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Norm attainment and structural properties in Orlicz spaces: A comprehensive study on strict convexity, duality, and optimization

Mogoi N. Evans and Robert Obogi*

Abstract. We investigate norm attainability and duality properties in Orlicz spaces, extending classical results from Banach and Hilbert spaces to a more general functional framework. We establish 14 fundamental theorems that characterize norm attainment in terms of strict convexity, uniform convexity, and weak convergence. We explore the duality structure of Orlicz spaces, highlighting key differences from L^p spaces and providing a variational characterization of the norm. We also discuss applications in optimization and variational problems, demonstrating the significance of norm-attaining functionals in these settings. Our findings contribute to a deeper understanding of Orlicz space geometry and its implications for functional analysis and applied mathematics.

Keywords: Orlicz spaces, norm attainability, duality properties, convexity, functional analysis
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1. Introduction

Orlicz spaces, introduced as a generalization of classical L^p spaces, provide a robust framework for analyzing norm structures and duality properties in functional analysis. These spaces, defined using Young functions, extend the flexibility of L^p

*Corresponding author

spaces by allowing more general growth conditions, as discussed in the foundational works of [5] and [10]. Their applications span various fields, including optimization, variational problems, and partial differential equations, as highlighted in [8] and [12]. In this paper, we investigate the attainability of norms in Orlicz spaces, establishing conditions under which norm-attaining functionals exist and exploring their implications in duality theory. Norm attainability has been extensively studied in Banach and Hilbert spaces, with well-established results linking norm attainment to strict convexity, uniform convexity, and smoothness, as seen in [3] and [6]. However, in Orlicz spaces, the interplay between the Young function and the norm structure introduces additional complexities that require a more nuanced approach, as noted in [10] and [8]. Our study extends classical results to the Orlicz setting, characterizing norm attainment in terms of strict convexity and weak convergence while also examining the role of compact operators in norm-attaining sequences, building on the framework established in [2] and [9]. The duality structure of Orlicz spaces plays a crucial role in our analysis. Unlike L^p spaces, where the dual space is well-defined as L^q with $\frac{1}{p} + \frac{1}{q} = 1$, Orlicz spaces require the use of complementary Young functions to describe their dual spaces, as detailed in [5] and [10]. We explore how this structure affects norm-attaining functionals and establish a variational characterization of the norm in Orlicz spaces, drawing on insights from [12] and [7]. Additionally, we present applications in optimization and variational problems, demonstrating the practical significance of our theoretical findings, as supported by recent advancements in [1] and [4]. Our work also connects to broader themes in functional analysis, such as the role of unconditional convergence in topological vector spaces, as discussed in [11], and the interplay between probability and Banach spaces, as explored in [6]. Furthermore, the historical development of inequalities and embeddings in Orlicz spaces, as seen in [13] and [12], provides a rich context for our results. By synthesizing these classical and modern perspectives, we contribute to a deeper understanding of norm attainability and duality in Orlicz spaces, offering new insights and applications in functional analysis and beyond.

2. Preliminaries

Before discussing our main results, we provide a brief overview of Orlicz spaces and their fundamental properties.

Young Functions and Orlicz Spaces. A function $\Phi : \mathbb{R} \to [0, \infty)$ is called a Young function if it is convex, even, and satisfies $\Phi(0) = 0$ with $\lim_{x\to\infty} \Phi(x) = \infty$. The Orlicz space $L^\Phi(\Omega)$ associated with a measure space (Ω, Σ, μ) is defined as:

$$L^\Phi(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \int_\Omega \Phi(|f(x)|)\, d\mu(x) < \infty \right\}.$$

This generalization allows for more flexibility in function growth rates compared to the standard L^p spaces.

Norms in Orlicz Spaces. The norm in an Orlicz space is typically defined using the Luxemburg norm:

$$\|f\|_\Phi = \inf \left\{ k > 0 \;\middle|\; \int_\Omega \Phi\left(\frac{|f(x)|}{k}\right) d\mu(x) \leq 1 \right\}.$$

Another equivalent norm is the Orlicz norm, which is defined using modular functionals. The choice of norm affects norm-attainability properties, as we shall explore in subsequent sections.

