Mathematical Analysis and its Contemporary Applications
Volume 7, Issue 2, 2025, 47–55
doi: 10.30495/maca.2025.2054727.1131
ISSN 2716-9898

Lie group analysis of partial differential equation with mixed double fractional derivative

Yapeng Shi, Yuqiang Feng, and Jicheng Yu*

Abstract. Recently, the theory of Lie symmetry group has been extended to some time-fractional partial differential equations with the mixed derivative of Riemann-Liouville time-fractional derivative and integer-order x-derivative. This paper is the first to apply the Lie group method to fractional-order partial differential equations with time-space mixed derivatives. It discusses the Lie symmetry of this nonlinear partial differential equation and obtains the exact solutions of the equation.

Keywords: Lie group analysis, fractional partial differential equations, time-space mixed fractional derivatives, exact solution

2020 Mathematics Subject Classification: Primary: 76M60, 70H33; Secondary: 37C79, 34K37

1. Introduction

Fractional differential equations offer a more versatile and comprehensive framework than traditional integer-order counterparts, enabling the depiction of systems' historical dependence and nonlocal characteristics through the incorporation of fractional derivatives. This feature enhances our ability to address the complexity of dynamic systems. Therefore, fractional differential equations are widely used in physics, engineering, biology, financial mathematics and other disciplines, and become an effective means to accurately describe the dynamic behavior of the system

*Corresponding author

[2, 6, 10, 17]. Over the span of several decades, mathematicians have devised intricate methodologies to tackle both analytical and numerical solutions of fractional differential equations. These encompass diverse techniques such as the Adomian decomposition method [3], finite difference method [14], homotopy perturbation method [15], homotopy analysis method [4], variational iteration method [16], Lie group method [5, 11], invariant subspace method [18], and several others. Notably, the Lie group method emerges as a pivotal tool in the pursuit of exact solutions for differential equations, underscoring its significance among these sophisticated approaches.

Currently, the Lie group method has been successfully applied in identifying group invariance and simplifying a range of nonlinear evolution equations [1, 8, 12, 13, 19, 20, 21]. Nevertheless, the fractional partial differential equations examined by these researchers primarily belong to the categories of pure time fractional order, pure space fractional order, or a combination of both. Recently, Professor Zhang's team studied invariant analysis of the time-fractional b-family peakon equations [22] and Lie symmetries of the time-fractional (2+1)-dimensional Hirota–Satsuma–Ito equations [23]. The specific forms of these two equations are as follows:

$$\partial_t^\alpha u - \partial_t^\alpha (u_{xx}) + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \tag{1.1}$$

$$\begin{cases} u_y = v_x, \\ \partial_t^\alpha u = \omega_x, \\ \partial_t^\alpha v + \partial_t^\alpha (u_{xx}) + 3(u\omega)_x + \gamma u_x = 0. \end{cases} \tag{1.2}$$

By observing equations (1.1) and (1.2), it can be found that their equations contain mixed derivatives of fractional order and integer order. The Lie group method is further extended. Nevertheless, the realm of Lie symmetry analysis for partial differential equations involving mixed time-fractional and spatial-fractional derivatives remains largely unexplored. In addition, when the spatial fractional derivative and the temporal fractional derivative function are simultaneously, the mixed derivative may have different physical meanings.

In this paper, we mainly study a variant of the fractional partial differential equation mentioned in [9], whose specific form is as follows

$$\frac{\partial^\alpha f}{\partial t^\alpha} = (\frac{\partial^\beta f}{\partial x^\beta})^2 - f(\frac{\partial^\beta f}{\partial x^\beta}) + \frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\beta}{\partial x^\beta} f), \alpha, \beta \in (0,1), \tag{1.3}$$

where f is a function of t and x, and $\frac{\partial^\alpha f}{\partial t^\alpha}$ and $\frac{\partial^\beta f}{\partial x^\beta}$ represent Riemann-Liouville fractional derivatives, respectively. The fractional derivative is defined [10] as follows:

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-s)^{n-\alpha-1} f(s,x)ds, 0 \le n-1 < \alpha < n, \tag{1.4}$$

$$\frac{\partial^\beta f}{\partial x^\beta} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_0^x (x-s)^{n-\beta-1} f(t,s)ds, 0 \le n-1 < \beta < n, \tag{1.5}$$

where n is a positive integer and $\Gamma(y) = \int_0^\infty e^{-y}t^{y-1}dt$ is a gamma function. We observe that equation (1.3) incorporates mixed derivatives of double fractional order, a realm that has yet to be explored through the lens of Lie group theory in existing literature. Consequently, the cornerstone contribution of this paper lies in demonstrating the efficacy of the Lie group method in tackling this novel class of double fractional mixed derivative differential equations.

