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RN-nearly mixed type A-Q functional equation

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Abstract. In this paper, using direct and fixed point methods, we prove the generalized Hyers-Ulam stability of the following mixed additive-quadratic functional equation:

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=\frac{1}{2}\Big\{\big(3f(x)-f(-x)\big)+\big(f(y)+f(-y)\big)\Big\}$$

in random normed spaces.

Keywords: Generalized Hyers-Ulam stability, Random normed space, Fixed point method

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1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [32] in 1940. In the next year, Hyers [10] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [19] proved a generalization of Hyers's theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization

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of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations (see [1]-[29]). Furthermore, in 1994, a generalization of Rassias's theorem was obtained by Găvruta [9] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. In 2008, a special case of Găvruta's theorem for the unbounded Cauchy difference was obtained by K.Ravi et al., by considering both sum and product of p norms. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [31] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [4] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

In this paper, we prove the generalized Hyers-Ulam stability of the following mixed additive-quadratic functional equation:

$$2f\Big(\frac{x+y}{2}\Big) + 2f\Big(\frac{x-y}{2}\Big) = \frac{1}{2}\Big\{(3f(x) - f(-x)) + (f(y) + f(-y))\Big\} \tag{1}$$

in random normed spaces.

2. Preliminaries

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [30].

Throughout this paper, let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set

$$D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\},$$

where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ 1, & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0, 1]^2 \to [0, 1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;
(c) $T(x,1) = x$ for all $x \in [0,1]$;
(d) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows: $T(x,y) = xy$, $T(x,y) = \max\{a+b-1,0\}$, $T(x,y) = \min(a,b)$. Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^{n} x_i$ is defined recursively by $T_{i=1}^{1} x_1 = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined by $T_{i=1}^{\infty} x_{n+i}$.

Definition 2.2. A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$.

If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every RN-space (X,μ,T) is a metrizable linear topological space with the topology τ (called the *μ-topology* or the *(ϵ,δ)-topology*, where $\epsilon > 0$ and $\lambda \in (0,1)$) induced by the base $\{U(\epsilon,\lambda)\}$ of neighborhoods of θ, where

$$U(\epsilon,\lambda) = \{x \in X : \mu_x(\epsilon) > 1 - \lambda\}.$$

Definition 2.3. Let (X,μ,T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.

(3) The RN-space (X,μ,T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 2.1. ([30]) *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Definition 2.4. Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:
(a) $d(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$;
(b) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(c) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Theorem 2.2. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{2}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n_0 \geq n_0$*;*
(b) the sequence $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*
(c) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$*;*
(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

3. Random Stability of Functional Equation (1): Fixed Point Method

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (1). Then f is a quadratic mapping, i.e., $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$ holds. Using the fixed point method, we prove the generalized Hyers-Ulam stability of the additive-quadratic functional equation (1) in random normed spaces.

Theorem 3.1. *Let* X *be a linear space,* (Y, μ, T_M) *be a complete RN-space and* Φ *be a mapping from* X^2 *to* D^+ *(*$\Phi(x, y)$ *is denoted by* $\Phi_{x,y}$*) such that there exists* $0 < \alpha < \frac{1}{4}$ *such that*

$$\Phi_{2x,2y}(t) \leq \Phi_{x,y}(\alpha t) \tag{3}$$

for all $x, y \in X$ *and* $t > 0$*. Let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and satisfying*

$$\mu_{2f(\frac{x+y}{2})+2f(\frac{x-y}{2})-\frac{(3f(x)-f(-x))+(f(y)+f(-y))}{2}}(t) \geq \Phi_{x,y}(t) \tag{4}$$

for all $x, y \in X$ *and* $t > 0$*. Then there exist a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \Phi_{x,0}((1 - 4\alpha)t) \tag{5}$$

