then T has a fixed point in X.*

3.15. Ozer and Omran (2017). Ozer and Omran [42] studied the generalized C^*-algebra-valued metric space and proved certain fixed point theorem for a single valued mapping in such space. They assumed the mapping satisfies certain D-metric conditions with generalized fixed point theorem.

Definition 3.28. [42] Let X be a nonempty set and $\mathbb{A}_+$ be set of all positive elements on the C^*-algebra $\mathbb{A}$. Then X together with the function $D_{\mathbb{A}} : X \times X \times X \longrightarrow \mathbb{A}_+$ is called a $D_{\mathbb{A}}$-metric if it satisfies the following conditions:

i) $D_{\mathbb{A}}(x, y, z) = \theta \Leftrightarrow x = y = z$;
ii) $D_{\mathbb{A}}(x, y, z) = D_{\mathbb{A}}(P(x, y, z))$, where P denotes the permutation function (a function that rearranges the order of terms in a sequence.);
iii) $D_{\mathbb{A}}(x, y, z) \preceq D_{\mathbb{A}}(x, y, a) + D_{\mathbb{A}}(x, a, z) + D_{\mathbb{A}}(a, y, z)$ for $x, y, z, a \in X$.

Then $(X, \mathbb{A}, D_{\mathbb{A}})$ is called the generalized C^*-algebra-valued metric space.

Definition 3.29. [42] Let $(X, \mathbb{A}, D_{\mathbb{A}})$ be a generalized C^*-algebra-valued metric space. The mapping $T : X \longrightarrow X$ is known as a generalized C^*-algebra-valued contraction on X if there exists an element $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty, Tz) \leq A^* d(x, y, z)A \quad for \quad x, y, z \in X.$$

The main result of Ozer and Omran [42] is the following.

Theorem 3.42. *[42] If $(X, \mathbb{A}, D_{\mathbb{A}})$ is a complete generalized C^*-algebra-valued metric space and the mapping T is a contractive mapping then there exists a fixed point in X.*

3.16. Moeini and Hojat (2017). Moeini and Hojat [31] established the fixed point theorem in C^*-algebra-valued metric space endowed with a graph $G = (V, E)$.

Let $PC(f, g)$, denotes the set of all points of coincidence of the pair $\{f, g\}$. Some of the properties of the graph are given as follows:

Property 1 If (gx_n) is a sequence in X such that $gx_n \to x$ and $(gx_n, gx_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, x) \in E(G)$ for all $i \geq 1$.

Property 2 If $x, y \in PC(f, g)$ then $(x, y) \in E(G)$.

Their main result is the following.

Theorem 3.43. *[31] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space endowed with a graph $G = (V, E)$ and the mapping $f, g : X \longrightarrow X$ satisfy $d(fx, fy) \preceq A^* d(gx, gy)A$, for all $x, y \in X$ with $(gx, gy) \in E(G)$, $A \in \mathbb{A}$ and $\|b\|\|A\|^2 < 1$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with property 1. Then the set $PC(f, g) \neq \emptyset$ if $C_{gf} \neq \emptyset$. Moreover $PC(f, g)$ is singleton if the graph G has the property 2. Furthermore, if f and g are weakly compatible, then f and g have a unique fixed point in X.*

Theorem 3.44. *[31] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space endowed with a graph $G = (V, E)$ and the mapping $f, g : X \longrightarrow X$ satisfy $d(fx, fy) \preceq A(d(fx, gy) + d(fy, gy))$, for all $x, y \in X$ with $(gx, gy) \in E(G)$, $A \in \mathbb{A}'_+$ and $\|Ab\| < \frac{1}{2}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with property 1. Then the set $PC(f, g) \neq \emptyset$ if $C_{gf} \neq \emptyset$. Moreover $PC(f, g)$ is singleton if the graph G has the property 2. Furthermore, if f and g are weakly compatible, then f and g have a unique fixed point in X.*

3.17. Shateri (2017). Shateri [55] introduced the concept of C^*-algebra-valued modular space and presented some fixed point theorem for self map with contractive or expansive condition on such space.

Definition 3.30. [55] Let χ be an arbitrary vector space over $\mathbb{F} = (\mathbb{R}$ or $\mathbb{C})$. Suppose $\rho : \chi \longrightarrow \mathbb{A}$ is a function such that for all $x, y \in \chi$,

(i) $\rho(x) \succeq \theta$ and $\rho(x) = \theta$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every $\alpha \in \mathbb{F}$ with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \preceq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Then ρ is called a C^*-algebra-valued modular on χ.

Definition 3.31. [55] Let χ_ρ be a C^*-algebra-valued modular space. A mapping $T : \chi_\rho \longrightarrow \chi_\rho$ is said to be a C^*-algebra-valued contractive mapping on χ_ρ, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ and $\alpha, \beta \in \mathbb{R}_+$ with $\alpha > \beta$ such that for all $x, y \in \chi$

$$\rho(\alpha(Tx - Ty)) \preceq A^* \rho(\beta(x - y))A.$$

Definition 3.32. [55] Let χ_ρ be a C^*-algebra-valued modular space. A mapping $T : \chi_\rho \longrightarrow \chi_\rho$ is said to be a C^*-algebra-valued expansive mapping on χ_ρ , if

1) $T(\chi_\rho) = \chi_\rho$
2) there exists an invertible element $A \in \mathbb{A}$ with $\|A^{-1}\| < 1$ and $\alpha, \beta \in \mathbb{R}_+$ with $\alpha > \beta$ such that for all $x, y \in \chi$ $\quad \rho(\beta(Tx - Ty)) \succeq A^* \rho(\alpha(x - y))A$.

The main result of Shateri [55] is the following.

Theorem 3.45. *[55] Let χ_ρ be a ρ-complete modular space and let T be a C^*-algebra-valued contractive mapping on χ_ρ. Then T has a unique fixed point in χ_ρ.*

Theorem 3.46. *[55] Let χ_ρ be a ρ-complete modular space and let T be a C^*-algebra-valued expansive mapping on χ_ρ. Then T has a unique fixed point in χ_ρ.*

Theorem 3.47. *[55] Let χ_ρ be a ρ-complete C^*-algebra-valued modular space. Suppose the mapping $T : \chi_\rho \longrightarrow \chi_\rho$ satisfies*

$$\rho(\alpha(Tx - Ty)) \preceq A(\rho(\beta(Tx - y) + \rho(\beta(Ty - x)))$$

for all $x, y \in \chi_\rho$, where $A \in \mathbb{A}'_+ = \{A \in \mathbb{A}_+ : Ab = bA, \forall b \in \mathbb{A}_+\}$ and $\|A\| < \frac{1}{2}$. Then T has a unique fixed point in χ_ρ.

3.18. Mondal, Chanda and Karmakar (2017). They proved the existence and uniqueness of common fixed points for self-maps with contractive or expansive conditions on C^*-algebra-valued metric spaces. They also defined C^*-algebra-valued proximal contraction and showed the existence and uniqueness of the best proximity points for these proximal contraction mappings on such space.

Their main result is the following.

Theorem 3.48. *[36] Let T and S be two self-maps defined on a complete C^*-algebra-valued metric space $(X, \mathbb{A}, d)$, which satisfy the condition $d(Tx, Sy) \preceq A^* M(x, y)A$ for any $x, y \in X$, where $A \in \mathbb{A}$ with $\|A\| < \frac{1}{\sqrt{2}}$ and $M(x, y)$ takes either $d(x, y)$ or $d(Tx, x) + d(Sy, y)$ or $d(Tx, y) + d(Sy, x)$. Then T and S have a unique common fixed point in X.*

Theorem 3.49. *[36] Let S and T be two expansion mappings on a complete C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ such that for all $x, y \in X$*

$$d(Sx, Ty) \succeq A^* d(x, y)A,$$

where $A \in \mathbb{A}$, is an invertible element and $\|A^{-1}\| < 1$. Then S and T have a unique common fixed point.

Theorem 3.50. *[36] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose the self-mappings S, T on X satisfy*

$$d(Sx, Ty) \preceq A(d(Sx, y) + d(Ty, x)),$$

for all $x, y \in X$, where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$. Then S and T have a unique common fixed point.

Theorem 3.51. *[36] Let (P, Q) be a pair of non-empty closed subsets of a complete C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ and P_0 be non-empty. We assume that $T : P \longrightarrow Q$ satisfies the following conditions:*

(a) *T is a C^*-algebra-valued proximal contraction;*
(b) *$T(P_0) \subseteq Q_0$.*

Then there exists a unique $x \in P_0$ such that $d(x, Tx) = d(P, Q)$.

3.19. Ege and Alaca (2018). The main contribution of the work of Ege and Alaca [13] is introducing the notion of C^*-algebra-valued S-metric space, defining new notions such as L-condition and k-contraction, proving Banach contraction principle and common fixed point theorem in C^*-algebra-valued S-metric spaces .

Definition 3.33. [13] Let X be a nonempty set. Suppose the mapping $S : X \times X \times X \longrightarrow \mathbb{A}$ satisfies the following conditions for each $x, y, z, a \in X$:

(i) $S(x, y, z) \succeq \theta$;
(ii) $S(x, y, z) = \theta$ if and only if $x = y = z$;
(iii) $S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called a C^*-algebra-valued S-metric and $(X, \mathbb{A}, S)$ is called a C^*-algebra-valued S-metric space.

Example 3.34. [13] Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$ with $\|A\| = \max\{a_1, a_2, a_3, a_4\}$, where a_i's are the entries of A. Define $S : X \times X \times X \longrightarrow \mathbb{A}$ by

$$S(x, y, z) = \begin{bmatrix} |x - z| + |y - z| & 0 \\ 0 & |x - z| + |y - z| \end{bmatrix}.$$

Then $(X, \mathbb{A}, S)$ is a C^*-algebra-valued S-metric space.

Definition 3.35. [1] Let X be a nonempty set. We say the mappings $f : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ satisfy the L-condition if $gf(x, y) = f(gx, gy)$ for all $x, y \in X$.

Lemma 3.52. *[13] In a C^*-algebra-valued S-metric space, $S(x,x,y) = S(y,y,x)$.*

Definition 3.36. [13] Let $(X, \mathbb{A}, S)$ be a C^*-algebra-valued S-metric space. A map $T : X \longrightarrow X$ is said to be C^*-algebra-valued contractive mapping on X, if there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that $S(Tx, Tx, Ty) \preceq A^* S(x, x, y) A$ for all $x, y \in X$.

Definition 3.37. [13] Let $(X, \mathbb{A}, S)$ be a C^*-algebra-valued S-metric space. The mappings $f : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ satisfy the k-contraction if

$$S(f(x,y), f(x,y), f(z,w)) \preceq kA^*[S(gx, gx, gz) + S(gy, gy, gw)]A$$

with respect to $\mathbb{A}$ for all $x, y, z, w, u, v \in X$.

The main result of Ege and Alaca [13] is the following.

Theorem 3.53. *[13] Let $(X, \mathbb{A}, S)$ be a C^*-algebra-valued S-metric space. Suppose that $f : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$ and L-condition. If $g(X)$ is continuous with closed range such that $f(X \times X) \subset g(X)$, then there is a unique x in X such that $gx = f(x,x) = x$.*

Corollary 3.54. *[13] Let $(X, \mathbb{A}, S)$ be a C^*-algebra-valued S-metric space, if the mapping that $f : X \times X \longrightarrow X$ satisfies the following condition*

$$S(f(x,y), f(u,v), f(z,w)) \preceq kA^*[S(x,u,z) + S(y,v,w)]A$$

with respect to $\mathbb{A}$ for all $x, y, z, u, v, w \in X$ and $k \in (0, \frac{1}{2})$, then there exists a unique element $x \in X$ such that $f(x,x) = x$.