Complementary Young Functions and Duality. The dual space $(L^\Phi)^*$ is characterized using the complementary Young function Ψ, defined via the Legendre transform:

$$\Psi(y) = \sup_{x \geq 0} \{xy - \Phi(x)\}.$$

For a function $f \in L^\Phi$, the dual pairing between L^Φ and L^Ψ is given by:

$$\langle f, g \rangle = \int_\Omega f(x)g(x)\, d\mu(x).$$

This formulation is crucial in studying norm attainment, as it provides necessary conditions for the existence of norm-attaining functionals.

3. Main Results and Discussions

Theorem 3.1. *Let (Ω, Σ, μ) be a measure space, and let $L^\Phi(\Omega)$ be an Orlicz space with a Young function Φ. If Φ is strictly convex, then every norm-attaining sequence in L^Φ converges strongly to a norm-attaining function.*

Proof. Let $\{f_n\} \subset L^\Phi(\Omega)$ be a norm-attaining sequence, meaning there exists a functional $F \in (L^\Phi)^*$ such that

$$\lim_{n\to\infty} F(f_n) = \sup_{\|g\|_\Phi = 1} F(g).$$

Since Φ is strictly convex, the Orlicz space L^Φ is uniformly convex, ensuring that every weakly convergent sequence is strongly convergent. By the Banach-Alaoglu theorem, the sequence $\{f_n\}$ has a weakly convergent subsequence, say $f_{n_k} \rightharpoonup f$ in L^Φ.

Now, by the strict convexity of Φ, norm convergence follows from weak convergence, implying

$$\|f_{n_k} - f\|_\Phi \to 0 \quad \text{as } k \to \infty.$$

Thus, f is the strong limit of $\{f_n\}$. Since functionals are continuous with respect to strong convergence, it follows that

$$F(f) = \sup_{\|g\|_\Phi = 1} F(g),$$

showing that f is a norm-attaining function. This completes the proof. □

Example 3.1. Consider the Orlicz space $L^\Phi(\mathbb{R})$ where the Young function is given by

$$\Phi(t) = e^{|t|} - 1.$$

This function is strictly convex, ensuring that L^Φ is uniformly convex.

Define the sequence of functions

$$f_n(x) = \frac{e^x}{n} \chi_{[0, \ln n]}(x),$$

where $\chi_{[0,\ln n]}(x)$ is the characteristic function of the interval $[0, \ln n]$. For a functional $F(g) = \int_{\mathbb{R}} g(x) e^{-x}\, dx$, we see that $F(f_n)$ is maximized for large n, making $\{f_n\}$ a norm-attaining sequence. Since L^Φ is uniformly convex, weak convergence of $\{f_n\}$ implies strong convergence. In this case, f_n converges strongly to

$$f(x) = 0,$$

which is the norm-attaining function in L^Φ. This demonstrates the theorem's assertion that every norm-attaining sequence converges strongly to a norm-attaining function.

Theorem 3.2. *If Φ is a strictly convex Young function, then for any nonzero functional in the dual space $(L^\Phi)^*$, there exists a unique function $f \in L^\Phi$ such that $|f|_\Phi = 1$ and the supremum in the norm definition is attained.*

Proof. Let $F \in (L^\Phi)^*$ be a nonzero functional, given by

$$F(f) = \int_\Omega f g \, d\mu$$

for some function $g \in L^\Psi(\Omega)$, where Ψ is the Young function complementary to Φ. The norm of F is given by

$$\|F\| = \sup_{\|f\|_\Phi = 1} \left| \int_\Omega f g \, d\mu \right|.$$

By the strict convexity of Φ, the unit sphere in L^Φ contains at most one function attaining the supremum. That is, if both f_1 and f_2 attained the supremum, then strict convexity would imply that any convex combination of them also attains the supremum, contradicting the uniqueness property of strict convexity. Thus, there exists a unique function f such that $\|f\|_\Phi = 1$ and $F(f) = \sup_{\|g\|_\Phi = 1} F(g)$. This proves the result. □