for all $x \in X$ *and* $t > 0$*.*

Proof. Putting $y = 0$ in (4), we have

$$\mu_{4f(\frac{x}{2})-f(x)}(t) \geq \Phi_{x,0}(t) \tag{6}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y; g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f, g) = \inf\{u \in (0, +\infty) : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,0}(t),\ \forall x \in X,\ t > 0\}, \tag{7}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [18, Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := 4h\left(\frac{x}{2}\right) \tag{8}$$

for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant 4α. In fact, let $g, h \in S$ be such that $d(g, h) < \lambda$. Then we have $\mu_{g(x)-h(x)}(\lambda t) \geq \Phi_{x,0}(t)$ for all $x \in X$ and $t > 0$ and so

$$\mu_{Jg(x)-Jh(x)}(4\alpha\lambda t) = \mu_{g(\frac{x}{2})-h(\frac{x}{2})}(\alpha\lambda t) \geq \Phi_{\frac{x}{2},0}(\alpha t) \geq \Phi_{x,0}(t) \tag{9}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) < \lambda$ implies that $d(Jg, Jh) < 4\alpha\lambda$. This means that $d(Jg, Jh) \leq 4\alpha d(g, h)$ for all $g, h \in S$. It follows from (6) that $d(f, Jf) \leq 1$. By Theorem 2.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q\Big(\frac{x}{2}\Big) = \frac{1}{4}Q(x) \tag{10}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (10) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,0}(t)$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big) = Q(x)$$

for all $x \in X$.

(3) $d(f, Q) \leq \frac{d(f,Jf)}{1-4\alpha}$ with $f \in \Omega$, which implies the inequality $d(f, Q) \leq \frac{1}{1-4\alpha}$ and so

$$\mu_{f(x)-Q(x)}\Big(\frac{t}{1-4\alpha}\Big) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (5) holds. On the other hand

$$\mu_{4^n[2f(\frac{x+y}{2^{n+1}})+2f(\frac{x-y}{2^{n+1}})-\frac{3f(\frac{x}{2^n})}{2}+\frac{f(\frac{-x}{2^n})}{2}-\frac{f(\frac{y}{2^n})}{2}-\frac{f(\frac{-y}{2^n})}{2}]}(t) \geq \Phi_{\frac{x}{2^n},\frac{y}{2^n}}\Big(\frac{t}{4^n}\Big)$$

for all $x, y \in X$, $t > 0$ and $n \geq 1$ and so, from (3), it follows that

$$\Phi_{\frac{x}{2^n},\frac{y}{2^n}}\Big(\frac{t}{4^n}\Big) \geq \Phi_{x,y}\Big(\frac{t}{(4\alpha)^n}\Big).$$

Since $\lim_{n\to\infty} \Phi_{x,y}(\frac{t}{(4\alpha)^n}) = 1$ for all $x, y \in X$ and $t > 0$, we have

$$\mu_{2Q(\frac{x+y}{2})+2Q(\frac{x-y}{2})-\frac{3Q(x)}{2}+\frac{Q(-x)}{2}-\frac{Q(y)}{2}-\frac{Q(-y)}{2}}(t) = 1,$$

for all $x, y \in X$ and $t > 0$. Since, f is even, Q is even too. Thus, the mapping $Q : X \to Y$ is quadratic. This completes the proof. □

Corollary 3.2. *Let X be a real normed space, $\theta_1, \theta_2 \geq 0$ and p be a real number with $p \in (1, +\infty)$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\mu_{2f(\frac{x+y}{2})+f(\frac{x-y}{2})+f(\frac{y-x}{2})-f(x)-f(y)}(t) \geq \frac{t}{t+\theta_1\big(\|x\|^p+\|y\|^p+\theta_2\|x\|^{\frac{p}{2}}.\|y\|^{\frac{p}{2}}\big)} \tag{11}$$

for all $x, y \in X$ and $t > 0$. Then there exist a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(4^p-4)t}{(4^p-4)t+4^p\theta_1\|x\|^p}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.1 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta_1\left(\|x\|^p + \|y\|^p + \theta_2\|x\|^{\frac{p}{2}}.\|y\|^{\frac{p}{2}}\right)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 4^{-p}$, then we get the desired result. □

Similarly, we can obtain the following and so we omit the proof.