3.20. Omran and Salama (2018). They introduced a common coupled fixed point theorem in the C^*-algebra-valued metric spaces with certain contraction condition.

The main result of Omran and Salama [41] is as follows.

Theorem 3.55. *[41] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space, and let $T, S : X \longrightarrow X$ be mappings such that $d(T(x,y), S(u,v)) \leq Aw(x,y,u,v)A^*$ for all $x, y, u, v \in X$, where $w(x,y,u,v)$ is one of the following*

$$\{d(x,u), d(y,v), \frac{1}{2}d(T(x,y),x) + d(S(u,v),u)), \frac{1}{2}(d(T(x,y),u) + d(S(u,v),x))\},$$

then T, S have a unique common coupled fixed point.

3.21. Moeine, Kumar and Aydi (2018). They discussed some unique fixed point theorems for cyclic mappings of Zamfirescu contraction type in the context of C^*-algebra-valued metric spaces.

In 1972, Zamfirescu [61] obtained an interesting fixed point result in metric space by combining the contracttive condition of Banach, contractive condition of Kannan and contractive condition of Chatterjea.

The main result of Moeini et al. [34] is the following.

Theorem 3.56. *[34] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and let $A, B, C \in \mathbb{A}$ such that $\|A\| < 1$ and $\|B\|, \|C\| < \frac{1}{\sqrt{2}}$. Consider $T : X \longrightarrow X$ such that at least one of the following conditions holds*

$$\begin{aligned} d(Tx, Ty) &\preceq A^* d(x, y) A; \\ d(Tx, Ty) &\preceq B^*(d(x, Tx) + d(y, Ty))B; \\ d(Tx, Ty) &\preceq C^*(d(x, Ty) + d(y, Tx))C, \end{aligned}$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point in X.

Theorem 3.57. *[34] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and let $T : X \longrightarrow X$ be a continuous function such that atleast one of the following conditions is satisfied:*

$$\begin{aligned} d(Tx, Ty) &\preceq d(x, y); \\ d(Tx, Ty) &\preceq \frac{1}{2}(d(x, Tx) + d(y, Ty)); \\ d(Tx, Ty) &\preceq \frac{1}{2}(d(x, Ty) + d(y, Tx)), \end{aligned}$$

for all $x, y \in X$ with $x \neq y$. If for some x_0, the sequence $(T^n x_0)$ has a limit point $z \in X$, then z is the unique fixed point of T.

3.22. Kalaivani and Kalpana(2018). Kalaivani and Kalpana [19] established the fixed point theorems for self-map with contractive conditions in a C^*-algebra-valued S-metric space.

Their main result is the following.

Theorem 3.58. *[19] Let $(X, \mathbb{A}, S)$ be a complete C^*-algebra-valued S-metric space. Suppose that the mapping $T : X \longrightarrow X$ satisfies $S(Tx, Tx, Ty) \preceq A^* S(x, x, y)A$, where $A \in \mathbb{A}'_+$ with $\|A\| < 1$, for all $x, y \in X$. Then there exists a unique fixed point in X.*

Theorem 3.59. *[19] Let $(X, \mathbb{A}, S)$ be a complete C^*-algebra-valued S-metric space. Suppose that the mapping $T : X \longrightarrow X$ satisfies*

$$S(Tx, Tx, Ty) \preceq A(S(Tx, Tx, y) + S(Ty, Ty, x))$$

where $A \in \mathbb{A}'_+$ with $\|A\| < \frac{1}{2}$, for all $x, y \in X$. Then there exists a unique fixed point in X.

3.23. Roy and Saha(2018). Roy and Saha [50] established the fixed points of generalized contractive mappings and n-times reasonable expansive mappings over a C^*-algebra-valued metric space.

Definition 3.38. [50] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space and T be a self map on X. Then T is said to be orbitally continuous at $u \in X$ if for any $x \in X$ $\|d(T^{n_i} x, u)\| \to 0$ as $i \to \infty$ implies $\|d(T^{n_i+1} x, Tu)\| \to 0$ as $i \to \infty$.

Lemma 3.60. *[50] Every lower semi-continuous function ϕ from a complete C^*-algebra-valued metric space X into $\mathbb{A}_+$ has a d-point p in X, that is we get*

$$\phi(p) - \phi(x) \preceq d(p,x) \quad \text{and} \quad \phi(p) - \phi(x) \neq d(p,x)$$

for each point $x(\neq p) \in X$.

Definition 3.39. [50] A mapping $T : X \to X$ is called C^*-algebra-valued n-times reasonable expansive mapping of metric-2 type if there exists a fixed element(constant) $h \succeq I, h \neq I$ such that

$$d^2(T^{n-1}x, T^{n-1}y) \succeq h \inf\{d^2(x,y), d(x,y).d(x,Tx), d(x,Tx).d(T^{n-2}y, T^{n-1}y),$$
$$d^2(x,Tx), d(T^{n-2}y, T^{n-1}y).d(x, T^{n-1}y), d(T^{n-2}y, T^{n-1}y).d(T^{n-2}y, T^{n-1}x)\}$$

for all $x, y \in X(n \geq 2, n \in \mathbb{N})$.

The main result of Roy and Saha [50] is the following.

Theorem 3.61. *[50] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ be a C^*-algebra-valued Hardy-Rogers type mapping that is for any $x, y \in X$*

$$d(Tx,Ty) \preceq a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(y,Tx)] + cd(x,y)$$

where $a, b, c \in \mathbb{A}'_+$ and $2\|a\| + 2\|b\| + \|c\| < 1$. If $X = \overline{O(x_0)}$ for some $x_0 \in X$, where $O(x_0) = \{x_0, T(x_0), T^2(x_0), ...\}$ and $\phi : X \to \mathbb{A}_+$ is defined by $\phi(x) = [I - (2a + 2b + c)]^{-1}[I - (a+b)]d(x,Tx)$, $\forall x \in X$ then by defining the relation $\ll \phi \in X$ we get T has a fixed point in X which is also unique.

Corollary 3.62.

(a) If in Theorem 3.61, $a = b = \theta$ then T will reduce to a Banach contraction mapping and for this case ϕ is given by $\phi(x) = (I - c)^{-1}d(x,Tx)$, for all $x \in X$ consequently T will have a unique fixed point in X
(b) If in Theorem 3.61 we put $b = c = \theta$ then T will reduce to a Kannan type mapping and for this case ϕ is given by $\phi(x) = (I - 2a)^{-1}(I - a)d(x,Tx)$, for all $x \in X$ and hence T has a unique fixed point in X.
(c) If in Theorem 3.61 we put $a = c = \theta$ then T will reduce to a Chatterjea type mapping and for this case ϕ is given by $\phi(x) = (I - 2b)^{-1}(I - b)d(x,Tx)$, for all $x \in X$. So T must have a unique fixed point in X.

Theorem 3.63. *[50] Let $(X, \mathbb{A}, d)$ be a complete commutative C^*-algebra-valued metric space. If $T : X \to X$ is a continuous and surjective C^*-algebra-valued n-times reasonable expansive mapping which satisfies*

$$d(T^n x, T^n y) \succeq h \inf\{d(x,y), d(y, T^n y)\} \quad \forall x, y \in X(n \geq 2, n \in \mathbb{N})$$

and the fixed element (constant) $h \succeq I$, $h \neq I$ with $(h - I)$ is invertible, then T has a fixed point in X.

3.24. Senapati and Dey (2018). Senapati and Dey [52] studied the work of Xin et al. [46] and noticed that the common fixed point results of this article do not produce any new result in literature. In fact the main results of this article coincide with some consequences of previous published results.

Theorem 3.64. *[52] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. Suppose that two mappings $T, S : X \longrightarrow X$ satisfy $d(Tx, S) \preceq A^* d(x, y)A$ for any $x, y \in X$ and $A \in \mathbb{A}$ with $\|A\| < 1$. Then $Tx = Sx$ for all $x \in X$.*

Remark 3.40. [52] From the above theorem, Senapati and Dey [52] observed that Theorem 3.6 of [46] does not give anything new and it coincides with Theorem 2.1 of the work of Ma et al. [25]. Also Kadelburg and Radenovic [18] proved that fixed point results in this space are the direct consequences of metric fixed point results.

3.25. Kalpana and Tasneem (2019). They Introduced the concept of C^*-algebra-valued rectangular b-metric spaces as a generalization of C^*-algebra-valued b-metric spaces. They proved the analogue of Banach contraction principle and Kannan's fixed point theorem in such space.

Definition 3.41. [20] Let X be a nonempty set and $A \in \mathbb{A}'$ such that $b \succeq I$. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}$ satisfies:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ iff $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq b[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y, u, v \in X$ and for all distinct points $u, v \in X - \{x, y\}$.

Then d is called a C^*-algebra-valued rectangular b-metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued rectangular b-metric space.

The main result of Kalpana and Tasneem [20] is the following.

Theorem 3.65. *[20] If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued rectangular b-metric spaces and $T : X \to X$ is a contractive mapping, then there exists a unique fixed point in X.*

Theorem 3.66. *[20] Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued rectangular b-metric space. Suppose the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \preceq A(d(Tx, x) + d(Ty, y))(\forall x, y \in X)$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

3.26. Chandok, Kumar and Park (2019). Chandok et al. [8] initiated the notion of C^*-algebra-valued partial metric space which is more general than partial metric space. Some fixed point results using C-class functions on such spaces were obtained.

Definition 3.42. [32] Suppose that $\mathbb{A}$ is a unital C^*-algebra, then a continuous function $F : \mathbb{A}_+ \times \mathbb{A}_+ \longrightarrow \mathbb{A}$ is called a C_*-class function if for any $A, B \in \mathbb{A}_+$, the following conditions hold

(1) $F(A, B) \preceq A$,
(2) $F(A, B) = A$ implies that either $A = \theta$ or $B = \theta$.

The letter C_* denotes the class of all C_*-class functions.

Let Ψ be the set of all continuous functions $\Psi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfying the following conditions:

(i) Ψ is continuous and nondecreasing,
(ii) $\Psi(A) = \theta \Leftrightarrow A = \theta$.

Definition 3.43. [8] Let X be a nonempty set. A function $p : X \times X \longrightarrow \mathbb{A}$ is called a C^*-algebra-valued partial metric on X if the following conditions are satisfied:

(p1) $\theta \preceq p(x, y)$ for all $x, y \in X$ and $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;
(p2) $p(x, x) \preceq p(x, y)$ for all $x, y \in X$;
(p3) $p(x, y) = p(y, x)$ for all $x, y \in X$;
(p4) $p(x, y) \preceq p(x, z) + p(z, y) + p(z, z)$ for all $x, y \in X$.

Then the pair $(X, \mathbb{A}, p)$ is called a C^*-algebra-valued partial metric space.

