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Norm-attainment in locally convex spaces: Weak-* topology, inductive limits, and reflexivity

Evans N. Mogoi and Priscah Moraa Ohuru*

Abstract. We characterize norm-attaining functionals in locally convex spaces (LCS), with particular focus on three fundamental aspects: the weak-* (weak-star) topology in dual spaces, inductive limits (including LF-spaces and DF-spaces), and reflexivity conditions. Our main results establish that (1) norm-attainment in the weak-* dual coincides precisely with the canonical embedding $X \hookrightarrow X^{**}$; (2) strict inductive limits (such as $\mathcal{D}(\mathbb{R})$) permit non-attaining functionals, whereas Montel spaces ensure universal attainment; and (3) both barrelledness and reflexivity conditions recover norm-attainment through weak-* continuity. This work extends classical Banach space techniques to general LCS settings, revealing the crucial interplay between compactness properties and approximation methods in determining norm-attainment behavior.

Keywords: Norm-attainment, Weak-* topology, Locally convex spaces, Inductive limits, Montel spaces, Reflexivity, Dual spaces

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1. Introduction

The study of norm-attaining functionals has been a central theme in functional analysis since the foundational work of [9] on reflexivity and weak compactness. While the classical theory in Banach spaces is well-developed [2, 12], the behavior of norm-attaining functionals in general locally convex spaces (LCS) presents rich and

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*Corresponding author

nuanced phenomena that warrant systematic investigation. This paper provides a comprehensive analysis of norm-attainment across different LCS topologies, unifying and extending classical Banach space results to broader settings. Norm-attainment in locally convex spaces (LCS) is fundamental in various applications. In Partial Differential Equations, solutions in the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ often do not attain their norm, which poses challenges to the use of variational methods. In Quantum Physics, rigged Hilbert spaces, which are constructed through inductive limits, benefit from properties that ensure observable measurements correspond to norm-attaining functionals. In Signal Processing, the stability and robustness of frame operators defined on Schwartz spaces depend critically on the attainment of norms. Our work builds upon several key strands of research. The duality between norm-attainment and weak-* topologies, first explored in [1] and [14], finds new depth in our characterization of when functionals in X^{**} attain their norm on $(X^*, \text{weak-}^*)$ (Theorem 1). The critical role of compactness, particularly in Montel spaces as studied in [10] and [15], emerges in our proof that all continuous functionals attain their norms in such spaces (Theorem 3). The pathology of norm-attainment in inductive limits, including LF-spaces like $\mathcal{D}(\mathbb{R})$, contrasts sharply with the Banach space case [3], while our DF-space results (Theorem 6) complement recent advances in operator theory [5, 8]. The connection between barrelledness and norm-attainment (Theorem 4) draws inspiration from [16] and [11], while our reflexivity results (Theorem 5) extend the classical theory of [13] and [4]. Recent work by [6] on Orlicz spaces and [7] on frame operators has demonstrated the continued vitality of norm-attainment research. Our paper contributes to this ongoing conversation by:

- Establishing complete characterizations of norm-attainment in weak-* topologies
- Revealing the structural reasons for non-attainment in inductive limits
- Identifying precise conditions (Montel property, reflexivity) that guarantee attainment
- Providing concrete examples in distribution spaces and duals of Frechet spaces

Our results collectively advance the understanding of how topological, geometric, and algebraic properties interact to determine norm-attainment behavior beyond Banach spaces.

Relation to Prior Work. Our work bridges classical Banach space results with the locally convex setting. While the Bishop-Phelps theorem guarantees density of norm-attaining functionals in Banach spaces, this fails for general locally convex spaces as demonstrated by the Dirac delta functional (Example 2). However, Theorem 4 reveals that barrelledness restores a density property analogous to Bishop-Phelps. The celebrated James' theorem [9] on reflexivity extends to our setting (Theorem 6), but with a crucial distinction: weak compactness replaces norm compactness in the characterization. Furthermore, while recent advances in operator ideals [5] established norm-attainment criteria for compact operators, our Theorem 5 demonstrates that their adjoints in DF-spaces exhibit fundamentally different behavior, with explicit examples of non-attainment.

Preliminaries

This section establishes notation, key definitions, and foundational results in locally convex spaces (LCS) essential for our analysis of norm-attainment. We assume familiarity with basic functional analysis as in [14, 2].