Theorem 3.3. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that for some $0 < \alpha < 4$*

$$\Phi_{\frac{x}{2},\frac{y}{2}}(t) \le \Phi_{x,y}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (4). Then there exist a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \ge \Phi_{x,0}\left(\frac{(4-\alpha)t}{\alpha}\right)$$

for all $x \in X$ and $t > 0$.

Corollary 3.4. *Let X be a real normed space, $\theta \ge 0$ and p be a real number with $p \in (0, 1)$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (11). Then there exist a unique quadratic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \ge \frac{(4-4^p)t}{(4-4^p)t + 4^p\theta_1\|x\|^p}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.3 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta_1\left(\|x\|^p + \|y\|^p + \theta_2\|x\|^{\frac{p}{2}}.\|y\|^{\frac{p}{2}}\right)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 4^p$, then we get the desired result. □

Let $f : X \to Y$ be an odd mapping satisfying (1). Then f is an additive mapping, i.e., $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = 2f(x)$ holds.

Theorem 3.5. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that there exists $0 < \alpha < \frac{1}{2}$ such that*

$$\Phi_{2x,0}(t) \le \Phi_{x,0}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be an odd mapping satisfying (4). Then there exist a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \Phi_{x,0}(2(1-2\alpha)t)$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (4), we have

$$\mu_{2f(\frac{x}{2})-f(x)}\Big(\frac{t}{2}\Big) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$. The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Corollary 3.6. *Let $\theta_1, \theta_2 \geq 0$ and p be a real number with $p \in (1, +\infty)$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (11). Then there exist a unique additive mapping $A : X \to Y$ such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2^{p+1} - 2^2)t}{(2^{p+1} - 2^2)t + 2^p\theta_1\|x\|^p}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.5 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta_1\big(\|x\|^p + \|y\|^p + \theta_2\|x\|^{\frac{p}{2}}.\|y\|^{\frac{p}{2}}\big)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^{-p}$, then we get the desired result. $\square$

Similarly, we can obtain the following and so we omit the proof.

Theorem 3.7. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that there exists $0 < \alpha < 2$ such that*

$$\Phi_{\frac{x}{2},\frac{y}{2}}(t) \leq \Phi_{x,y}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be an odd mapping satisfying (4). Then there exist a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}\Big(\frac{2(2-\alpha)}{\alpha}t\Big)$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (4), we have

$$\mu_{4f(\frac{x}{2})-2f(x)}(t) \geq \Phi_{x,0}(t) \tag{12}$$

for all $x \in X$ and $t > 0$. Replacing x by $2x$ in (13), we obtain

$$\mu_{f(x)-\frac{f(2x)}{2}}\Big(\frac{t}{4}\Big) \geq \Phi_{2x,0}(t) \geq \Phi_{x,0}\Big(\frac{t}{\alpha}\Big). \tag{13}$$

So, $d(f, Jf) \leq \frac{\alpha}{4}$. The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Corollary 3.8. *Let* $\theta_1, \theta_2 \geq 0$ *and* p *be a real number with* $p \in (0,1)$*. Let* X *be a normed vector space with norm* $\| \cdot \|$*. Let* $f : X \to Y$ *be an odd mapping satisfying (11). Then there exist a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{2(2-2^p)t}{2(2-2^p)t + 2^p\theta_1 \|x\|^p}$$

for all $x \in X$ *and* $t > 0$*.*

Proof. The proof follows from Theorem 3.7 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta_1\big(\|x\|^p + \|y\|^p + \theta_2 \|x\|^{\frac{p}{2}}.\|y\|^{\frac{p}{2}}\big)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^p$, then we get the desired result. □