Example 3.44. [8] Let $X = [0, 1]$ and $x \in \mathbb{A}$ be a nonzero element. Define $p(s, t) = \max\{1 + s, 1 + t\}xx^*$. Then $p : X \times X \longrightarrow \mathbb{A}$ is a C^*-algebra-valued partial metric. But $p : X \times X \longrightarrow \mathbb{A}$ is not a C^*-algebra-valued metric, since $p(s, s) = (1 + s)xx^* \neq \theta$

The main result of Chandok et al. [8] is the following.

Theorem 3.67. *[8] Let $(X, \mathbb{A}, p)$ be a C^*-algebra-valued partial metric space and $T : X \to X$ be a self mapping satisfying*

$$\psi(p(Tx, Ty)) \preceq F_*(\psi(p(x, y)), \phi(p(x, y))) \quad \forall \quad x, y \in X,$$

where $\psi, \phi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

Corollary 3.68. *[8] Let $(X, \mathbb{A}, p)$ be a C^*-algebra-valued partial metric space and $T : X \to X$ be a self mapping satisfying*

$$\psi(p(Tx, Ty)) \preceq \psi(p(x, y)) - \phi(p(x, y)) \quad for \quad all \quad x, y \in X,$$

where $\psi, \phi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

3.27. Ozer and Omran (2019). Ozer and Omran [44] established a new coupled fixed point theorem for C^*-algebra-valued b-metric spaces. Their main result is as follows.

Theorem 3.69. *[44] Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric spaces. Suppose the mapping $T : X \times X \to X$ satisfies*

$$d(T(x, y), T(u, v)) \leq b[Ad(x, u)A^* + Bd(y, v)B^*],$$

for $x, y, u, v \in X$ and $A, B \in \mathbb{A}$, such that $\|A\| \leq \frac{1}{2}$ and $\|B\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

Corollary 3.70. *[44] Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric spaces. Suppose the mapping $T : X \times X \to X$ satisfies*

$$d(T(x, y), T(u, v)) \leq A[d(x, u) + d(y, v)]A^*,$$

for $x, y, u, v \in X$ and $A \in \mathbb{A}$ where $\|A\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

3.28. Moeini et al. (2019). Moeini et al. [32] initiated the concept of C^*-algebra-valued M-metric spaces generalizing the M-metric spaces. Some fixed point theorems are also established via C_*-class functions in such spaces.

Definition 3.45. [32] Let X be a non empty set. A function $m : X \times X \longrightarrow \mathbb{A}$ is called a C^*-algebra-valued M-metric if the following conditions are satisfied:

(cm1) $\theta \preceq m(x, y)$ for all $x, y \in X$ and $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$,
(cm2) $m(x, x)$ and $m(y, y)$ be comparable for all $x, y \in X$,
(cm3) $m_{xy} \preceq m(x, y)$ for all $x, y \in X$, where $m_{xy} = \min\{m(x, x), m(y, y)\}$,
(cm4) $m(x, y) = m(y, x)$ for all $x, y \in X$,
(cm5) $(m(x, y) - m_{xy}) \preceq (m(x, z) - m_{xz}) + (m(z, y) - m_{zy})$ for all $x, y, z \in X$.

Then the pair $(X, \mathbb{A}, m)$ is called a C^*-algebra-valued M-metric space.

Remark 3.46. [32] Let $(X, \mathbb{A}, m)$ is called a C^*-algebra-valued M-metric space. Define M_{xy} by $M_{xy} = \max\{m(x, x), m(y, y)\}$, for every $x, y, z \in X$, we have

(i) $\theta \preceq M_{xy} + m_{xy} = m(x, x) + m(y, y)$;
(ii) $\theta \preceq M_{xy} - m_{xy} = (m(x, x) - m(y, y)) \vee (m(y, y) - m(x, x))$;
(iii) $M_{xy} - m_{xy} \preceq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

Remark 3.47. According to the definition of a C^*-algebra-valued partial metric and C^*-algebra-valued M-metric,

(i) if $p(x, x) = \min\{p(x, x), p(y, y)\}$ then, $(p2)$ is replaced by $\min\{p(x, x), p(y, y)\} \leq p(x, y)$, that is, condition $(cm3)$.
(ii) If $p(x, x) = \min\{p(x, x), p(y, y)\}$ and $p(z, z) = \min\{p(z, z), p(y, y)\}$, we improve condition $(p4)$ by subtracting $p(x, x)$ on both sides to the form of $(cm5)$.

Thus, every C^*-algebra-valued partial metric is C^*-algebra-valued m-metric, but the converse is not true as in the following examples.

Example 3.48. Let $X = [0, \infty)$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$. Define

$$m(x, y) = \begin{bmatrix} \frac{x+y}{2} & 0 \\ 0 & \frac{x+y}{2} \end{bmatrix},$$

it is easy to see that $m(x, y)$ is a C^*-algebra-valued m-metric, but not C^*-algebra-valued partial metric since, $m(3, 3) > m(1, 3)$; that is, condition $(p2)$ fails.

From Example 3.7, Remark 3.80 and Example 3.8, we obtain the following relation:

C^*-algebra-valued metric $\Longrightarrow$ C^*-algebra-valued partial metric $\Longrightarrow$ C^*-algebra-valued m-metric

The main result of Moeini et al. [32] is the following.

Theorem 3.71. *[32] Let $(X, \mathbb{A}, m)$ be a C^*-algebra-valued M-metric space and $T : X \longrightarrow X$ be a self-mapping satisfying*

$$\psi(m(Tx, Ty)) \preceq F_*(\psi(m(x, y)), \phi(m(x, y))) \quad for \quad all \quad x, y \in X,$$

where $\psi, \phi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

Corollary 3.72. *[32] Let $(X, \mathbb{A}, m)$ be a complete C^*-algebra-valued M-metric space and $T : X \longrightarrow X$ be a self-mapping satisfying*

$$\psi(m(Tx, Ty)) \preceq \psi(m(x, y)) - \phi(m(x, y)) \quad for \quad all \quad x, y \in X,$$

where $\psi, \phi \in \Psi$. Then T has a unique fixed point.

3.29. Prasad et al. (2019). Prasad et al. [45] established some results on coincidence point and common fixed point theorems for a hybrid pair of single valued and multivalued mappings in complete C^*-algebra-valued fuzzy soft metric spaces.

Definition 3.49. [45] Let $C \subseteq E$ and $\overline{E}$ be the absolute fuzzy soft set that is $F_E(e) = \overline{1}$ for all $e \in E$. Let $\mathbb{A}$ denote the C^*-algebra. The C^*-algebra-valued fuzzy soft metric using fuzzy soft points is defined as a mapping $\overline{d} : \overline{E} \times \overline{E} \longrightarrow \mathbb{A}$ satisfying the following conditions.

(M0) $\theta \preceq \overline{d}(F_{e_1}, F_{e_2})$ for all $F_{e_1}, F_{e_2} \in \overline{E}$.
(M1) $\overline{d}(F_{e_1}, F_{e_2}) = \theta \Leftrightarrow F_{e_1} = F_{e_2}$
(M2) $\overline{d}(F_{e_1}, F_{e_2}) = \overline{d}(F_{e_2}, F_{e_1})$
(M3) $\overline{d}(F_{e_1}, F_{e_3}) \preceq \overline{d}(F_{e_1}, F_{e_2}) + (F_{e_2}, F_{e_3})\ \forall F_{e_1}, F_{e_2}, F_{e_3} \in \overline{E}$.

The fuzzy soft set $\overline{E}$ with the C^*-algebra-valued fuzzy soft metric $\overline{d}$ is called the C^*-algebra-valued fuzzy soft metric space. It is denoted by $(\overline{E}, \mathbb{A}, \overline{d})$.

Definition 3.50. [45] Let $(\overline{E}, \mathbb{A}, \overline{d})$ be a C^*-algebra-valued fuzzy soft metric space. Suppose $CB(\overline{E})$ be a class of all nonempty closed and bounded subsets of $\overline{E}$. For a points $F_{e_1}, F_{e_2} \in \overline{E}$ and $\overline{X}, \overline{Y} \in CB(\overline{E})$, define $\overline{D}_{\mathbb{A}}(F_{e_1}, \overline{Y}) = \inf_{G_{e_1} \in \overline{Y}} \overline{d}(F_{e_1}, G_{e_1})$. Let $H_{\mathbb{A}}$ be the Hausdorff C^*-algebra-valued fuzzy soft metric induced by the C^*-algebra-valued fuzzy soft metric $\overline{d}$ on $\overline{E}$ that is

$$H_{\mathbb{A}}(\overline{X}, \overline{Y}) = \max\left\{\sup \overline{D}_{\mathbb{A}}(F_{e_1}, \overline{Y}), \sup \overline{D}_{\mathbb{A}}(\overline{X}, G_{e_1})\right\}, \text{ for all } \overline{X}, \overline{Y} \in CB(\overline{E})$$

Theorem 3.73. *[45] Let $(\overline{E}, \mathbb{A}, \overline{d})$ be a complete C^*-algebra-valued fuzzy soft metric space, and $T\overline{E} \longrightarrow CB(\overline{E})$ be a multivalued map satisfying*

$$H_{\mathbb{A}}(TF_{e_1}, TF_{e_2}) \preceq a^*\overline{d}(F_{e_1}, F_{e_2})a$$

for all $F_{e_1}, F_{e_2} \in \overline{E}$, where $a \in \mathbb{A}$ with $\|a\| < 1$. Then T has a unique fixed point in $\overline{E}$.

Prasad et al. [45] also proved the couple fixed point result as follows.

Theorem 3.74. *[45] Let $(\overline{E}, \mathbb{A}, \overline{d})$ be a C^*-algebra-valued fuzzy soft metric space. Suppose $S, T : \overline{E} \times \overline{E} \longrightarrow \overline{E}$ and $f, g : \overline{E} \longrightarrow \overline{E}$ be mappings satisfying:*

1. *$S(\overline{E} \times \overline{E}) \subseteq g(\overline{E})$ and $T(\overline{E} \times \overline{E}) \subseteq f(\overline{E})$;*
2. *$\{S, f\}$ and $\{T, g\}$ are ω-compatible pairs;*
3. *one of $f(\overline{E})$ or $g(\overline{E})$ is complete C^*-algebra-valued fuzzy soft metric of $\overline{E}$;*
4. *$\overline{d}(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2}) \preceq a^*\overline{d}(fF_{e_1}, gF_{e_2})a + a^*\overline{d}(fG_{e_1}, gG_{e_2})a$, for all F_{e_1}, F_{e_2}, G_{e_1}, $G_{e_2} \in \overline{E}$,*

where $a \in \mathbb{A}$ with $\|\sqrt{2a}\| < 1$. Then S, T, f and g have a unique common coupled fixed point in $\overline{E} \times \overline{E}$.

3.30. Omran and Ozer (2020). Omran and Ozer [40] proved the coupled fixed point theorem in C^*-algebra-valued metric spaces which get values in non-comutative operators. They showed the existence and uniqueness of coupled fixed point in such space.

Their main result are as follows.

Theorem 3.75. *[40] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. The mapping $T : X \times X \longrightarrow X$ satisfies*

$$d(T(x, y), T(u, v)) \leq Ad(x, u)A^* + Bd(y, v)B^*$$

for $x, y, u, v \in X$ and $A, B \in \mathbb{A}'_+$, such that $\|A\| \leq \frac{1}{2}$ and $\|B\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

Corollary 3.76. *[40] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that the mapping $T : X \times X \longrightarrow X$ satisfies*

$$d(T(x, y), T(u, v)) \leq C[d(x, u) + d(y, v)]C^*$$

for $x, y, u, v \in X$ and $C \in \mathbb{A}'_+$, such that $\|C\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

3.31. Moeini, Isik and Aydi (2020). They initiated the concept of C^*-algebra-valued G_b-metric spaces and proved the fixed point theorems for Banach and Kannan types via C_*-class functions.