Locally Convex Spaces and Seminorms.

Definition 1.1 ([1]). A **locally convex space** (LCS) is a Hausdorff topological vector space X whose topology is generated by a family of continuous seminorms $\{p_\alpha\}_{\alpha \in I}$. The **dual space** X^* consists of all continuous linear functionals on X.

Remark 1.2. Unlike Banach spaces, LCS topologies need not be normable. Key examples include:

- Frechet spaces: Complete LCS with topology induced by a countable family of seminorms
- LF-spaces: Strict inductive limits of Frechet spaces (e.g., $\mathcal{D}(\mathbb{R}^n)$)
- Montel spaces: LCS where every closed bounded set is compact

Dual Space Topologies.

Definition 1.3 ([11, 1]). For an LCS X, the dual X^* carries several important topologies:

- **Weak-* topology** $\sigma(X^*, X)$: Coarsest topology making all evaluations $\hat{x} : f \mapsto f(x)$ continuous
- **Strong topology** $\beta(X^*, X)$: Generated by seminorms $p_B(f) = \sup_{x \in B} |f(x)|$ for bounded $B \subset X$
- **Mackey topology** $\tau(X^*, X)$: Uniform convergence on weakly compact convex sets

Norm-Attainment Concepts.

Definition 1.4 ([9, 13]). For an LCS X with continuous seminorm p, a functional $\phi \in X^*$ **attains its norm** (relative to p) if:

$$\exists x_0 \in X \text{ with } p(x_0) \leq 1 \text{ such that } |\phi(x_0)| = \sup_{p(x) \leq 1} |\phi(x)|$$

Key Classes of LCS.

Definition 1.5 ([10, 11])**.** An LCS X is:

- **Barrelled**: Every closed absorbing absolutely convex set is a neighborhood of 0
- **Reflexive**: The canonical embedding $\kappa : X \hookrightarrow X^{**}$ is a topological isomorphism
- **DF-space**: Strong dual of a Frechet space with fundamental sequence of bounded sets

Critical Theorems.

Theorem 1.1 (Hahn-Banach for LCS)**.** *Let X be an LCS, $Y \subset X$ a subspace, and $\psi : Y \to \mathbb{K}$ continuous. Then ψ extends to a continuous linear functional on X.*

Theorem 1.2 (Banach-Alaoglu in LCS)**.** *For any neighborhood U of 0 in X, the polar $U^\circ \subset X^*$ is weak-* compact.*

Theorem 1.3 (Mackey-Arens)**.** *A locally convex topology τ on X is compatible with duality (i.e., $(X,\tau)^* = X^*$) iff $\sigma(X, X^*) \subseteq \tau \subseteq \tau(X, X^*)$.*

Inductive and Projective Limits.

Definition 1.6. An **inductive limit** $X = \varinjlim X_n$ of LCS is strict if:

- Each $X_n \hookrightarrow X_{n+1}$ is an embedding of closed subspaces
- X carries the finest LCS topology making all inclusions $X_n \hookrightarrow X$ continuous

Proposition 1.4 (Boundedness in LF-spaces)**.** *In a strict LF-space $X = \varinjlim X_n$, a set B is bounded iff $B \subset X_n$ for some n and is bounded there.*

Notational Conventions. For clarity:

- $\mathscr{E}(\Omega)$ denotes smooth functions with compact convergence (Montel spaces),
- $\mathcal{D}(\Omega)$ is the space of test functions (LF-spaces),
- Duals are always endowed with the strong topology $\beta(X^*, X)$ unless specified.

Main Results and Discussions

Theorem 1.5 (Norm-Attainment in Weak-* Topology)**.** *Let X be a Banach space and X^* its dual equipped with the weak-* topology. A functional $\phi \in X^{**}$ attains its norm on $(X^*, \text{weak-}^*)$ if and only if $\phi \in \kappa(X)$, where $\kappa : X \hookrightarrow X^{**}$ is the canonical embedding.*

Proof. We prove both directions separately.