Theorem 3.3. *Let $T : L^\Phi \to L^\Psi$ be a compact linear operator between two Orlicz spaces. If Φ and Ψ are strictly convex, then T attains its norm.*

Proof. The operator norm of T is defined as

$$\|T\| = \sup_{\|f\|_\Phi \leq 1} \|Tf\|_\Psi.$$

Since T is compact, the image of the unit ball in L^Φ under T is relatively compact in L^Ψ. Hence, there exists a maximizing sequence $\{f_n\}$ in L^Φ such that

$$\|f_n\|_\Phi \leq 1, \quad \text{and} \quad \|Tf_n\|_\Psi \to \|T\|.$$

By the compactness of T, the sequence $\{Tf_n\}$ has a convergent subsequence $Tf_{n_k} \to g$ in L^Ψ. Since Φ and Ψ are strictly convex, L^Φ and L^Ψ are reflexive, ensuring that $\{f_n\}$ has a weakly convergent subsequence $f_{n_k} \rightharpoonup f$ in L^Φ. Now, by the continuity of T, we obtain

$$Tf_{n_k} \to Tf.$$

Since norm convergence follows from weak convergence in strictly convex spaces, it follows that $\|Tf\|_\Psi = \|T\|$, proving that T attains its norm at f. □

Theorem 3.4. *The dual space of L^Φ is given by L^Ψ, where Ψ is the complementary Young function of Φ. Moreover, the pairing $\langle f, g\rangle = \int_\Omega fg d\mu$ defines a norm attainment criterion.*

Proof. Let Φ be a Young function, and define its complementary function Ψ by the Legendre transform

$$\Psi(y) = \sup_{x>0}\{xy - \Phi(x)\}.$$

The Orlicz space L^Φ consists of all measurable functions f for which

$$\int_\Omega \Phi(|f|) d\mu < \infty.$$

By the standard duality theory of Orlicz spaces, the dual space $(L^\Phi)^*$ is given by L^Ψ, where the norm is given by

$$\|g\|_{L^\Psi} = \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|g|}{\lambda}\right) d\mu \leq 1\right\}.$$

The pairing

$$\langle f, g\rangle = \int_\Omega fg d\mu$$

is well-defined due to Young's inequality:

$$|fg| \leq \Phi(|f|) + \Psi(|g|).$$

Norm attainability follows from the fact that if g is a norm-attaining functional, then there exists $f \in L^\Phi$ with $\|f\|_{L^\Phi} = 1$ such that

$$\|g\|_{L^\Psi} = \sup_{\|f\|_{L^\Phi} \leq 1} |\langle f, g\rangle|,$$

which follows from the strict convexity of the integral functional. □

Example 3.2. Consider the Orlicz space $L^\Phi(\mathbb{R})$ where the Young function is given by

$$\Phi(x) = e^{|x|} - 1.$$

The complementary Young function is found using the Legendre transform:

$$\Psi(y) = (1 + y)\ln(1 + y) - y.$$

Thus, the dual space $(L^\Phi)^*$ is given by the Orlicz space $L^\Psi(\mathbb{R})$, consisting of all measurable functions g for which

$$\int_{\mathbb{R}} \Psi(|g|) d\mu < \infty.$$

To illustrate norm attainability, consider the functions

$$f(x) = e^{-|x|} \quad \text{and} \quad g(x) = \ln(1 + |x|).$$

We verify that $f \in L^\Phi$ and $g \in L^\Psi$ satisfy the pairing

$$\langle f, g\rangle = \int_{\mathbb{R}} e^{-|x|} \ln(1 + |x|) d\mu.$$

By Young's inequality,

$$e^{-|x|} \ln(1 + |x|) \leq \Phi(e^{-|x|}) + \Psi(\ln(1 + |x|)).$$

Since both terms on the right-hand side are integrable, we conclude that g attains its norm in L^Ψ, demonstrating norm attainability in this Orlicz setting.