Definition 3.51. [33] Let $\mathbb{A}$ be a unital C^*-algebra and X be a nonempty set. Let $b \in \mathbb{A}$ be such that $\|b\| \geq 1$. A mapping $G : X \times X \times X \longrightarrow \mathbb{A}_+$ is said to be a C^*-algebra-valued G_b-metric on X if

Gb1) $G(x, y, z) = \theta$ if $x = y = z$;
Gb2) $\theta \prec G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
Gb3) $G(x, x, y) \preceq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
Gb4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z;
Gb5) $G(x, y, z) \preceq b(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$.

The triplet $(X, \mathbb{A}, G)$ is called a C^*-algebra-valued G_b-metric space.

The main result of Moeini et al. [33] is the following.

Theorem 3.77. *[33] Let $(X, \mathbb{A}, G)$ be a complete C^*-algebra-valued G_b-metric space with $b = (c.1_{\mathbb{A}}) > 1_{\mathbb{A}}$ and $T : X \longrightarrow X$ be such that*

$$\sigma((c^{\varepsilon}.1_{\mathbb{A}})G(Tx, Ty, Tz)) \preceq F_* \left(\sigma(G(x, y, z)), \vartheta(G(x, y, z))\right))$$

for all $x, y, z \in X$, where $F_* \in C_*$, $\sigma, \vartheta \in \Sigma$ and $\varepsilon \in (1, \infty)$. Then T possesses a unique fixed point.

Theorem 3.78. *[33] Let $(X, \mathbb{A}, G)$ be a complete C^*-algebra-valued G_b-metric space. Let $T : X \longrightarrow X$ verifies for all $x, y \in X$,*

$$\sigma(G(Tx, Ty, Ty)) \preceq F_*(\sigma(m(x, y)), \vartheta(m(x, y))),$$

where $F_* \in C_*$, $x, \vartheta \in \Sigma$, and

$$m(x, y) = c \left(G(x, Tx, Tx) + G(y, Ty, Ty)\right),$$

where $c \in \mathbb{A}'_+$ and $\|c\| < \frac{1}{2}$. Then T possesses a unique fixed point.

3.32. Mlaiki, Asim and Imdad(2020). Mlaiki et al. [30] proved the fixed point results by enlarging the class of C^*-algebra-valued partial metric spaces as well as the class of C^*-algebra-valued b-metric spaces by introducing the class of C^*-algebra-valued partial b-metric space.

Definition 3.52. [30] Let $X \neq \emptyset$ and $b \in \mathbb{A}$ such that $b < I$. A mapping $d : X \times X \longrightarrow \mathbb{A}$ is called a C^*-algebra-valued partial b-metric on X, if it satisfies the following for all $x, y, z \in X$:

(i) $d(x, y) \succeq \theta$ and $x = y \Leftrightarrow d(x, x) = d(y, y) = d(x, y)$;
(ii) $d(x, x) \preceq d(y, x)$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, y) \preceq b[d(x, z) + d(z, y)] - d(z, z)$.

The triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued Partial b-metric space.

Note that, every C^*-algebra-valued b-metric space is a C^*-algebra-valued partial b-metric space with zero self distance and every C^*-algebra-valued partial metric space is a C^*-algebra-valued partial b-metric space with $b = I$, but the converse is not true.

Example 3.53. [30] Let $X = [0, 1)$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{C})$, the class of bounded and linear operators on a Hilbert space $\mathbb{C}^2$. Define $d : X \times X \longrightarrow \mathbb{A}$ by

$$d(x, y) = \begin{bmatrix} |x-y|^p & 0 \\ 0 & \alpha|x-y|^p \end{bmatrix} + \begin{bmatrix} \max\{x, y\}^p & 0 \\ 0 & \alpha \max\{x, y\}^p \end{bmatrix}$$

for all $x, y \in X$, $\alpha \geq 0$ and $p \geq 1$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued partial b-metric space with coefficient $b = 2^{p-1}I$. However, it is easy to see that $(X, \mathbb{A}, d)$ is neither a C^*-algebra-valued b-metric space nor a C^*-algebra-valued partial metric space.

Definition 3.54. [30] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued partial b-metric space. A mapping $T : X \longrightarrow X$ is said to be C^*_b-contraction if there exists $A \in \mathbb{A}$ with $\|bA\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y) A, \quad \forall \quad x, y \in X.$$

The main result of Mlaiki et al. [30] is the following.

Theorem 3.79. *[30] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued partial b-metric space and $T : X \longrightarrow X$ be a C^*_b-contraction. Then T has a unique fixed point $x \in X$ such that $d(x, x) = \theta$.*

Corollary 3.80. *[30] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued partial b-metric space and $T : X \longrightarrow X$ be a C^*_b-contraction. Afterwards, T has a unique fixed point $x \in X$.*

Corollary 3.81. *[30] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued partial b-metric space and $T : X \longrightarrow X$ be a C^*_b-contraction. Afterwards, T has a unique fixed point $x \in X$, such that $d(x, x) = \theta$.*

3.33. Asim and Imdad(2020). Asim and Imdad [2] introduced the notion of C^*-algebra-valued extended b-metric spaces and used it to prove an analogue of Banach Contraction Principle.

Definition 3.55. [2] Let $X \neq \emptyset$ and $E : X \times X \longrightarrow \mathbb{A}'$. The mapping $d_E : X \times X \longrightarrow \mathbb{A}$ is called a C^*-algebra-valued extended b-metric on X, if it satisfies the following (for all $x, y, z \in X$):

(1) $d_E(x, y) \succeq \theta$ and $d_E(x, y) = \theta$ iff $x = y$;
(2) $d_E(x, y) = d_E(x, y)$;

(3) $d_E(x,y) \preceq E(x,y)[d_E(x,z)+d_E(z,y)]$.

The triplet $(X, \mathbb{A}, d_E)$ is called a C^*-algebra-valued extended b-metric space.

Remark 3.56. [2] Observe that, if $E(x,y) = b \geq I$, then $(X, \mathbb{A}, d_E)$ reduces to a C^*-algebra-valued b-metric space.

C^*-algebra-valued metric space $\Longrightarrow$ C^*-algebra-valued b-metric space $\Longrightarrow$ C^*-algebra-valued extended b-metric space

Their main result is as follows.

Theorem 3.82. *[2] Let $(X, \mathbb{A}, d_E)$ be a complete C^*-algebra-valued extended b-metric space and $T : X \longrightarrow X$ satisfies the following:*

$$d_E(Tx, Ty) \preceq A^* d_E(x,y) A, \quad \forall x, y \in X$$

where $A \in \mathbb{A}$ with $\|A\| < 1$ and $\lim\limits_{n,m \to \infty} E(x_n, x_m)\|A\| \prec I$. Then T has a unique fixed point $x \in X$.

Corollary 3.83. *[2] Theorem 3.13 of Ma et al. [25] is immediate from theorem 3.98*

3.34. Asim and Imdad (2020) II. Asim and Imdad II [3] introduced the class of C^*-algebra-valued symmetric spaces and proved some fixed point results in such space.

Definition 3.57. [3] Suppose X is a non-empty set. The mapping $d : X \times X \longrightarrow \mathbb{A}$ is called a C^*-algebra-valued symmetric on X, if it satisfies the following for all $x, y \in X$:

(i) $d(x,y) \succeq \theta$ and $d(x,y) = \theta$ iff $x = y$;
(ii) $d(x,y) = d(y,x)$.

The triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued symmetric space.

Definition 3.58. [3] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued symmetric space. A mapping $T : X \longrightarrow X$ is said to be Kannan-Ciri'c type C^*-contraction if there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that (for all $x, y \in X$)

$$d(Tx, Ty) \preceq A^* \max\{d(x, Tx), d(y, Ty)\} A$$

The main result of Asim and Imdad II [3] is the following.

Theorem 3.84. *[3] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued symmetric space and $T : X \longrightarrow X$. Suppose that T is a Kannan-Ciri'c type C^*-contraction and T is continuous. Then T has a unique fixed point $x \in X$.*

3.35. Kari, Rossafi and Massit (2021). They discussed the existence and uniqueness of fixed points for a self-mapping defined on a C^*-algebra-valued rectangular quasi-metric space.

Definition 3.59. [22] Let X be a non empty set. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}_+$ satisfies:

(i) $d(x, y) = \theta$ if and only if $x = y$; and $\theta \preceq d(x, y)$ for all $x, y \in X$;
(ii) $d(x, y) \preceq d(x, u) + d(u, v) + d(v, y)$ for all $x, u, v, y \in X$ and for all distinct points $u, v \in X \times X$.

Then $(X, \mathbb{A}_+, d)$ is called a C^*-algebra-valued rectangular quasi-metric space.

Remark 3.60. [22] The C^*-algebra-valued rectangular quasi-metric space generalise the C^*-algebra-valued metric space and C^*-algebra-valued rectangular metric space.

The main result of Kari et al. [22] is the following.

Theorem 3.85. *[22] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued rectangular quasi-metric space and let $T : X \longrightarrow X$ is a C^*-algebra-valued contractive mapping on X, then there exists a unique fixed point in X.*

Theorem 3.86. *[22] Let $(X\mathbb{A}, d)$ be a complete C^*-algebra-valued rectangular quasi-metric space and let $T : X \longrightarrow X$ be a C^*-algebra-valued Kannan-type mapping on X. Then there exists a unique fixed point in X.*

3.36. Massit and Rossafi (2021). Massit and Rossafi [29] extended the notion of (ϕ, F)-contraction to C^*-algebra-valued metric spaces and established the existence and uniqueness of fixed point.

Definition 3.61. [29] Suppose that P and Q are C^*-algebras. A mapping $\phi : P \longrightarrow Q$ is said to be a $*$-homomorphism if:

(i) $\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in P$,
(ii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in P$,
(iii) $\phi(x^*) = \phi(x)^*$ for all $x \in P$,
(iv) ϕ maps the unit in P to the unit in Q.

Definition 3.62. [59] Let (X, d) be a complete metric space. A mapping $T : X \longrightarrow X$ is called an (ϕ, F)-contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\phi \in \Phi$ such that

$$(d(Tx, Ty) > 0 \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \leq F(d(x, y))$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Definition 3.63. [29] Let $F : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ be a function satisfying:

(i) F is continuous and nondecreasing.

(ii) $F(t) = \theta$ if and only if $t = \theta$.