The subsequent sections of this article are structured as follows. Section 2 delves into the challenges encountered when applying Lie group methods directly and outlines our innovative strategies to overcome these obstacles. Concluding the article, we provide a comprehensive summary of our research findings.

2. Lie Group method

Let us posit that equation (1.4) admits a one-parameter Lie transformation group of the following form

$$\begin{aligned} t^* &= t + \varepsilon\tau(t,x,f) + O(\varepsilon^2), \\ x^* &= x + \varepsilon\xi(t,x,f) + O(\varepsilon^2), \\ f^* &= f + \varepsilon\eta(t,x,f) + O(\varepsilon^2), \end{aligned} \tag{2.1}$$

where τ, ξ and η are infinitesimals. The infinitesimal generator pertinent to the extension is denoted by

$$\operatorname{Pr} V = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial f} + \eta_t^\alpha\frac{\partial}{\partial(\partial_t^\alpha f)} + \eta_x^\beta\frac{\partial}{\partial(\partial_x^\beta f)} + \eta_{t,x}^{\alpha,\beta}\frac{\partial}{\partial(\partial_t^\alpha\partial_x^\beta f)},$$

where $\eta_t^\alpha, \eta_x^\beta$ and $\eta_{t,x}^{\alpha,\beta}$ are extended infinitesimals and $\partial_t^\alpha f = \frac{\partial^\alpha f}{\partial t^\alpha}$. While the explicit forms of η_t^α and η_x^β are well-established (as detailed in [7]), the corresponding expression for the infinitesimal $\eta_{t,x}^{\alpha,\beta}$ remains elusive. The challenge in deriving this expression poses a significant obstacle to the continued application of the Lie group method. In the following, we elucidate our approach to overcoming this problem.

First, we introduce a new dependent variable $g(t,x)$ and define it as $g(t,x) = \frac{\partial^\beta f}{\partial x^\beta}$, then the original equation (1.3) becomes

$$\begin{cases} \frac{\partial^\alpha f}{\partial t^\alpha} = g^2 - fg + \frac{\partial^\alpha g}{\partial t^\alpha}, \\ \frac{\partial^\beta f}{\partial x^\beta} = g. \end{cases} \tag{2.2}$$

Next, we can use the general Lie symmetry framework of fractional partial differential equations to obtain the symmetry of equation (2.2), and then obtain some exact solutions of the original equation (1.3), the steps are as follows.

Utilizing the Lie group method, we can define a one-parameter Lie group that operates on an infinitesimal transformation, encompassing both an independent variable and a dependent variable, denoted as

$$\begin{aligned} t^* &= t + \varepsilon\tau(t,x,f,g) + O(\varepsilon^2),\\ x^* &= x + \varepsilon\xi(t,x,f,g) + O(\varepsilon^2),\\ f^* &= f + \varepsilon\eta(t,x,f,g) + O(\varepsilon^2),\\ g* &= g + \varepsilon\theta(t,x,f,g) + O(\varepsilon^2), \end{aligned} \tag{2.3}$$

where $t^*, x^*, f^*, g^*, \tau, \xi, \eta$ and θ represent real functions that depend on t, x, f and g. The ε is a parameter of an infinitesimal transformation, and $O(\varepsilon^2)$ denotes a term that is infinitesimal in nature, being of the same order as ε^2.

The Lie point symmetric generator associated with the one-parameter Lie group defined in (2.3) can be succinctly represented in vector form as

$$V = \tau(t,x,f,g)\frac{\partial}{\partial t} + \xi(t,x,f,g)\frac{\partial}{\partial x} + \eta(t,x,f,g)\frac{\partial}{\partial f} + \theta(t,x,f,g)\frac{\partial}{\partial g}, \tag{2.4}$$

where the functions τ, ξ, f and g must satisfy the conditions

$$\begin{cases} \operatorname{Pr} V(E_1)|_{E_1=0} = 0,\\ \operatorname{Pr} V(E_2)|_{E_2=0} = 0, \end{cases} \tag{2.5}$$

with

$$\begin{cases} E_1 = \frac{\partial^\alpha f}{\partial t^\alpha} - g^2 + fg - \frac{\partial^\alpha g}{\partial t^\alpha},\\ E_2 = \frac{\partial^\beta f}{\partial x^\beta} - g. \end{cases} \tag{2.6}$$