Theorem 3.5. *If Φ is uniformly convex, then L^Φ is reflexive, ensuring the existence of norm-attaining functionals.*

Proof. A Banach space X is reflexive if and only if its unit ball is weakly compact. Uniform convexity ensures that for any sequence $\{f_n\}$ in L^Φ with $\|f_n\| \to 1$, we have

$$\|f_n + g_n\| \to 2$$

for any sequence $\{g_n\}$ satisfying $\|g_n\| \to 1$. This property implies that L^Φ is both strictly convex and reflexive. Since L^Φ is reflexive, every bounded linear functional on L^Φ attains its norm by the James' theorem, i.e., for every $g \in (L^\Phi)^*$, there exists $f \in L^\Phi$ with $\|f\|_{L^\Phi} = 1$ such that

$$\|g\| = \langle f, g\rangle.$$

This ensures norm attainability in L^Φ. □

Theorem 3.6. *Let K be a closed convex subset of L^Φ. If Φ is strictly convex, then every function in L^Φ has a unique best approximation in K.*

Proof. Since K is a closed and convex subset of the strictly convex Banach space L^Φ, the strict convexity of Φ ensures the uniqueness of the projection.

For any $f \in L^\Phi$, we seek $g^* \in K$ such that

$$\|f - g^*\| = \inf_{g \in K} \|f - g\|.$$

The existence follows from the standard projection theorem in Banach spaces. Uniqueness follows from strict convexity: Suppose $g_1, g_2 \in K$ are two best approximations, meaning

$$\|f - g_1\| = \|f - g_2\| = d.$$

For any $\lambda \in (0, 1)$, consider $g_\lambda = \lambda g_1 + (1 - \lambda) g_2$. By convexity, $g_\lambda \in K$, and by strict convexity,

$$\|f - g_\lambda\| < \lambda \|f - g_1\| + (1 - \lambda)\|f - g_2\| = d.$$

This contradicts the minimality of g_1 and g_2, proving uniqueness. □

Theorem 3.7. *If Φ is strictly convex, then the unit ball in L^Φ has no non-trivial extreme points, ensuring norm uniqueness.*

Proof. Let $f \in L^\Phi$ be an extreme point of the unit ball, i.e., $\|f\|_\Phi \leq 1$. Suppose there exist distinct functions $g, h \in L^\Phi$ such that

$$f = \frac{g + h}{2}, \quad \text{and} \quad \|g\|_\Phi \leq 1, \quad \|h\|_\Phi \leq 1.$$

By the definition of extreme points, this implies $g = h = f$. Since Φ is strictly convex, the Luxemburg norm

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}$$

is also strictly convex. This means that for any non-trivial convex combination of g and h, strict inequality must hold:

$$\|\lambda g + (1 - \lambda) h\|_\Phi < \lambda \|g\|_\Phi + (1 - \lambda)\|h\|_\Phi.$$

This contradicts our assumption that f is an extreme point, proving that non-trivial extreme points do not exist. □

Theorem 3.8. *The Hahn-Banach theorem holds in L^Φ with respect to the Luxemburg norm, guaranteeing the extension of bounded linear functionals.*

Proof. Let $X = L^\Phi$ be an Orlicz space with the Luxemburg norm $\| \cdot \|_\Phi$. Suppose we have a linear functional φ defined on a subspace $M \subseteq X$ such that it is bounded, i.e., there exists a constant $C > 0$ such that

$$|\varphi(f)| \leq C \|f\|_\Phi, \quad \forall f \in M.$$

Our goal is to extend φ to all of X while preserving linearity and boundedness. Define the Minkowski functional associated with the norm:

$$p(f) = \inf \left\{ \lambda > 0 : \int \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

Since Orlicz spaces are Banach spaces under the Luxemburg norm, the classical Hahn-Banach theorem applies. Thus, there exists an extension $\tilde{\varphi} : X \to \mathbb{R}$ satisfying

$$|\tilde{\varphi}(f)| \leq C\|f\|_\Phi, \quad \forall f \in X.$$

This proves the existence of a norm-preserving extension of φ to the whole space. □

Theorem 3.9. *If L^Φ is smooth, then every bounded linear functional on L^Φ attains its norm at some function in L^Φ.*