($\Rightarrow$) Suppose $\phi \in X^{**}$ attains its norm at some $f \in X^*$ with $\|f\| = 1$, meaning $|\phi(f)| = \|\phi\|$. By the Banach-Alaoglu theorem, the closed unit ball B_{X^*} is weak-* compact. Since ϕ is weak-* continuous (by definition of the weak-* topology on X^{**}), it attains its supremum on B_{X^*}. Now consider Goldstine's theorem, which states that $\kappa(X)$ is weak-* dense in X^{**}. If $\phi \notin \kappa(X)$, then by Hahn-Banach separation, there would exist a weak-* neighborhood where ϕ cannot attain its norm, contradicting our assumption. Thus $\phi \in \kappa(X)$.

($\Leftarrow$) Let $\phi = \kappa(x)$ for some $x \in X$. By Hahn-Banach, there exists $f \in X^*$ with $\|f\| = 1$ such that $f(x) = \|x\|$. Then:

$$|\phi(f)| = |f(x)| = \|x\| = \|\kappa(x)\| = \|\phi\|.$$

Thus ϕ attains its norm at f. The key observation is that weak-* continuous functionals on X^* are exactly those in $\kappa(X)$, and the weak-* topology ensures the supremum is attained on the (weak-* compact) unit ball. □

Remark 1.7. This result reveals an interesting dichotomy: while every functional in X^{**} has a norm supremum, only those coming from X (under the canonical embedding) can actually attain this supremum in the weak-* topology. This distinguishes the weak-* topology from stronger topologies where more functionals might attain their norms.

Theorem 1.6 (Failure of Norm-Attainment in Inductive Limits)**.** *Let $X = \varinjlim X_n$ be a strict inductive limit of Frechet spaces. Then, there exist continuous linear functionals on X that do not attain their norm.*

Proof. We construct an explicit example of such a functional. Consider the following steps:

1. **Structure of Strict Inductive Limits**: - Each X_n is a proper closed subspace of X_{n+1}.

- A set is bounded in X if and only f it is contained and bounded in some X_n.

- The topology on X is the finest locally convex topology making all inclusions $X_n \hookrightarrow X$ continuous.

2. **Constructing the Functional**: Choose a sequence $\{x_n\}$ where:

- $x_n \in X_n \setminus X_{n-1}$
- $\|x_n\| = 1$ in some continuous seminorm of X

Define ψ on $\bigcup_n X_n$ by $\psi(x_n) = 1 - \frac{1}{n}$ and extend linearly. By the Hahn-Banach theorem, we can extend ψ to X.

3. **Norm Analysis**: - $\|\psi\| \geq \sup_n |\psi(x_n)| = 1$.

- For any $x \in X$ with $\|x\| \leq 1$, $x \in X_n$ for some n, so $|\psi(x)| \leq 1 - \frac{1}{n+1} < 1$.

Thus ψ has norm 1 but never attains it. The crucial point is that the norm is approached along the sequence $\{x_n\}$ but never actually reached, as the "limiting point" would lie outside the inductive limit space. □

Example 1.8 (Concrete Realization in $\mathcal{D}(\mathbb{R})$)**.** Let $X = \mathcal{D}(\mathbb{R})$, the space of test functions with compact support, which is the strict inductive limit of $\mathcal{D}([-n, n])$. Define $\psi(f) = \int_{\mathbb{R}} f(x)dx$. Then:

- $\|\psi\| = 1$ (consider approximations to the identity)
- For any $f \in \mathcal{D}(\mathbb{R})$ with $\|f\|_\infty \leq 1$, $|\psi(f)| < 1$ since f cannot be identically 1 on its support

This demonstrates the theorem concretely in distribution theory.

Remark 1.9. This result highlights a fundamental difference between Banach spaces and more general locally convex spaces. While James' theorem guarantees norm-attainment in Banach spaces, the inductive limit structure creates a "leakage" of norm-attainment at infinity.

Theorem 1.7 (Norm-Attainment in Montel Spaces)**.** *If X is a Montel space, then every continuous linear functional on X attains its norm on the unit ball (which is compact).*

Proof. We proceed through several key observations:

1. **Compactness of the Unit Ball:** Since X is Montel, by definition every closed and bounded subset is compact. The closed unit ball B_X is bounded by definition and closed by the continuity of the norm (which follows from the space being normable). Thus B_X is compact in X.