The notation $\operatorname{Pr} V$ represents the extension of V, which is defined as

$$\operatorname{Pr} V = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial f} + \theta\frac{\partial}{\partial g} + \eta_t^\alpha\frac{\partial}{\partial(\partial_t^\alpha f)} + \theta_t^\alpha\frac{\partial}{\partial(\partial_t^\alpha g)} + \eta_x^\beta\frac{\partial}{\partial(\partial_x^\beta f)}, \tag{2.7}$$

$$\begin{aligned} \eta_t^\alpha =& \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha f}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_f}{\partial t^\alpha} + \left(\eta_g\frac{\partial^\alpha g}{\partial t^\alpha} - g\frac{\partial^\alpha \eta_g}{\partial t^\alpha}\right)\\ &+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_f}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(f) + \mu_1 + \mu_2\\ &+ \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_g}{\partial t^n}D_t^{\alpha-n}(g) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(f_x), \end{aligned} \tag{2.8}$$

with

$$\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-f)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m f^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial f^k},$$

$$\mu_2 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-g)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m g^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial g^k},$$

$$\begin{aligned}\theta_t^\alpha =&\frac{\partial^\alpha \theta}{\partial t^\alpha} + (\theta_g - \alpha D_t(\tau))\frac{\partial^\alpha g}{\partial t^\alpha} - g\frac{\partial^\alpha \theta_g}{\partial t^\alpha} + \left(\theta_f\frac{\partial^\alpha f}{\partial t^\alpha} - f\frac{\partial^\alpha \theta_f}{\partial t^\alpha}\right)\\ &+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \theta_g}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(g) + \mu_3 + \mu_4 \\ &+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\theta_f}{\partial t^n}D_t^{\alpha-n}(f) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(g_x),\end{aligned} \tag{2.9}$$

with

$$\mu_3 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-f)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m f^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\theta}{\partial t^{n-m}\partial f^k},$$
$$\mu_4 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-g)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m g^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\theta}{\partial t^{n-m}\partial g^k},$$

and

$$\begin{aligned}\eta_x^\beta =&\frac{\partial^\beta \eta}{\partial x^\beta} + (\eta_f - \beta D_x(\xi))\frac{\partial^\beta f}{\partial x^\beta} - f\frac{\partial^\beta \eta_f}{\partial x^\beta} + \left(\eta_g\frac{\partial^\beta g}{\partial x^\beta} - g\frac{\partial^\beta \eta_g}{\partial x^\beta}\right)\\ &+\sum_{n=1}^{\infty}\left[\binom{\beta}{n}\frac{\partial^n \eta_f}{\partial x^n} - \binom{\beta}{n+1}D_x^{n+1}(\xi)\right]D_x^{\beta-n}(f) + \mu_5 + \mu_6 \\ &+\sum_{n=1}^{\infty}\binom{\beta}{n}\frac{\partial^n\eta_g}{\partial x^n}D_x^{\beta-n}(g) - \sum_{n=1}^{\infty}\binom{\beta}{n}D_x^n(\tau)D_x^{\beta-n}(f_t),\end{aligned} \tag{2.10}$$

with

$$\mu_5 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{x^{n-\beta}(-f)^r}{k!\Gamma(n+1-\beta)}\frac{\partial^m f^{k-r}}{\partial x^m}\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial f^k},$$
$$\mu_6 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{x^{n-\beta}(-g)^r}{k!\Gamma(n+1-\beta)}\frac{\partial^m g^{k-r}}{\partial x^m}\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial g^k},$$

where D_t and D_x denote the total derivative operators, while D_t^α represents the fractional total derivative operator, as per the reference [11]. Additionally, $\frac{\partial}{\partial t}$ signifies the first-order partial derivative with respect to the variable t. Given that the fractional partial derivative in equations (1.4) and (1.5) has a lower bound of zero, ensuring that $t = 0$ and $x = 0$ remain invariant under the one-parameter Lie transformation group (2.3) necessitates $\tau(t, x, f, g)|_{t=0} = 0$ and $\xi(t, x, f, g)|_{x=0} = 0$. Furthermore, analyzing the expressions for $\mu_1, \mu_2, ..., \mu_6$, if the infinitesimals η, θ exhibit a linear relationship with the variables f and g, it follows that $\mu_1 = \mu_2 = \cdots = \mu_6 = 0$,ultimately leading to the condition, $\frac{\partial^2\eta}{\partial f^2} = \frac{\partial^2\eta}{\partial g^2} = \frac{\partial^2\theta}{\partial f^2} = \frac{\partial^2\theta}{\partial g^2} = 0$.