1. A mapping $T : X \longrightarrow X$ is said to be a (ϕ, F) C^*-algebra-valued contraction of type(I) if there exists $\phi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ a $*$ -homomorphism such that

$$\forall x, y \in X \quad (d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(d(x, y))$$

2. A mapping $T : X \longrightarrow X$ is said to be a (ϕ, F) C^*-algebra-valued contraction of type(II) if there exists $\phi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ a $*$- homomorphism satisfying:
   (a) $\phi(a) \prec a$ for $a \in \mathbb{A}_+$
   (b) Either $\phi(a) \preceq d(x, y)$ or $d(x, y) \preceq \phi(a)$, where $a \in \mathbb{A}_+$ and $x, y \in X$
   (c) $F(a) \prec \phi(a)$ such that

$$(d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(M(x, y))$$

   where $M(x, y) = a_1 d(x, y) + a_2[d(Tx, y) + d(Ty, x)] + a_3[d(Tx, x) + d(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$, $a_1 + 2a_2 + 2a_3 \leq 1$
3. T is said to be (ϕ, F)-Kannan-type C^*-algebra-valued contraction if there exists ϕ satisfying (a),(b) and (c) such that $d(Tx, Ty) \succeq \theta$ we have

$$F(d(Tx, Ty) + \phi(d(x, y)) \preceq F\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right).$$

4. T is said to be (ϕ, F)-Reich-type C^*-algebra-valued contraction if there exists ϕ satisfy (a),(b) and (c) such that $(d(Tx, Ty) \succeq \theta$, we have

$$F(d(Tx, Ty) + \phi(d(x, y)) \preceq F\left(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}\right).$$

Theorem 3.87. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and let $T : X \longrightarrow X$ be a (ϕ, F)-contraction mapping of type(I). Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is convergent to x^*.*

Theorem 3.88. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Let $T : X \longrightarrow X$ be a (ϕ, F)-contraction mapping of type (II), Then, T has a fixed point.*

Theorem 3.89. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Let $T : X \longrightarrow X$ be a (ϕ, F)-Kannan-type C^*-algebra-valued contraction. Then T has a unique fixed point.*

Theorem 3.90. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Let $T : X \longrightarrow X$ be a (ϕ, F)- Reich-type C^*-algebra-valued contraction. Then T has a unique fixed point.*

3.37. Kumar et al. (2021). Kumar et al. [24] proved some fixed point results using C_*-class function within the context of C^*-algebra-valued metric space.

The main result of Kumar et al. [24] is the following.

Theorem 3.91. *[24] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ be a self mapping satisfying the following*

$$\varphi(d(Tx, Ty)) \preceq F_*(\varphi(d(x, y)), \phi(d(x, y))), \quad \forall x, y \in X,$$

where $\varphi, \phi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

Corollary 3.92. *[24] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ be a self mapping satisfying the following*

$$\varphi(d(Tx, Ty)) \preceq \varphi(d(x, y)) - \phi(d(x, y)), \quad \forall x, y \in X,$$

where $\varphi, \phi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

3.38. Omran and Masmali (2021). Omran and Masmali [39] introduced a version of α-admissible on C^*-algebra-valued b-metric space and proved some Banach and common fixed point theorems using α-admissible.

Definition 3.64. [39] Let X be a non-empty set and $\alpha_{\mathbb{A}} : X \times X \longrightarrow \mathbb{A}'_+$ be a function, we say that the self map T is $\alpha_{\mathbb{A}}$-admissible if $(x, y) \in X \times X$, $\alpha_{\mathbb{A}}(x, y) \succeq I \Rightarrow \alpha_{\mathbb{A}}(Tx, Ty) \succeq I$, where I is the unity of $\mathbb{A}$.

Definition 3.65. [39] let $(T, S) : X \longrightarrow X$ be a continuous self mappings on X. $\alpha_{\mathbb{A}} : X \times X \longrightarrow \mathbb{A}_+$. (T, S) are said to be common $\alpha_{\mathbb{A}}$-admissible if for any $x_0 \in X$,

$$\alpha_{\mathbb{A}}(x_0, y) \succeq I \Rightarrow \alpha_{\mathbb{A}}(Tx_0, Sy) \succeq I \Rightarrow \alpha_{\mathbb{A}}(T^2x_0, S^2y) \succeq I$$

The main result of Omran and Masmali [39] is the following.

Theorem 3.93. *[39] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra- valued b-metric space, with $b \succeq I_{\mathbb{A}}$, $b \in \mathbb{A}'$, $\|b\|\|A\|^2 < 1$. Suppose that $T : X \longrightarrow X$ be a generalised Lipschitz contraction satisfying the following conditions:*

(i) *T is $\alpha_{\mathbb{A}}$-admissible;*
(ii) *There exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I$;*
(iii) *T is continuous.*

Then T has a fixed point.

Theorem 3.94. *[39] Let $(X, \mathbb{A}, d)$ be complete C^*-algebra- valued b-metric space and $T, S : X \longrightarrow X$ such that*

$$\alpha_{\mathbb{A}}(x, y)d(Tx, Sy) \preceq A^*d(x, y)A,$$

and $\|A\| < 1$, $\|b\|\|A\|^2 < 1$ and the following conditions are satisfies:

(i) *(T, S) are common $\alpha_{\mathbb{A}}$-admissible.*
(ii) *there exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, y) \succeq I \Rightarrow \alpha_{\mathbb{A}}(Tx_0, Sy) \succeq I$.*
(iii) *T and S are continuous and have a common fixed point in X.*

3.39. Mustafa, Omran and Nguyen (2021). Mustafa et al. [38] established fixed point theorems using ψ-contractive mapping in C^*-algebra-valued b-metric space.

Their main result is the following.

Theorem 3.95. *[38] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Let $T : X \longrightarrow X$ satisfies the following condition:*

$$d(Tx, Ty) \preceq A^* d(x, y)A - \psi(d(x, y))$$

where ψ is $*$-homomorphism, $\lim\limits_{d\to\infty} \psi(d) = \infty$ and $\|b\|\|A\|^2 < 1$. Then, T has a fixed point.

Theorem 3.96. *[38] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Let $T : X \longrightarrow X$ be a contractive mapping function:*

$$\psi(d(Tx, Ty)) \preceq \phi(d(x, y)),$$

where ψ is $*$-homomorphism and $\phi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ is a continuous function with the constraint $\psi(d) < \phi(d)$. Then, T has a fixed point.

Theorem 3.97. *[38] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Let $T : X \longrightarrow X$ be a contractive mapping function and $\psi(d(Tx, Ty)) \preceq \psi(M(x, y)) - \phi(d(x, y))$, and*

$$M(x, y) = a_1 d(x, y) + a_2[d(Tx, y) + d(Ty, x)] + a_3[d(Tx, x) + d(Ty, y)]$$

where $b \in \mathbb{A}_+$, $a_1, a_2, a_3 \geq 0$, $a_1 + 2a_2 b + 2a_3 \leq 1$, ψ and ϕ are $*$-homomorphisms and with the constraint $\psi(d) < \phi(d)$. Then, T has a fixed point.

3.40. Tomar and Joshi (2021). With the help of examples, Tomar and Joshi [58] pointed out that the C^*-algebra-valued metric space is more general and results in this space are proper generalizations of the corresponding results in the literature in standard metric spaces.
They pointed out that functions have different nature in different spaces and the results in C^*-algebra-valued metric space can not be reduced to their metric counterparts unless C^*-algebra $\mathbb{A}$ is the set of real number $\mathbb{R}$.

3.41. Rossafi, Massit and Kabaj (2022). Rossafi et al. [29] extended the notion of (ψ, MF)-contraction to the frame work of C^*-algebra-valued metric spaces and established the existence and uniqueness of fixed point.

Definition 3.66. [29] Let $F : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ be a function satisfying:

(i) F is continuous and nondecreasing.
(ii) $F(t) = \theta$ if and only if $t = \theta$.

A mapping $T : X \longrightarrow X$ is said to be a (ψ, MF) C^*-algebra-valued contraction if there exists $\phi : \mathbb{A}_+ \longrightarrow \mathbb{A}$ an $*$-homomorphism such that

$$\forall x, y \in X; M(Tx, Ty) \succeq \theta \Rightarrow F(M(Tx, Ty)) + \phi(M(x, y)) \preceq F(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Theorem 3.98. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and let $T : X \longrightarrow X$ be a (ψ, MF)-C^*-algebra-valued contraction mapping. Then T has a unique fixed point.*

Theorem 3.99. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and let $T : X \longrightarrow X$ be a (ψ, MF) C^*-algebra-valued contraction of Hardy Rogers type where $M(x, y) = \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)$ and $\alpha_i \geq 0$, $i \in \{1, 2, 3, 4, 5\}$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$. Then T has a unique fixed point in X.*

As a corollary, Rossafi et al. [29] showed that theorem 3.98 can be proved using (ψ, MF)-Chaterjea type, (ψ, MF)-Kannan-type and (ψ, MF)-Riech-type C^*-algebra-valued contraction.

3.42. Rashwan, Omran and Fangary (2022). Rashwan et al. [47] obtained the Kannan and Chatterjee type fixed point theorems and their extension for a self mappings in a complete C^*-algebra-valued b-metric space by using positive functions on C^*-algebras.

Their main result is the following.

Theorem 3.100. *[47] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition*

$$d(Tx, Ty) \preceq \psi(d(Tx, x) + d(Ty, y)),$$

where $\psi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfy the condition $\|\psi\| < \frac{1}{2}$. Then T has a unique fixed point.

Theorem 3.101. *[47] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition*

$$d(Tx, Ty) \preceq \psi\left(\frac{d(x, y)}{2} + \frac{d(Tx, x) + d(Ty, y)}{2}\right),$$

where $\psi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ satisfy the condition $\|\psi\| < \frac{1}{4}$. Then T has a unique fixed point.

Theorem 3.102. *[47] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition*

$$d(Tx, Ty) \preceq \psi(d(x, y) + d(Tx, y) + d(Ty, x)),$$

where $\psi : \mathbb{A}_+ \longrightarrow \mathbb{A}_+$ *satisfy the conditions* $\psi(b) \in \mathbb{A}'_+$ *for all* $b \in \mathbb{A}'_+$ *and* $\|b\psi(b)\psi\| < \frac{1}{2}$. *Then* T *has a unique fixed point.*

3.43. Mohanta and Biswas(2022). Mohanta and Biswas [35] obtained some coincidence point and common fixed point results in C^*-algebra-valued partial metric space. Their main result is as follows.

Theorem 3.103. *[35] Let* $(X, \mathbb{A}, p)$ *be a* 0*-complete* C^**-algebra-valued partial metric space and* $T : X \longrightarrow X$ *be a mapping. If there exists an* $A \in \mathbb{A}$ *with* $\|A\| < 1$ *such that* $p(Tx, Ty) \preceq A^*p(x, y)A$ *for all* $x, y \in X$*, then* T *has a unique fixed point* x *(say) in* X *and* $p(x, x) = \theta$.

Theorem 3.104. *[35] Let* $(X, \mathbb{A}, p)$ *be a* 0*-complete* C^**-algebra-valued partial metric space and the mapping* $T : X \longrightarrow X$ *be such that*

$$p(Tx, Ty) \preceq A[p(Tx, x) + p(Ty, y)]$$

for all $x, y \in X$*, where* $A \in \mathbb{A}'_+$ *with* $\|A\| < \frac{1}{2}$. *Then,* T *has a unique fixed point* x *(say) in* X *and* $p(x, x) = \theta$.

Remark 3.67. [35] Theorem 3.116 is a generalization of Kannan's fixed point theorem in metric spaces to C^*-algebra-valued partial metric spaces.

Theorem 3.105. *[35] Let* $(X, \mathbb{A}, p)$ *be a* C^**-algebra-valued partial metric space and the mappings* $T, S : X \longrightarrow X$ *satisfy the following condition*

$$p(Tx, Ty) \preceq A^*p(Sx, Sy)A$$

for all $x, y \in X$*, where* $A \in \mathbb{A}$ *with* $\|A\| < 1$. *If* $T(X) \subseteq S(X)$ *and* $S(X)$ *is a* 0*-complete subspace of* X*, then* T *and* S *have a unique point of coincidence* x *(say) in* $S(X)$ *with* $p(x, x) = \theta$. *Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point in* $S(X)$.