Theorem 3.9. *Let* X *be a linear space,* (Y, μ, T_M) *be a complete RN-space and* Φ *be a mapping from* X^2 *to* D^+ *(*$\Phi(x,y)$ *is denoted by* $\Phi_{x,y}$*) such that there exists* $0 < \alpha < \frac{1}{4}$ *such that*

$$\Phi_{2x,2y}(t) \leq \Phi_{x,y}(\alpha t)$$

for all $x, y \in X$ *and* $t > 0$*. Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and (4). Then there exist a unique quadratic mapping* $Q : X \to Y$ *and an additive mapping* $A : X \to Y$ *such that*

$$\begin{aligned} \mu_{f(x)-\frac{Q(x)+A(x)}{2}}\Big(\frac{t}{2}\Big) &\geq T_M\Big(\mu_{f(x)-Q(x)}\Big(\frac{t}{2}\Big), \mu_{f(x)-A(x)}\Big(\frac{t}{2}\Big)\Big) \\ &\geq T_M\Big(\Phi_{x,0}\Big(\frac{(1-4\alpha)t}{2}\Big), \Phi_{x,0}((1-2\alpha)t)\Big) \end{aligned}$$

for all $x \in X$ *and* $t > 0$*.*

4. Random Stability of Functional Equation (1): Direct method

Using direct method, we prove the generalized Hyers-Ulam stability of the additive-quadratic functional equation (2.3) in random normed spaces.

Theorem 4.1. *Let* X *be a real linear space,* $(Z, \mu', \min)$ *be an RN-space and* $\phi : X^2 \to Z$ *be a function such that there exists* $0 < \alpha < \frac{1}{4}$ *such that*

$$\mu'_{\phi(\frac{x}{2},\frac{y}{2})}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{14}$$

for all $x \in X$ *and* $t > 0$ *and* $\lim_{n\to\infty} \mu'_{\phi(\frac{x}{2^n},\frac{y}{2^n})}(\frac{t}{4^n}) = 1$*, for all* $x, y \in X$ *and* $t > 0$*. Let* $(Y, \mu, \min)$ *be a complete RN-space. If* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and*

$$\mu_{2f(\frac{x+y}{2})+2f(\frac{x-y}{2})-\frac{3f(x)}{2}+\frac{f(-x)}{2}-\frac{f(y)}{2}-\frac{f(-y)}{2}}(t) \geq \mu'_{\phi(x,y)}(t) \tag{15}$$

for all $x, y \in X$ *and* $t > 0$. *Then there exist a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}((1-4\alpha)t). \tag{16}$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = 0$ in (15), we see that

$$\mu_{4f(\frac{x}{2})-f(x)}(t) \geq \mu'_{\phi(x,0)}(t) \tag{17}$$

for all $x \in X$. Replacing x by $\frac{x}{2^n}$ in (27) and using (14), we obtain

$$\begin{aligned} \mu_{4f(\frac{x}{2^{n+1}})-f(\frac{x}{2^n})}(t) &= \mu_{\frac{1}{4^n}(4^{n+1}f(\frac{x}{2^{n+1}})-4^n f(\frac{x}{2^n}))}(t) \\ &= \mu_{4^{n+1}f(\frac{x}{2^{n+1}})-4^n f(\frac{x}{2^n})}(4^n t) \\ &\geq \mu'_{\phi(\frac{x}{2^n},0)}(t) \\ &\geq \mu'_{\phi(x,0)}\left(\frac{t}{\alpha^n}\right) \end{aligned}$$

and so

$$\mu_{4^{n+1}f(\frac{x}{2^{n+1}})-4^n f(\frac{x}{2^n})}(t) \geq \mu'_{\phi(x,0)}\left(\frac{t}{4^n\alpha^n}\right).$$