Proof. Let $\varphi : L^\Phi \to \mathbb{R}$ be a bounded linear functional. The norm of φ is given by

$$\|\varphi\| = \sup_{\|f\|_\Phi \leq 1} |\varphi(f)|.$$

Since L^Φ is smooth, the norm $\|\cdot\|_\Phi$ is Frechet differentiable. By the Riesz representation theorem for Orlicz spaces, there exists a unique function $g \in L^\Psi$, where Ψ is the complementary Young function of Φ, such that

$$\varphi(f) = \int f g \, dx, \quad \forall f \in L^\Phi.$$

Since the norm is smooth, there exists some $f_0 \in L^\Phi$ with $\|f_0\|_\Phi = 1$ such that

$$\varphi(f_0) = \sup_{\|f\|_\Phi \leq 1} |\varphi(f)|.$$

Thus, φ attains its norm at f_0, proving the theorem. □

Theorem 3.10. *A family of functionals $\{T_n\}$ in $(L^\Phi)^*$ is uniformly bounded if and only if the corresponding norm-attaining sequence in L^Φ remains bounded.*

Proof. Assume $\{T_n\}$ is uniformly bounded. That is, there exists $C > 0$ such that $\|T_n\| \leq C$ for all n. By the definition of the dual norm, we have:

$$|T_n(f)| \leq \|T_n\|\|f\| \leq C\|f\|, \quad \forall f \in L^\Phi.$$

If $\{f_n\} \subset L^\Phi$ is a norm-attaining sequence, then there exists $g \in L^\Phi$ such that

$$T_n(f_n) = \|T_n\|\|f_n\|.$$

Since $\|T_n\|$ is bounded by C, it follows that $\|f_n\|$ must also be bounded; otherwise, the equality above would contradict the uniform boundedness of $\{T_n\}$.

Conversely, suppose $\{f_n\}$ is bounded, meaning there exists $M > 0$ such that $\|f_n\| \leq M$ for all n. Since T_n is a norm-attaining functional, we have:

$$|T_n(f_n)| = \|T_n\|\|f_n\| \leq M\|T_n\|.$$

Thus, if $\{T_n\}$ were unbounded, there would exist subsequences with $\|T_n\| \to \infty$, contradicting the boundedness of $|T_n(f_n)|$. Hence, $\{T_n\}$ is uniformly bounded. □

Theorem 3.11. *If Φ is strictly convex, then the norm function on L^Φ is Frechet differentiable at every nonzero point.*

PROOF. Let $f \in L^\Phi$ with $\|f\| > 0$. The Frechet differentiability of the norm function at f means that there exists a unique continuous linear functional T_f such that:

$$\lim_{\|h\|\to 0} \frac{\|f+h\| - \|f\| - T_f(h)}{\|h\|} = 0.$$

Since Φ is strictly convex, the norm in L^Φ is also strictly convex, implying that every supporting hyperplane at f is unique. In particular, the duality mapping

$$J(f) = \{T_f \in (L^\Phi)^* : T_f(f) = \|f\|^2, \|T_f\| = \|f\|\}$$

selects a unique functional T_f satisfying the above limit condition. This establishes Frechet differentiability. □

Theorem 3.12. *A sequence $\{f_n\}$ in L^Φ attains its norm in the weak topology if and only if it converges strongly to a norm-attaining function.*

PROOF. ($\Rightarrow$) Suppose $\{f_n\}$ attains its norm in the weak topology, meaning there exists $f \in L^\Phi$ such that $f_n \rightharpoonup f$ and

$$\|f\| = \lim_{n\to\infty} \|f_n\|.$$

Since weak convergence preserves norms in reflexive Banach spaces and L^Φ is reflexive under appropriate conditions (e.g., when Φ satisfies the Δ_2-condition), it follows that $f_n \to f$ strongly in norm. ($\Leftarrow$) Suppose $f_n \to f$ strongly, meaning $\|f_n - f\| \to 0$. Since norm convergence implies weak convergence, we also have $f_n \rightharpoonup f$. Moreover, norm attainment ensures that

$$\|f\| = \lim_{n\to\infty} \|f_n\|,$$

showing that f_n attains its norm in the weak topology. □

Theorem 3.13. *Let $J(f) = \int_\Omega F(f)d\mu$ be a functional on L^Φ, where F is a convex function. If J is coercive and weakly lower semicontinuous, then it attains its minimum in L^Φ.*

PROOF. Since $J(f) = \int_\Omega F(f)d\mu$ is coercive, it follows that for any sequence $\{f_n\} \subset L^\Phi$ such that $\|f_n\|_\Phi \to \infty$, we have $J(f_n) \to \infty$. This ensures that minimizing sequences remain bounded in L^Φ.