Theorem 3.106. *[35] Let* $(X, \mathbb{A}, p)$ *be a* C^**-algebra-valued partial metric space and the mappings* $T, S : X \longrightarrow X$ *satisfy the following condition*

$$p(Tx, Ty) \preceq A[p(Tx, Sy) + p(Ty, Sx)]$$

for all $x, y \in X$*, where* $A \in \mathbb{A}'_+$ *and* $\|A\| < \frac{1}{2}$. *If* $T(X) \subseteq S(X)$ *and* $S(X)$ *is a* 0*-complete subspace of* X*, then* T *and* S *have a unique point of coincidence* x *(say) in* $S(X)$ *with* $p(x, x) = \theta$. *Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point in* $S(X)$.

3.44. Rossafi, Kari and Massit (2022). Rossafi et al. [49] extended a version of $\alpha-\psi$-contraction to C^*-algebra-valued rectangular b-metric spaces and established the existence and uniqueness of fixed point.

Definition 3.68. [49] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b- rectangular metric space and $T : X \longrightarrow X$ is mapping, we say that T is an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : X \times X \longrightarrow \mathbb{A}_+$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \preceq \psi(d(x, y)), \quad \forall x, y \in X$$

The main result of Rossafi et al. [49] is the following.

Theorem 3.107. *[49] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b- rectangular metric space and $T : X \longrightarrow X$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*
(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \succeq I$,*
(iii) *for all $x, y \in X$,there exists $z \in X$ such that $\alpha(x, z) \succeq I$ and $\alpha(y, z) \succeq I$,*
(iv) *T is continuous.*

Then, T has a unique fixed point in X.

Theorem 3.108. *[49]Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b-rectangular metric space and $T : X \longrightarrow X$ be an $\alpha - \psi$-contractive mapping of Kannan type i.e,*

$$\alpha(x, y)d(Tx, Ty) \preceq \psi(d(Tx, x) + d(Ty, y))$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\alpha : X \times X \longrightarrow \mathbb{A}_+$ and the following conditions hold:

(i) *T is α-admissible,*
(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \succeq I$,*
(iii) *T is continuous.*

Then, T has a unique fixed point in X.

3.45. Malhotra, Kumar and Park (2022). They introduced the notion of C^*-algebra-valued $\mathcal{R}$-metric space and C^*-algebra-valued $\mathcal{R}$-contractive map along with some fixed point results.

Definition 3.69. [27] For a non-empty set X together with a unital C^*-algebra $\mathbb{A}$ and binary relation $\mathcal{R}$, define $d : X \times X \longrightarrow \mathbb{A}$. Then $(X, \mathbb{A}, d, \mathcal{R})$ is called C^*-algebra-valued $\mathcal{R}$-metric space if following are satisfied:

(1) $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space;
(2) $\mathcal{R}$ is a binary relation on X.

Definition 3.70. [27] For a C^*-algebra-valued $\mathcal{R}$-metric space $(X, \mathbb{A}, d, \mathcal{R})$, a self map $T : X \longrightarrow X$ is said to be C^*-algebra-valued $\mathcal{R}$-contractive map if for all $x, y \in X$ with $(x, y) \in \mathcal{R}$, there exists an $A \in \mathbb{A}$ where $\|A\| < 1$ such that $d(Tx, Ty) \preceq A^* d(x, y)A$.

The main result of Malhotra et al. [27] is the following.

Theorem 3.109. *[27] Let $(X, \mathbb{A}, d, \mathcal{R})$ be a C^*-algebra-valued $\mathcal{R}$-metric space and let Y be a complete C^*-algebra-valued $\mathcal{R}$-subspace of X. If $T : X \longrightarrow X$ is a self map on X such that:*

i) $T(X) \subseteq Y$;
ii) *T is $\mathcal{R}$-preserving;*
iii) *There exists some $x_0 \in X$ such that $(x_0, y) \in \mathcal{R}$ for all $y \in T(X)$;*
iv) *T is C^*-algebra-valued $\mathcal{R}$-contractive map;*
v) *Either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self closed on Y.*

Then T possesses a unique fixed point.

Theorem 3.110. *[27] Let $(X, \mathbb{A}, d, \mathcal{R})$ be a C^*-algebra-valued $\mathcal{R}$-metric space and let Y be a complete C^*-algebra-valued $\mathcal{R}$-subspace of X. If $T : X \longrightarrow X$ is a self map on X such that:*

i) $T(X) \subseteq Y$;
ii) *T is $\mathcal{R}$-preserving;*
iii) *There exists some $x_0 \in X$ such that $(x_0, y) \in \mathcal{R}$ for all $y \in T(X)$;*
iv *For all $x, y \in X$ with $(x, y) \in \mathcal{R}$, there exists an $A \in \mathbb{A}'_+$, where $\|A\| < \frac{1}{2}$ such that*
$$d(Tx, Ty) \preceq A(d(Tx, x) + d(Ty, y));$$
v) *Either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self closed on Y.*

Then T possesses a unique fixed point.

3.46. Mani et al. (2022). Mani et al. [28] initiated the concept of C^*-algebra-valued bipolar metric space and proved coupled fixed point theorems in such space.

Definition 3.71. [28] Let $\mathbb{A}$ be a C^*-algebra, and X, Y be two non-void sets. A mapping $\rho : X \times Y \longrightarrow \mathbb{A}_+$ be a function such that

(a) $\rho(x, y) = 0$ iff $x = y$, for all $(x, y) \in X \times Y$.
(b) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X \cap Y$.
(c) $\rho(x, y) \leq \rho(x, z) + \rho(x_1, z) + \rho(x_1, y)$, for all $x, x_1 \in X$ and $z, y \in Y$.

The 4-tuple $(X, Y, \mathbb{A}, \rho)$ is called a C^*-algebra-valued bipolar metric space.

The main result of Mani et al. [28] is the following.

Theorem 3.111. *[28] . Let $(X, Y, \mathbb{A}, \rho)$ be a complete C^*-algebra-valued bipolar metric space. Suppose $T : (X^2, Y^2, \mathbb{A}, \rho) \rightrightarrows (X, Y, \mathbb{A}, \rho)$ is a covariant mapping such that*
$$\rho(T(x, y), T(u, v)) \preceq A^*\rho(x, u)A + A^*\rho(y, v)A \quad \forall x, y \in X, u, v \in Y,$$
where $A \in \mathbb{A}$ with $2\|A\|^2 < 1$. Then the function $T : X^2 \cup Y^2 \longrightarrow X \cup Y$ has a unique coupled fixed point.

Theorem 3.112. *[28] Let $(X, Y, \mathbb{A}, \rho)$ be a complete C^*-algebra-valued bipolar metric space. Suppose $T : (X \times Y, Y \times X, \mathbb{A}, \rho) \rightrightarrows (X, Y, \mathbb{A}, \rho)$ is a covariant mapping such that*

$$\rho(T(x,u), T(v,y)) \preceq A^*\rho(x,v)A + A^*\rho(y,u)A \quad \forall x, y \in X, u, v \in Y$$

where $A \in \mathbb{A}$ with $2\|A\|^2 < 1$. Then the function $T : (X \times Y) \cup (Y \times X) \longrightarrow X \cup Y$ has a unique coupled fixed point.

3.47. Bouftouh, Kabbaj, Abdeljawad and Mukheimer (2022). They initiated the notion of C^*-algebra-valued b-asymmetric metric spaces and established several fixed point theorems.

Definition 3.72. [7] Let X be a nonempty set and $b \in \mathbb{A}_+$ where $b \succeq 1_{\mathbb{A}}$. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}$ satisfies:

(i) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta \Leftrightarrow x = y$;
(ii) $d(x,y) \preceq b[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued b-asymmetric metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued b-asymmetric metric space.

Their main result is as follows.

Theorem 3.113. *[7] If $(X, \mathbb{A}, d)$ is a b-complete C^*-algebra-valued b-asymmetric metric space and $T : X \longrightarrow X$ is a C^*-algebra-valued contractive mapping on X, then T admit a unique fixed point in X.*

3.48. Maheswari et al. (2022). Maheswari et al. [26] proved some common coupled fixed point theorems on complete C^*-algebra-valued partial b-metric spaces. Their main result is the following.

Theorem 3.114. *[26] Let $(X, \mathbb{A}, \rho)$ be a complete C^*-algebra-valued partial b-metric space with coefficient s. Suppose that the mappings $T : X \times X \longrightarrow X$ and $S : X \longrightarrow X$ satisfy the following condition:*

$$\rho(T(x,y), T(u,v)) \preceq A^*\rho(Sx, Su)A + A^*\rho(Sy, Sv)A \quad for \quad any \quad x, y, u, v \in X,$$

where $A \in \mathbb{A}$ with $\|A\| < \frac{1}{\sqrt{2}}$ and $\|b\|\|\sqrt{2A}\|^2 < 1$. If $T(X \times X) \subseteq S(X)$ and $S(X)$ is complete in X, then T and S have a coupled coincidence point and $\rho(Sx, Sx) = \theta$ and $\rho(Sy, Sy) = \theta$. Moreover, if T and S are ω-compatible, then they have a unique common coupled fixed point in X.

Theorem 3.115. *[26] Let $(X, \mathbb{A}, \rho)$ be a complete C^*-algebra-valued partial b-metric space with coefficient s. Suppose that the mappings $T : X \times X \longrightarrow X$ and $S : X \longrightarrow X$ satisfy the following condition:*

$$\rho(T(x,y), T(u,v)) \preceq a\rho(T(x,y), Su) + b\rho(T(u,v), Sx)$$

for any $x, y, u, v \in X$, where $a, b \in \mathbb{A}'_+$ with $\|a\| + \|b\| < 1$ and $\|sa\| + \|sb\| < 1$. If $T(X \times X) \subseteq S(X)$ and $S(X)$ is complete in X, then T and S have a coupled coincidence point and $\rho(Sx, Sx) = \theta$ and $\rho(Sy, Sy) = \theta$. Moreover, if T and S are ω-compatible, then they have a unique common coupled fixed point in X.

3.49. Shagari et al. (2022). Shagari et al. [53] introduced the concepts of C^*-algebra-valued F-contractions and C^*-algebra-valued F-Suzuki contractions and proved the existence of fixed points for such mappings.

Definition 3.73. [53] Let $F : \mathbb{A}_+ \longrightarrow \mathbb{A}$ be a mapping satisfying the following assumptions:

(A1) F is $\preceq$-increasing;
(A2) for every sequence $\{A_n\}_{n\in\mathbb{N}} \in \mathbb{A}_+$,

$$\lim_{n\to\infty} A_n = \theta \text{ if and only if } \lim_{n\to\infty} \|F(A_n)\| = \infty;$$

(A3) there exists $k \in (0, 1)$ such that $\lim_{A\to\theta} A^k F(A) = \theta$.

Denote the family of mappings obeying (A1)-(A3) by $\mathbb{A}_F$.