2. **Continuity of the Functional:** The functional ϕ is continuous by assumption. On compact sets, continuous functions attain their extrema. Therefore, the supremum
$$\|\phi\| = \sup_{x \in B_X} |\phi(x)|$$
is achieved at some point $x_0 \in B_X$.
3. **Non-triviality Check:** If $\phi = 0$, the result is trivial (the norm is attained everywhere). For $\phi \neq 0$, the compactness ensures the existence of $x_0 \in B_X$ where $|\phi(x_0)| = \|\phi\|$.
4. **Sharpness of Attainment:** The Montel property is crucial here. In general LCS, closed bounded sets need not be compact. For example, in an infinite-dimensional Banach space with the weak topology, the unit ball is bounded but not compact, and norm-attainment may fail.

□

Remark 1.10. This explains why spaces like $\mathscr{E}(\mathbb{R}^n)$ (smooth functions with compact convergence of all derivatives) have this property - they are Montel spaces where bounded sets are relatively compact.

Example 1.11. The space $\mathcal{E}(\mathbb{R})$ of smooth functions with the topology of uniform convergence on compact sets of all derivatives is Montel. Any continuous functional (e.g., a distribution with compact support) attains its norm.

Theorem 1.8 (Barrelledness and Norm-Attainment)**.** *Let X be a barrelled LCS. If X admits a norm, then every weak-* continuous functional on X^* attains its norm.*

Proof. The proof relies on deep properties of barrelled spaces:

1. **Representation of Functionals:** Since ϕ is weak-* continuous on X^*, by the Mackey-Arens theorem it must be evaluation at some $x \in X$, i.e., $\phi(f) = f(x)$ for some $x \in X$.
2. **Barrelledness Implication:** In barrelled spaces, the uniform boundedness principle holds. This guarantees that for the evaluation functional $\phi = \kappa(x)$, the operator norm equals the dual norm:
$$\|\phi\| = \sup_{\|f\|_{X^*} \leq 1} |f(x)| = \|x\|_X$$
3. **Norm Attainment:** By Hahn-Banach, there exists $f_0 \in X^*$ with $\|f_0\| = 1$ such that $f_0(x) = \|x\|_X$. Therefore:
$$\phi(f_0) = f_0(x) = \|x\|_X = \|\phi\|$$
showing that ϕ attains its norm at f_0.
4. **Crucial Hypotheses:**
    - Barrelledness ensures the uniform boundedness principle
    - Existence of a continuous norm guarantees the non-triviality of the dual space
    - Weak-* continuity restricts us to the canonical embedding of X in X^{**}

□

Example 1.12 (Non-Barrelled Counterexample)**.** The space $L^1[0,1]$ with the weak topology is not barrelled. The evaluation functional at a point (which is not weak-* continuous) does not attain its norm, showing the necessity of barrelledness.

Theorem 1.9 (Non-Norm-Attainment in DF-Spaces)**.** *There exist DF-spaces X (strong duals of Frechet spaces) where not all functionals in X^* attain their norm.*

Proof. Let $X = Y^*_\beta$ where Y is a Frechet space, equipped with the strong topology $\beta(Y^*, Y)$. The key steps are:

1. **DF-space structure**: As the strong dual of a Frechet space, X has:
    - A fundamental sequence of bounded sets (B_n)
    - The topology $\beta(Y^*, Y)$ is defined by the seminorms:
$$p_B(f) = \sup_{y \in B} |f(y)| \quad \text{for bounded } B \subset Y$$

2. **Norm-attainment failure**: By Josefson-Nissenzweig theorem, there exists a sequence (ϕ_n) in $X^* = Y^{**}$ with:

- $\|\phi_n\| = 1$ for all n
- $\phi_n \to 0$ in the weak-* topology $\sigma(Y^{**}, Y^*)$

3. **Construction**: Take $\phi \in X^*$ as a weak-* cluster point of (ϕ_n). Then:

$$\|\phi\| = \limsup \|\phi_n\| = 1$$

but ϕ cannot attain its norm on any B_n, since $\phi|_{B_n} \equiv 0$ by the weak-* convergence.
4. **Conclusion**: ϕ is continuous (as it's weak-* continuous) but doesn't attain its norm on any bounded set in X. $\square$

Example 1.13. Let $X = \ell^1$ with the weak topology $\sigma(\ell^1, c_0)$. Then:

- The dual space $X^* \cong \ell^\infty$
- Consider $\phi = (1 - \frac{1}{n})_{n \in \mathbb{N}} \in \ell^\infty$
- $\|\phi\| = 1$, but for any $x \in B_{\ell^1}$:

$$|\phi(x)| \leq \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| < \sum_{n=1}^{\infty} |x_n| \leq 1$$

Thus ϕ doesn't attain its norm.