By the Banach-Alaoglu theorem, there exists a weakly convergent subsequence $f_{n_k} \rightharpoonup f^* \in L^\Phi$. Since J is weakly lower semicontinuous, we obtain

$$J(f^*) \le \liminf_{k\to\infty} J(f_{n_k}).$$

Thus, f^* attains the minimum of J, proving the theorem. □

Example 3.3. Consider the Orlicz space $L^\Phi(\mathbb{R})$ defined using the Young function $\Phi(t) = e^{|t|} - 1$, which characterizes an exponential Orlicz space. Define the functional

$$J(f) = \int_{\mathbb{R}} |f(x)|^p e^{-|x|} dx,$$

where $1 < p < \infty$. The function $F(f) = |f|^p$ is convex, ensuring that $J(f)$ retains convexity. Furthermore, coercivity holds because if $\|f_n\|_\Phi \to \infty$, then $J(f_n) \to \infty$, preventing minimizing sequences from escaping to infinity. Since $J(f)$ is defined as an integral functional, its weak lower semicontinuity follows directly from Fatou's lemma, which guarantees that for any weakly convergent sequence $\{f_n\}$ in L^Φ, we have

$$J(f^*) \leq \liminf_{n\to\infty} J(f_n).$$

This ensures the existence of a minimizer $f^* \in L^\Phi(\mathbb{R})$, verifying the attainability of the minimum and confirming the theorem.

Theorem 3.14. *The norm in L^Φ can be characterized variationally by*

$$\|f\|_\Phi = \sup\left\{|\langle f, g\rangle| : g \in L^\Psi, \|g\|_\Psi \leq 1\right\},$$

ensuring norm attainability.

Proof. By the definition of the norm in Orlicz spaces, we have

$$\|f\|_\Phi = \sup\left\{|\langle f, g\rangle| : g \in L^\Psi, \|g\|_\Psi \leq 1\right\}.$$

We first show that the supremum is attained. Consider the functional $\Lambda_g(f) = \langle f, g\rangle$ on L^Φ. By Holder's inequality for Orlicz spaces, we get

$$|\langle f, g\rangle| \leq \|f\|_\Phi \|g\|_\Psi.$$

If equality holds for some $g^* \in L^\Psi$ with $\|g^*\|_\Psi \leq 1$, then the supremum is attained, implying that

$$\|f\|_\Phi = |\langle f, g^*\rangle|.$$

To construct such a function g^*, we consider the modular function $\rho_\Phi(f) = \int_\Omega \Phi(f) d\mu$. If $\rho_\Phi(f) < \infty$, we define

$$g^* = \frac{\Phi'(f)}{\|\Phi'(f)\|_\Psi}.$$

This function belongs to L^Ψ with $\|g^*\|_\Psi \leq 1$ and satisfies

$$\|f\|_\Phi = \langle f, g^*\rangle.$$

Hence, norm attainability is guaranteed, completing the proof. □

Conclusion

In this paper, we investigate norm attainability and duality properties in Orlicz spaces, generalising classical results from Banach and Hilbert spaces to this broader functional framework. By establishing connections between norm attainment, strict convexity, weak convergence, and the duality structure defined by Young functions and their complements, we have provided a comprehensive characterization of these spaces. Our findings have important implications for optimization, variational problems, and the geometric understanding of Orlicz spaces. Future research may explore weighted Orlicz spaces, non-reflexive settings, and applications in applied mathematics, as well as further investigate the role of compact operators and duality in norm-attaining functionals.

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Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, Kenya
Email address: mogoievans4020@gmail.com

Department of Mathematics and Actuarial Science, Kisii University, Kenya
Email address: robogi@kisiiuniversity.ac.ke