Upon expanding expression (2.5) and isolating the derivatives of u and v, we arrive at an overdetermined system involving the variables τ, ξ, η and θ. By meticulously solving this intricate system, we confirm

$$\tau = -\frac{C_1}{\alpha}t, \xi = 0, \eta = C_1 f, \theta = C_1 g. \tag{2.11}$$

Consequently, the system (2.2) accommodates an infinitesimal generator characterized by

$$V = -\frac{t}{\alpha}\frac{\partial}{\partial t} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \tag{2.12}$$

leading to a corresponding characteristic equation

$$\frac{dt}{-\frac{t}{\alpha}} = \frac{df}{f} = \frac{dg}{g}.$$

Upon solving the characteristic equation, we procure invariant solutions expressed as $f = t^{-\alpha}\phi(z)$ and $g = t^{-\alpha}\psi(z)$, where $z = x$ serves as the invariant. By substituting these invariant solutions into the system of equations (2.2), we can effectively transform the system into a simplified form comprising fractional ordinary differential equations, i.e.

$$\begin{cases} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\phi(z) = \psi^2(z) - \phi(z)\psi(z) + \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\psi(z), \\ \frac{d^\beta \phi(z)}{dz^\beta} = \psi(z). \end{cases}$$

Employing the Laplace transform method to solve the fractional ordinary differential equation, we obtain two distinct sets of solutions as follows

$$\phi(z) = \psi(z) = kz^{\beta-1}E_{\beta,\beta}(z^\beta)$$

and

$$\phi(z) = -\frac{\Gamma(1-\alpha)}{\Gamma(1+\beta)\Gamma(1-2\alpha)}z^\beta, \psi(z) = -\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)},$$

where $k = \phi(0)$, $E_{\beta,\beta}(z)$ denotes the double Mittag function, which is defined as follows:

$$E_{\beta,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k\beta}}{\Gamma(k\beta+\beta)}.$$

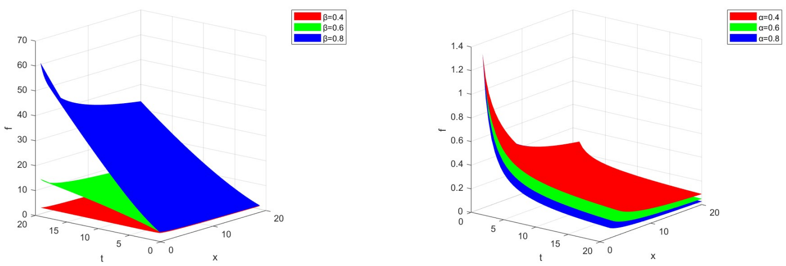
The exact solutions of the original equation (1.3) are therefore

$$f = kt^{-\alpha}x^{\beta-1}E_{\beta,\beta}(x^\beta), \tag{2.13}$$

and

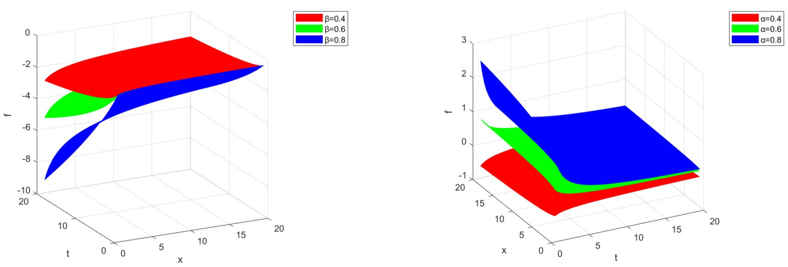
$$f = -\frac{\Gamma(1-\alpha)}{\Gamma(1+\beta)\Gamma(1-2\alpha)}t^{-\alpha}x^\beta. \tag{2.14}$$

To facilitate a more intuitive grasp of the solutions presented in (2.13) and (2.14), accompanying Figures 2 and 2 are provided for illustration.



(a) $\alpha = 0.2$ (b) $\beta = 0.2$

FIGURE 1. Dynamical profiles of the truncated solution (2.13) with $k = 1$.



(a) $\alpha = 0.2$ (b) $\beta = 0.2$

FIGURE 2. Dynamical profiles of solution (2.14).

3. Conclusion

In this paper, we examined fractional partial differential equations with mixed double fractional derivative types, as exemplified by equation (1.3). For the first time, we utilized the Lie group approach to derive the permissible symmetry generator for such fractional differential equations, achieving a symmetry reduction of the given equation. Subsequently, we procured the precise solutions to the equation through the application of the Laplace transform method, thereby validating the efficacy of the Lie group methodology.

Acknowledgment

The authors are deeply indebted to the referees for providing constructive comments and help in improving the contents of this article.

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Received: March 2025
Accepted: May 2025

School of Science, Wuhan University of Science and Technology, Wuhan 430081, Hubei, China.
Email address: sypwust@163.com

School of Science, Wuhan University of Science and Technology, Wuhan 430081, Hubei, China

Hubei Province Key Laboratory of Systems Science in Metallurgical Process, Wuhan 430081, Hubei, China
Email address: yqfeng6@126.com

School of Science, Wuhan University of Science and Technology, Wuhan 430081, Hubei, China.
Email address: yjicheng@126.com