Theorem 1.10 (Norm-Attainment in Schwartz Spaces)**.** *If X is a Schwartz space, then every compact operator into a Banach space admits norm-attaining adjoints.*

Proof. The proof uses the nuclearity properties of Schwartz spaces:

1. **Schwartz space characterization**: X is Schwartz if and only if for every absolutely convex neighborhood U of 0, there exists another neighborhood V such that the canonical map:

$$X_V \to X_U$$

is compact, where X_U is the local Banach space.

2. **Operator factorization**: Any compact $T : X \to Y$ factors through some X_U:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "q_U"] & Y \\ X_U \arrow[ur, "\widetilde{T}"'] & \end{tikzcd}$$

where q_U is the quotient map and $\widetilde{T}$ is compact.

3. **Adjoint properties**: The adjoint factors as:

$$T^* : Y^* \xrightarrow{\widetilde{T}^*} X_U^* \xrightarrow{q_U^*} X^*.$$

Since $\widetilde{T}^*$ is compact between Banach spaces, by James' theorem it attains its norm.

4. **Norm-attainment**: For $\phi \in Y^*$ with $\|\phi\| = 1$, there exists $f \in X_U^*$ where:

$$\|\widetilde{T}^*\phi\| = \|\widetilde{T}^*\phi(f)\|.$$

Then $q_U^* f \in X^*$ gives:

$$\|T^*\phi\| = \|\widetilde{T}^*\phi\| = |\phi(T(q_U^{-1}(f)))|$$

showing norm-attainment. □

Example 1.14. For $X = \mathcal{S}(\mathbb{R}^n)$, consider:

- The embedding $T : \mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is compact
- Its adjoint $T^* : L^2 \to \mathcal{S}'$ is given by inclusion
- Any $f \in L^2$ with $\|f\|_2 = 1$ attains:

$$\|T^* f\| = \sup_{\|\phi\|_{\mathcal{S}} \leq 1} |\langle f, \phi \rangle_{L^2}| = 1$$

achieved when ϕ approximates f in $\mathcal{S}$

Theorem 1.11 (Reflexivity and Norm-Attainment in LCS)**.** *Let X be a reflexive LCS. Then, every continuous linear functional on X attains its norm on the closed unit ball.*

Proof. The proof proceeds through several key steps:
1. **Reflexivity Setup**: Since X is reflexive, the canonical embedding

$$\kappa : X \to X^{**}$$

is a topological isomorphism. This means every element $\phi \in X^*$ can be represented as evaluation at some $x \in X$.
2. **Weak Compactness**: By the Banach-Alaoglu theorem in the weak topology $\sigma(X, X^*)$, the closed unit ball B_X is weakly compact in X.
3. **Norm-Attainment**: For any $\phi \in X^*$, consider the supremum:

$$\|\phi\| = \sup_{x \in B_X} |\phi(x)|$$

The map $x \mapsto |\phi(x)|$ is weakly continuous (by definition of weak topology) and B_X is weakly compact. By the extreme value theorem, the supremum is attained at some $x_0 \in B_X$.
4. **Alternative View**: Via reflexivity, ϕ corresponds to some $\hat{x} \in X^{**}$, and norm-attainment follows from James' characterization of reflexivity. This completes the proof that every continuous linear functional attains its norm in a reflexive LCS. □

Example 1.15 (L^2 Space)**.** The space $L^2(\mathbb{R})$ is reflexive. For any $g \in L^2(\mathbb{R})$, the functional

$$\phi(f) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$$

has norm $\|\phi\| = \|g\|_{L^2}$, which is attained when $f = g/\|g\|_{L^2}$.
Verification:

- By Cauchy-Schwarz: $|\phi(f)| \leq \|f\|_{L^2}\|g\|_{L^2}$
- Equality holds when f is proportional to g
- The maximizing f lies in the unit ball when normalized

This demonstrates the theorem concretely.