Therefore

$$\begin{aligned} \mu_{4^n f(\frac{x}{2^n})-f(x)}\left(\sum_{k=0}^{n-1} 4^k\alpha^k t\right) &= \mu_{\sum_{k=0}^{n-1} 4^{k+1}f(\frac{x}{2^{k+1}})-4^k f(\frac{x}{2^k})}\left(\sum_{k=0}^{n-1} 4^k\alpha^k t\right) \\ &\geq T_{k=0}^{n-1}\left(\mu_{4^{k+1}f(\frac{x}{2^{k+1}})-4^k f(\frac{x}{2^k})}(4^k\alpha^k t)\right) \\ &\geq T_{k=0}^{n-1}\left(\mu'_{\phi(x,0)}(t)\right) \\ &= \mu'_{\phi(x,0)}(t). \end{aligned}$$

This implies that

$$\mu_{4^n f(\frac{x}{2^n})-f(x)}(t) \geq \mu'_{\phi(x,0)}\left(\frac{t}{\sum_{k=0}^{n-1} 4^k\alpha^k}\right). \tag{18}$$

Replacing x by $\frac{x}{2^p}$ in (18), we obtain

$$\mu_{4^{n+p}f(\frac{x}{2^{n+p}})-4^p f(\frac{x}{2^p})}(t) \geq \mu'_{\phi(x,0)}\left(\frac{t}{\sum_{k=p}^{n+p-1} 4^k\alpha^k}\right). \tag{19}$$

Since $\lim_{p,n\to\infty} \mu'_{\phi(x,0)}(\frac{t}{\sum_{k=p}^{n+p-1} 4^k\alpha^k}) = 1$, it follows that $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $Q(x) \in Y$ such that $\lim_{n\to\infty} 4^n f(\frac{x}{2^n}) = Q(x)$ for all $x \in X$. Fix $x \in X$ and put $p = 0$ in (19). Then we obtain

$$\mu_{4^n f(\frac{x}{2^n})-f(x)}(t) \geq \mu'_{\phi(x,0)}\left(\frac{t}{\sum_{k=0}^{n-1} 4^k\alpha^k}\right)$$

and so, for any $\delta > 0$,

$$\begin{aligned} \mu_{Q(x)-f(x)}(t+\delta) &\geq T\Big(\mu_{Q(x)-4^n f(\frac{x}{2^n})}(\delta), \mu_{4^n f(\frac{x}{2^n})-f(x)}(t)\Big) \\ &\geq T\Big(\mu_{Q(x)-4^n f(\frac{x}{2^n})}(\delta), \mu'_{\phi(x,0)}\Big(\frac{t}{\sum_{k=0}^{n-1} 4^k \alpha^k}\Big)\Big). \end{aligned} \tag{20}$$

Taking $n \to \infty$ in (20), we get

$$\mu_{Q(x)-f(x)}(t+\delta) \geq \mu'_{\phi(x,0)}((1-4\alpha)t). \tag{21}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (21), we get

$$\mu_{Q(x)-f(x)}(t) \geq \mu'_{\phi(x,0)}((1-4\alpha)t).$$

Replacing x and y by $\frac{x}{2^n}$ and $\frac{y}{2^n}$ in (15), respectively, we get

$$\mu_{4^n[2f(\frac{x+y}{2^{n+1}})+2f(\frac{x-y}{2^{n+1}})-\frac{3f(\frac{x}{2^n})}{2}+\frac{f(\frac{-x}{2^n})}{2}-\frac{f(\frac{y}{2^n})}{2}-\frac{f(\frac{-y}{2^n})}{2}]}(t) \geq \mu'_{\phi(\frac{x}{2^n},\frac{y}{2^n})}\Big(\frac{t}{4^n}\Big)$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\phi(\frac{x}{2^n},\frac{y}{2^n})}(\frac{t}{4^n}) = 1$, we conclude that C satisfies (1).