Definition 3.74. [53] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $T : X \longrightarrow X$ is named a C^*-algebra-valued F-contraction if there exists an $A \in \mathbb{A}_+$ with $\|A\| < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \succ \theta \text{ implies } A + F(d(Tx, Ty)) \preceq F(A^* d(x, y)A), \text{ where } F \in \mathbb{A}_F.$$

Definition 3.75. [53] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $T : X \longrightarrow X$ is named a C^*-algebra-valued F-Suzuki contraction, if there exists $A \in \mathbb{A}_+$ with $\|A\| < \frac{1}{2}$ such that for all $x, y \in X$, $x \neq y$,

$$Ad(x, Tx) \preceq d(x, y) \text{ implies } A + F(d(Tx, Ty)) \preceq F(A^* d(x, y)A), \text{ where } F \in \mathbb{A}_F.$$

The main result of Shagari et al. [53] is the following.

Theorem 3.116. *[53] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ a C^*-algebra-valued F-contraction. Then, T has a unique fixed point in X.*

Theorem 3.117. *[53] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ a C^*-algebra-valued F-Suzuki contraction. Then T has a unique fixed point in X.*

Remark 3.76. [53] In Theorems 3.134 and 3.135, the completeness of X is necessary and if $\|A\| = 1$ (or $\|A\| = \frac{1}{2}$), the mapping T may not have a fixed point.

Definition 3.77. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $T : X \longrightarrow X$ is called a C^*-algebra-valued F-Contraction of Hardy Rogers type if there exists $A \in \mathbb{A}_+$ with $\|A\| < 1$ and $\alpha, \beta, \gamma \in \mathbb{A}'_+$ with $\|\alpha\| + 2\|\beta\| + 2\|\gamma\| \leq 1$ such that for all $x \in X$, $d(Tx, Ty) \succ \theta \Rightarrow$

$$A + F(d(Tx, Ty)) \preceq F\left(\alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)]\right). \tag{4}$$

Remark 3.78.

(i) If $\beta = \gamma = \theta$, Definition 3.77 reduces to Definition 3.74 of [53].
(ii) If $\alpha = \gamma = \theta$, Definition 3.77 reduces C^*-algebra-valued F-contraction of Kannan type.
(iii) If $\alpha = \beta = \theta$, Definition 3.77 reduces C^*-algebra-valued F-contraction of Chatterjea type.
(iv) If $\gamma = \theta$, Definition 3.77 reduces C^*-algebra-valued F-contraction of Reich type.

Theorem 3.118. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that $T : X \longrightarrow X$ is continuous and T is C^*-algebra-valued F-Contraction of Hardy Rogers type. Then there exists $x \in X$ such that $Tx = x$. Moreover, if either*

(i) $\|\alpha\| + 2\|\gamma\| < 1$, *or*
(ii) $\|\beta\| + \|\gamma\| < 1$ *and F is continuous, then the fixed point is unique.*

Proof. Choose $x_0 \in X$ and define a sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots x_{n+1} = Tx_n = T^{n+1}x_0$. If $\mathbb{A} = \theta$, then for all $n \in \mathbb{N}$ $d(x, y) = \theta$ and the result is trivial. Let $d_n = d(x_n, x_{n+1})$ $\forall n \in \mathbb{N} \cup \{0\}$.

Suppose $\mathbb{A} \neq \theta$ and that $\theta \prec d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n)$, $n \in \mathbb{N}$. Using the contractive condition 4 with $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned}
& A + F(d(Tx_{n-1}, Tx_n)) \\
\preceq\; & F\left(\alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + \gamma[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n])\right) \\
=\; & F\left(\alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \gamma[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\right) \\
\preceq\; & F\left(\alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right) \\
=\; & F\left((\alpha + \beta + \gamma) d(x_{n-1}, x_n) + (\beta + \gamma) d(x_{n+1}, x_n)\right).
\end{aligned}$$

By the monotonicity of F, we have

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) \preceq (\alpha + \beta + \gamma) d(x_{n-1}, x_n) + (\beta + \gamma) d(x_{n+1}, x_n)$$

$$(I - \beta - \gamma) d(x_n, x_{n+1}) \preceq (\alpha + \beta + \gamma) d(x_{n-1}, x_n).$$

Since $\|\alpha\| + 2\|\beta\| + 2\|\gamma\| = 1$, then $\frac{\|\alpha\| + 2\|\beta\| + 2\|\gamma\|}{\|I\| - \|\beta\| - \|\gamma\|} \leq 1$. So,

$$d(x_n, x_{n+1}) \preceq \frac{\alpha + \beta + \gamma}{I - \beta - \gamma} d(x_{n-1}, x_n) = d(x_{n-1}, x_n).$$

Now, $A + F(d_n) \preceq F(d_{n-1})$, for all $n \in \mathbb{N}$, which implies

$$\begin{aligned} F(d_n) &\preceq F(d_{n-1}) - A \\ &\preceq F(d_{n-2}) - 2A \preceq ... \\ &\preceq F(d_0) - nA \end{aligned}$$

$$F(d_n) \preceq F(d_0) - nA, \quad \forall n \in \mathbb{N} \tag{5}$$

Taking the norm on both sides, we obtain, $\|F(d_n)\| \longrightarrow \infty$ as $n \longrightarrow \infty$. By $A2$,

$$\lim_{n\to\infty} d_n = \theta, \tag{6}$$

and from $A3$

$$\lim_{n\to\infty} \|d_n^k F(d_n)\| = 0 \quad for \quad k \in (0,1). \tag{7}$$

By (5), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} d_n^k F(d_n) - d_n^k F(d_0) &\preceq d_n^k[F(d_0) - nA] - d_n^k F(d_0) \\ &= -n d_n^k A \preceq \theta. \end{aligned}$$

Taking the limit on both sides and using (6) and (7) in the above inequality, we obtain

$$\lim_{n\to\infty} \|d_n^k F(d_n)\| = 0.$$

Hence, there exists an $n_0 \in \mathbb{N}$ with $\theta \prec n d_n^k A \preceq I$, for all $n \geq n_0$. Then,

$$\|d_n\| \leq \frac{1}{\|A\| n^{\frac{1}{k}}}, \quad \forall n \geq n_0.$$

So, for $n, p \in \mathbb{N}$ with $p > 1$, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &= d_n + d_{n+1} + ... + d_{n+p-1} \\ &= \sum_{i=n}^{n+p-1} d_i \preceq \left\| \sum_{i=n}^{\infty} d_i \right\| I \\ &\preceq \sum_{i=n}^{\infty} \|d_i\| I = \sum_{i=n}^{\infty} \left\| \frac{A^{-1}}{i^{\frac{1}{k}}} \right\| I \\ &\preceq \|A\|^{-1} \sum_{i=n}^{\infty} \left\| \frac{1}{i^{\frac{1}{k}}} \right\| I \\ &\preceq \sum_{i=n}^{\infty} \left\| \frac{1}{i^{\frac{1}{k}}} \right\| I. \end{aligned} \tag{8}$$

Clearly, the series in (8) is convergent, i.e. $\|d_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$ and by completeness of the space, there exists

$x \in X$ with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x$. Since T is continuous with respect to $\mathbb{A}$, then

$$\begin{aligned} d(x,Tx) &= \|d(x_n, \lim_{n\to\infty} Tx_n)\| \\ &= \| \lim_{n\to\infty} d(x_n, x_{n+1})\| \\ &= \lim_{n\to\infty} \|d(x_n, x_{n+1})\| \\ &= \|d(x,x)\| = 0. \end{aligned}$$

This implies $d(x,Tx) = \theta \Leftrightarrow Tx = x$. To show uniqueness, we assume $y(\neq x)$ is another fixed point of T. Then from (4),

$$\begin{aligned} A + F(d(x,y)) &= A + F(d(Tx,Ty)) \\ &\preceq F(\alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)]) \\ &= F(\alpha d(x,y) + \beta[d(x,x) + d(y,y)] + \gamma[d(x,y) + d(y,x)]) \\ &= F\left((\alpha + 2\gamma)d(x,y)\right). \end{aligned}$$

Since $\|\alpha\| + 2\|\gamma\| < 1$ and by $A1$,

$$\begin{aligned} \|d(x,y)\| &\leq \|(\alpha + 2\gamma)d(x,y)\| \\ &< \|d(x,y)\|, \end{aligned}$$

a contraction. Hence $x = y$. Similarly, suppose F is continuous and assume contrary that the fixed point is not unique, then

$$\begin{aligned} A + F(d(x_{n+1},x)) =& A + F(d(Tx_n,Tx)) \\ \preceq& F(\alpha d(x_n,x) + \beta[d(x_n,Tx_n) + d(x,Tx)] + \gamma[d(x_n,Tx) + d(x,Tx_n)]) \\ =& F(\alpha d(x_n,x) + \beta[d(x_n,x_{n+1}) + d(x,Tx)] + \gamma[d(x_n,Tx) + d(x_n,x_{n+1})]). \end{aligned}$$

Letting $n \longrightarrow \infty$ and given that $\|\alpha\| + \|\gamma\| < 1$, we obtain

$$\begin{aligned} A + F(d(x,Tx) &\preceq F((\beta + \gamma)d(x,Tx)) \\ &\preceq F(d(x,Tx), \end{aligned}$$

a contradiction. Hence the fixed point is unique. □

3.50. Shagari et al. (2022). The main contribution of the work of Shagari et al. [54] is introducing the concept of C^*-algebra-valued simulation functions and associated new classes of contractive mappings. The fixed point results in such space were obtained.

Definition 3.79. [54] Let $\mathbb{A}$ be a unital C^*-algebra and $f : \mathbb{A}_+^2 \longrightarrow \mathbb{A}$ be a mapping. Then, f is called a C^*-algebra-valued simulation function if it satisfies the following conditions:

(f1) f is increasing with respect to its second argument; that is, for every choice of $t \in \mathbb{A}_+$ such that for all $s, r \in \mathbb{A}_+$ with $s \preceq r$, we have $f(t,s) \preceq f(t,r)$;
(f2) $f(t,s) \preceq s - t$, for all $t, s \succeq \theta$;
(f3) If $\{t_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ are sequences in $\mathbb{A}_-$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n \succ \theta, \text{ then}$$
$$\limsup_{n\to\infty} f(t_n, s_n) \prec \theta,$$

where $\mathbb{A}_- = \{x \in \mathbb{A} : x \succ \theta\}$. We denote the family of functions satisfying $(f1) - (f3)$ by $\mathbb{Z}_{\mathbb{A}}$.

Definition 3.80. [54] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. We say that $T : X \longrightarrow X$ is a C^*-algebra-valued $\mathbb{Z}$-contractive mapping with respect to $f \in \mathbb{Z}_{\mathbb{A}}$, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that for all $x, y \in X$,

$$f(d(Tx, Ty), A^* d(x, y)A) \succeq \theta. \tag{9}$$

The main result of the work of Shagari et al. [54] is the following.

Theorem 3.119. *[54] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that the mapping $T : X \longrightarrow X$ satisfies the following conditions:*

$$f(d(Tx, Ty), A(d(Tx, y) + d(Ty, x))) \succeq \theta,$$

for all $x, y \in X$, where $f \in \mathbb{Z}_{\mathbb{A}}$, $A \in \mathbb{A}_+$ and $\|A\| < 1$. Then T has a unique fixed point in X.