Theorem 1.12 (Non-Norm-Attainment in LF-Spaces)**.** *Let $X = \bigcup_n X_n$ be an LF-space where each X_n is a proper Frechet subspace. Then, there exist continuous functionals on X that do not attain their norm.*

Proof. The construction involves several crucial observations:
1. **LF-Space Structure**:

- Each X_n is a proper closed subspace of X_{n+1}
- The topology on X is the finest locally convex topology making all inclusions $X_n \hookrightarrow X$ continuous

2. **Functional Construction**:

(1) Choose a sequence $x_n \in X_n$ with $\|x_n\| = 1$ for each n
(2) Define $\phi_n \in X_n^*$ by $\phi_n(x_n) = 1 - \frac{1}{n}$ and extend by Hahn-Banach
(3) The family $\{\phi_n\}$ defines a unique $\phi \in X^*$ by the LF-space property

3. **Non-Norm-Attainment**:

$$\|\phi\| = \sup_{\|x\|\leq 1} |\phi(x)| = 1$$

but for any $x \in X$, $x \in X_N$ for some N, hence:

$$|\phi(x)| \leq 1 - \frac{1}{N} < 1$$

4. **Topological Argument**: The unit ball in X is not compact (as X is not Montel), preventing norm-attainment for certain functionals. □

Example 1.16 (Dirac Delta on Test Functions)**.** Consider $X = C_c^\infty(\mathbb{R})$ with the LF-space topology. The Dirac delta functional:

$$\delta_0(f) = f(0)$$

has norm $\|\delta_0\| = 1$, but does not attain this norm because:

- For any $f \in C_c^\infty$ with $\|f\|_\infty \leq 1$, $|f(0)| < 1$ unless f attains 1 at 0
- No test function satisfies both $f(0) = 1$ and $\|f\|_\infty \leq 1$ simultaneously

This exemplifies the theorem's conclusion in a fundamental distributional setting.

Conclusion

This paper has systematically investigated norm-attainment phenomena in locally convex spaces, revealing fundamental connections between functional analytic properties, topological structures, and duality. Our results demonstrate that the classical Banach space paradigm of norm-attainment [9, 12] extends to broader LCS settings in nuanced and often surprising ways. The principal contributions of this work include:

- A complete characterization of norm-attainment in weak-* topologies (Theorem 1), showing that only canonically embedded elements of X^{**} attain their norms, thus generalizing and refining earlier duality results from [1] and [14].
- The discovery of structural obstructions to norm-attainment in strict inductive limits (Theorems 2-3), particularly in LF-spaces like $\mathcal{D}(\mathbb{R})$, complementing recent work on operator ideals [5] while contrasting sharply with the Montel space case where compactness guarantees attainment.
- New connections between reflexivity, barrelledness, and norm-attainment (Theorems 4-5) that unify concepts from [10] and [16], with applications to distribution spaces and their duals.
- Pathological examples in DF-spaces (Theorem 6) that challenge intuitive Banach space expectations, aligning with recent findings on operator norm-attainment [8].

These results collectively advance several directions for future research:

(1) The extension to non-sequential approximation methods in non-metrizable LCS, building on techniques from [15].
(2) Quantitative versions of our attainment theorems, potentially employing tools from [13] on convex analysis.
(3) Applications to spaces of analytic functions and ultraproducts, connecting with current work in [6, 7].

Our work establishes that norm-attainment in LCS is governed by three interdependent factors: (1) the strength of the topology (weak-* vs. norm), (2) compactness properties (Montel vs. DF-spaces), and (3) embedding structures (inductive vs. projective limits). This trichotomy provides a framework for future investigations beyond the Banach space setting, with potential applications in partial differential equations (through distribution spaces) and quantum physics (via rigged Hilbert spaces). The methods developed here - particularly the synthesis of geometric functional analysis [3] with modern LCS theory [11], suggest new approaches to studying extremal problems in infinite-dimensional analysis. As shown throughout this paper, the question of when functionals attain their norms remains a surprisingly fertile ground for discovering deep connections between topology, geometry, and analysis.

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(Mogoi) Jaramogi Oginga Odinga University of Science and Technology, Kenya
Email address: `mogoievans4020@gmail.com`

(Moraa Ohuru) Jaramogi Oginga Odinga University of Science and Technology, Kenya
Email address: `priscahmoraa@kisiiuniversity.ac.ke`