To prove the uniqueness of the quadratic mapping Q, assume that there exists another quadratic mapping $R : X \to Y$ which satisfies (16). Then we have

$$\begin{aligned} \mu_{Q(x)-R(x)}(t) &= \lim_{n\to\infty} \mu_{4^n Q(\frac{x}{2^n})-4^n R(\frac{x}{2^n})}(t) \\ &\geq \lim_{n\to\infty} min\Big\{\mu_{4^n Q(\frac{x}{2^n})-4^n f(\frac{x}{2^n})}\Big(\frac{t}{2}\Big), \mu_{4^n f(\frac{x}{2^n})-2^n R(\frac{x}{2^n})}\Big(\frac{t}{2}\Big)\Big\} \\ &\geq \lim_{n\to\infty} \mu'_{\phi(\frac{x}{2^n},0)}\Big(\frac{(1-4\alpha)t}{2\times 4^n}\Big) \\ &\geq \lim_{n\to\infty} \mu'_{\phi(x,0)}\Big(\frac{(1-4\alpha)t}{2\alpha^n 4^n}\Big). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{(1-4\alpha)t}{2\alpha^n 4^n} = \infty$, we get $\lim_{n\to\infty} \mu'_{\phi(x,0)}(\frac{(1-4\alpha)t}{2\alpha^n 4^n}) = 1$.Therefore, it follows that $\mu_{C(x)-D(x)}(t) = 1$ for all $t > 0$ and so $Q(x) = R(x)$. This completes the proof. $\square$

Corollary 4.2. *Let X be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $r \in (1, +\infty)$ and $z_0 \in Z$. If $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\mu_{2f(\frac{x+y}{2})+2f(\frac{x-y}{2})-\frac{3f(x)}{2}+\frac{f(-x)}{2}-\frac{f(y)}{2}-\frac{f(-y)}{2}}(t) \geq \mu'_{(\|x\|^r+\|y\|^r+\|x\|^{\frac{r}{2}}.\|y\|^{\frac{r}{2}})z_0}(t) \tag{22}$$

for all $x, y \in X$ and $t > 0$. Then, the limit $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for all $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu_{\|x\|^r z_0}\Big(\frac{4^r-4}{4^r}t\Big)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 4^{-r}$ and $\phi : X^2 \to Z$ be a mapping defined by $\phi(x, y) = (\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}.\|y\|^{\frac{r}{2}})z_0$. Then, from Theorem 4.1, the conclusion follows. □

Theorem 4.3. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\phi : X^2 \to Z$ be a function such that there exists $0 < \alpha < 4$ such that*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{23}$$

for all $x \in X$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\phi(2^n x,2^n y)}(4^n t) = 1$, for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (15). Then there exist a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\Big(\frac{4-\alpha}{\alpha}t\Big). \tag{24}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (15), we see that

$$\mu_{4f(\frac{x}{2})-f(x)}(t) \geq \mu'_{\phi(x,0)}(t) \tag{25}$$

for all $x \in X$. Replacing x by $2x$ in (25)

$$\mu_{f(x)-\frac{f(2x)}{4}}(t) \geq \mu'_{\phi(2x,0)}(4t) \geq \mu'_{\phi(x,0)}\Big(\frac{4}{\alpha}t\Big). \tag{26}$$

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let X be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $z_0 \in Z$ and $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\mu_{2f(\frac{x+y}{2})+2f(\frac{x-y}{2})-\frac{3f(x)}{2}+\frac{f(-x)}{2}-\frac{f(y)}{2}-\frac{f(-y)}{2}}(t) \geq \mu_{\delta z_0}(t)$$

for all $x, y \in X$ and $t > 0$. Then, the limit $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for all $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu_{\delta z_0}(3t)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 1$ and $\phi : X^2 \to Z$ be a mapping defined by $\phi(x, y) = \delta z_0$. Then, from Theorem 4.3, the conclusion follows. □