Theorem 3.120. *[54] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \longrightarrow X$ is a C^*-algebra-valued $\mathbb{Z}$-contractive mapping on X with respect to $f \in \mathbb{Z}_{\mathbb{A}}$. Then T has a unique fixed point in X.*

Remark 3.81. [54] As a consequence, Shagari et al. [54] proved that Theorem 2.1 of [25] can be obtained using Theorem 3.120.

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References

[1] J. M. Afra, *Fixed point type theorem in s-metric spaces*, Middle-East J. Sci. Res., **22**(6) (2014), 864–869.

[2] M. Asim and M. Imdad, *C^*-algebra valued extended b-metric spaces and fixed point results with an application*, U.P.B. Sci. Bull. **82**(1) (2020), 207–218.

[3] M. Asim and M. Imdad, *C^*-algebra valued symmetric spaces and fixed point results with an application.* Korean J. Math., **28**(1) (2020), 17–30.

[4] C. Bai, *Coupled fixed point theorems in C^*-algebra-valued b-metric spaces with application*, Fixed Point Theory Appl., **2016** (2016), no. 1, 70.

[5] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math., **3** (1922), 133–181.

[6] S. Batul and T. Kamran, *C^*-valued contractive type mappings*, Fixed Point Theory Appl., **2015** (2015), no. 1, 142.

[7] O. Bouftouh, S. Kabbaj, T. Abdeljawad, and A. Mukheimer, *On fixed point theorems in C^*-algebra valued b-asymmetric metric spaces*, AIMS Math., **7**(7) (2022), 11851–11861.

[8] S. Chandok, D. Kumar, and C. Park, *C^*-algebra-valued partial metric space and fixed point theorems*, Proc. Math. Sci., **129**(3) (2019), 37.

[9] L. B. Ciric, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273.

[10] S. Czerwick, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostraviensis, **1**(1) (1993), 5–11.

[11] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Spring. Berlin, 1998.

[12] N. V. Dung and V. T. Hang, *On relaxations of contraction constants and Caristi's theorem in b-metric spaces*, J. Fixed Point Theory Appl., **18**(2) (2016), 267–284.

[13] M. E. Ege and C. Alaca, *C^*-algebra-valued s-metric space*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Statis., **67** (2018), 165–177.

[14] N. Gholamian and M. Khanehgir, *On the fixed point theorem for C^*-algebra-valued 2-metric spaces*, Ann. Iran. Math. Conf., 2015.

[15] N. Gholamian, M. Khanehgir, and R. Allahyari, *Some fixed point theorems for C^*-algebra-valued b_2-metric spaces*, J. Math. Ext., **11**(2) (2017), 53–69.

[16] T. L. Hicks and B. E. Rhoades, *A Banach type fixed point theorem*, J. Math. Jpn., **24** (1979), 327–333.

[17] Z. Kadelburg, A. Nastasi, S. Radenovic, and P. Vetro, *Fixed points of contractions and cyclic contractions on C^*-algebra-valued b-metric spaces*, Adv. Oper. Theory, **1**(1) (2016), 92–103.

[18] Z. Kadelburg and S. Radenovic. *Fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts*, J. Fixed Point Theory Appl., **2016**(1) (2016), 53.

[19] C. Kalaivani and G. Kalpana, *Fixed point theorems in C^*-algebra-valued s-metric spaces with some applications*, U.P.B. Sci. Bull. Ser. A, **80** (2018), 93–102.

[20] G. Kalpana and Z. S. Tasneem, *C^* -algebra-valued rectangular b-metric spaces and some fixed point theorems*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**(2) (2019), 2198–2208.

[21] T. Kamran, M. Postolache, A. Ghiura, S. Batul, and R. Ali, *The Banach contraction principle in C^*-algebra-valued b-metric spaces with application*, Fixed Point Theory Appl., **2016**(1) (2016), 10.

[22] A. Kari, M. Rossafi, and H. Massati, *Some fixed point theorems on C^*-algebra valued rectangular quasi-metric spaces*, J. Math. Comput. Sci., **11**(6) 2021, 7459–7475.

[23] C. Klin-eam and P. Kaskasem, *Fixed point theorems for cyclic contractions in C^*-algebra-valued b-metric spaces*, J. Funct. Spaces, **2016**(1) (2016), 7827040.

[24] D. Kumar, D. Rishi, C. Park, and J. R. Lee, *On fixed point in C^*-algebra valued metric spaces using C_*-class function*, Int. J. Nonlinear Anal. Appl., **12**(2) (2021), 1157–1161.

[25] Z. Ma and L. Jiang, *C^*-algebra-valued b-metric spaces and related fixed point theorems*, J. Fixed Point Theory Appl., **2015**(1) (2015), 222.

[26] J. U. Maheswari, A. Anbarasan, M. Gunaseelan, V. Parvaneh, and S. H. Bonab, *Solving an integral equation via C^*-algebra-valued partial b-metrics*, J. Fixed Point Theory Algorithms Sci. Eng., **2022**(1) (2022), 18.

[27] A. Malhotra, D. Kumar, and C. Park, *C^*-algebra valued r-metric space and fixed point theorems*, AIMS Math., 7(4) (2022), 6550–6564.

[28] G. Mani, A. J. Gnanaprakasam, A. U. Haq, I. A. Baloch, and F. Jarad, *Coupled fixed point theorems on C^*-algebra valued bipolar metric spaces*, AIMS Math., **7**(5) (2022), 7552–7568.

[29] H. Massit and M. Rossafi, *Fixed point theorem for (ϕ, F)-contraction on C^*-algebra valued metric spaces*, Eur. J. Math. Appl., **2021**(1) (2021), 14.

[30] N. Mlaiki, M. Asim, and M. Imdad, *C^*-algebra valued partial b-metric spaces and fixed point results with an application*, Mathematics, **8**(8) (2020), 1381.

[31] B. Moeini and A. H. Ansari, *Common fixed point theorems in C^*-algebra-valued b-metric spaces endowed with a graph and applications*, arXiv preprint arXiv:1707.09906 (2017).

[32] B. Moeini, M. Asadi, H. Aydi, H. Alsamir, and M. S. Noorani, *C^*-algebra-valued m- metric space and some related fixed point results*, Ital. J. Pure Appl. Math., **41** (2019), 708–723.

[33] B. Moeini, H. Isik, and H. Aydi, *Related fixed point result via C_*-class functions on C^*-algebra valued g_b-metric spaces*, Carpath. Math. Publ., **12**(1) (2020), 94–106.

[34] B. Moeini, P. Kumar, and H. Aydi, *Zamfirescu type contraction on C^*-algebra-valued metric spaces*, J. Math. Anal., **9** (2018), 150–161.

[35] S. K. Mohanta and P. Biswas, *C^*-algebra valued partial metric space and some fixed point and coincidence point results*, Int. J. Nonlinear Analy. Appl., **13**(2) (2022), 1535–1551.

[36] S. Mondal, A. Chanda, and S. Karmakar, *Common fixed point and best proximity point theorems in C^*-algebra-valued metric space*, Int. J. Pure Applied Math., **115**(3) (2017), 477–496.

[37] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, London, 1990.

[38] R. Mustafa, S. Omran, and Q. N. Nguyen, *Fixed point theory using ψ contractive mapping in C^*-algebra valued b-metric space*, Mathematics, **9**(1) (2021), 92.

[39] S. Omran and I. Masmali, *α-admissible mapping in C^*-algebra-valued b-metric spaces and fixed point theorems*, AIMS Math., **9**(6) (2021):10192–10206.

[40] S. Omran and O. Ozer, *Determination of some result for couple fixed point theory in C^*-algebra valued metric spaces*, J. Indones. Math. Soc., **26**(2) (2020), 258–265.

[41] S. Omran and M. M. Salama, *Common coupled fixed point in C^*-algebras valued metric spaces*, Int. J. Applied Eng. Res., **13**(8) (2018), 5899–5903.

[42] O. Ozer and S. Omran, *On the generalized C^*-valued metric spaces related with Banach fixed point theory*, Int. J. Adv. Appl. Sci., **4**(2) (2017):35–37.

[43] O. Ozer and S. Omran, *Common fixed point in C^*-algebra b-valued metric space*, AIP Conf. Proc., Vol. 1773(75), 2016, pp. 1–6.

[44] O. Ozer and S. Omran, *A result on the coupled fixed point theorems in C^*-algebra valued b-metric spaces*, Italian J. Pure Appl. Math., 42 (2019),722–730.

[45] D. R. Prasad, G. N. Venkata, H. Isik, B. S. Rao, and G. A. Lakshmi, *C^*-algebra-valued fuzzy soft metric spaces and results for hybrid pair of mappings*, Axioms, **8**(3) (2019), 99.

[46] X. Qiaoling, J. Lining, and M. Zhenhua, *Common fixed point theorems in C^*-algebra-valued metric space*, J. Nonlinear Sci. Appl., **9**(6) (2016).

[47] R.A. Rashwan, S. Omran, and A. Fangary, *Kanan and Chatterjee's type fixed point theorems using Ψ-postive function in C^*-algebra valued b-metric spaces*, J. Math. Comput. Sci., **12** (2022), Article ID 63.

[48] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226** (1977), 257–290.

[49] M. Rossafi, A. Kari, and H. Massit, *On the α-ψ-contractive mappings in C^*-algebra valued b-rectangular metric spaces and fixed point theorems*, Eur. J. Math. Anal., **2**(1) (2022), 11.

[50] K. Roy and M. Saha, *Fixed point theorems for generalized contractive and expansive type mappings over a C^*-algebra valued metric space*, Sci. Stud. Res. Ser. Math. Inf., **28**(1) (2018), 115–130.

[51] S. Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag Berlin Heidelberg, New York, 1971.

[52] T. Senapati and L. K. Dey, *Remarks on common fixed point results in C^*-algebra-valued metric spaces*, J. Inf. Math. Sci., **10**(1-2) (2018), 333–337.

[53] M. S. Shagari, T. Alotaibi, O. S. Mohamed, A. O. Mustafa, and A. A. Bakery, *On existence results of Volterra-type integral equations via C^*-algebra-valued f-contractions*. AIMS Math., **8**(1) (2022), 1154–1171.

[54] M. S. Shagari, A. T. Imam, U. A. Danbaba, J. Yahaya, M. O. Oni, and A. A. Tijjani, *Existence of fixed points via C^*-algebra valued simulation functions with applications*, J. Anal., **31**(2) (2023), 1201–1221.

[55] T. L. Shateri, *C^*-algebra-valued modular spaces and fixed point theorems*, J. Fixed Point Theory Appl., **19**(2) (2017), 1551–1560.

[56] D. Shehwar, S. Batul, T. Kamran, and A. Ghiurad, *Caristi's fixed point theorem on C^*-algebra valued metric spaces*, J. Nonlinear Sci. Appl., **9**(2) (2016), 584–588.

[57] D. Shehwar and T. Kamran, *C^*-valued g-contractions and fixed point*. J. Ineq. Appl., **2015**(1) (2015), 304.

[58] A. Tomar and M. Joshi, *Note on results in C^*-algebra valued metric space*, Electronic J. Math. Anal. Appl., **9**(2) (2021), 262–264.

[59] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, J. Fixed Point Theory Appl., **2012**(1) (2012), 94.

[60] A. Zada, S. Saifullahi, and Z. Ma, *Common fixed point theorems for g-contraction in C^*-algebra-valued metric space*, Int. J. Anal. Appl., **11**(1) (2016), 23–27.

[61] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. (Basel), **298** (1972), 292–298.

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