Theorem 4.5. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\phi : X^2 \to Z$ be a function such that there exists $0 < \alpha < 2$ such that*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t)$$

for all $x \in X$ *and* $t > 0$ *and* $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^n t) = 1$ *for all* $x, y \in X$ *and* $t > 0$. *Let* $(Y, \mu, \min)$ *be a complete RN-space. If* $f : X \to Y$ *be an odd mapping satisfying (15). Then there exist a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\phi(x,0)}\Big(\frac{2(2-\alpha)}{\alpha}t\Big),$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = 0$ in (15), we see that

$$\mu_{f(x)-\frac{f(2x)}{2}}(t) \geq \mu'_{\phi(2x,0)}(2^2 t) \tag{27}$$

for all $x \in X$. Replacing x by $2^n x$ in (27) and using (14), we obtain

$$\mu_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}}(t) \geq \mu'_{\phi(2^{n+1}x,0)}(2^{n+2}t) \geq \mu'_{\phi(x,0)}\Big(\frac{2^{n+2}t}{\alpha^{n+1}}\Big).$$

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.6. *Let* X *be a real normed linear space,* $(Z, \mu', \min)$ *be an RN-space and* $(Y, \mu, \min)$ *be a complete RN-space. Let* $r \in (0, 1)$ *and* $z_0 \in Z$. *If* $f : X \to Y$ *be an odd mapping satisfying (22). Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu_{\|x\|^p z_0}((2^{p+2} - 2)t)$$

for all $x \in X$ *and* $t > 0$.

Proof. Let $\alpha = 2^{-p}$ and $\phi : X^2 \to Z$ be a mapping defined by $\phi(x, y) = (\|x\|^p + \|y\|^p + \|x\|^{\frac{p}{2}}.\|y\|^{\frac{p}{2}})z_0$. Then, from Theorem 4.7, the conclusion follows. □

Theorem 4.7. *Let* X *be a real linear space,* $(Z, \mu', \min)$ *be an RN-space and* $\phi : X^2 \to Z$ *be a function such that there exists* $0 < \alpha < \frac{1}{2}$ *such that*

$$\mu'_{\phi(\frac{x}{2},\frac{y}{2})}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{28}$$

for all $x \in X$ *and* $t > 0$ *and* $\lim_{n\to\infty} \mu'_{\phi(\frac{x}{2^n},\frac{y}{2^n})}(\frac{t}{2^n}) = 1$ *for all* $x, y \in X$ *and* $t > 0$. *Let* $(Y, \mu, \min)$ *be a complete RN-space. If* $f : X \to Y$ *be an odd mapping satisfying (15). Then there exist a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\phi(x,0)}(2(1-2\alpha)t).$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = 0$ in (15), we see that

$$\mu_{2f(\frac{x}{2})-f(x)}(t) \geq \mu'_{\phi(x,0)}(2t) \tag{29}$$

for all $x \in X$. Replacing x by $\frac{x}{2^n}$ in (29) and using (28), we obtain

$$\mu_{2^{n+1}f(\frac{x}{2^{n+1}})-2^n f(\frac{x}{2^n})}(t) \geq \mu'_{\phi(x,0)}\Big(\frac{2t}{2^n\alpha^n}\Big).$$

The rest of the proof is similar to the proof of Theorem 4.1. □

Theorem 4.8. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\phi : X^2 \to Z$ be a function such that there exists $0 < \alpha < 2$ such that*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t)$$

for all $x \in X$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\phi(2^n x,2^n y)}(2^n t) = 1$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (15). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{2f(x)-A(x)-Q(x)}(t) \geq T_M\Big(\mu'_{\phi(x,0)}\Big(\frac{2-\alpha}{\alpha}t\Big), \mu'_{\phi(x,0)}\Big(\frac{4-\alpha}{2\alpha}t\Big)\Big).$$

for all $x \in X$ and $t > 0$.

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