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Orthogonality of algebraic elementary operators when their numerical ranges are spheroidal

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Abstract. Characterizations involving algebraic elementary operators have been done over the years, for instance, orthogonality when the operators are induced by other different types of transformations. In particular, algebraic elementary operators induced by norm-attainable maps have not been characterized in terms of orthogonality when their numerical ranges have spheroid boundaries. In this note, we characterize algebraic elementary operators in terms of Birkhoff-James orthogonality when they are induced by norm-attainable maps and the boundaries of their numerical ranges are spheroidal in shape. We show that under the pheroidicity criterion for the numerical range boundary, various types of algebraic elementary operators satisfy Birkhoff-James orthogonality.

Keywords: Numerical range, Elementary operator, Orthogonal projection

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1. Introduction

Studies on operators and their properties have been carried out over decades with very fruitful results obtained (see [59]-[2] and the references therein for details). This work is dedicated to characterizing algebraic elementary operators(AEO) in terms of Birkhoff-James Orthogonality (BJO) [7]. We first consider the orthogonality aspect

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in general, then we narrow down to BJO. From the work of [9], various notions of orthogonality are given. Many researchers have henceforth studied these different notions of orthogonality in different spaces (see [30]-[34] and the references therein for details).

Choi and Kim [17] studied the norm-attainment of multilinear mappings in Hilbert spaces. They showed that the denseness of operators does not hold when the domain space is not null for an arbitrary range. This was a characterization in a general set-up about the identity operator [36].

Theorem 1.1. [17] *Every multilinear map attains its norm if it is the identity.*

Theorem 1.1 characterizes norm-attainment in terms of the identity. In our work, we consider AEO induced by other nontrivial operators like the unitary and isometries. Since multilinear operators are sometimes products of other operators, we found it interesting to study the product operator of other operators as suggested in [35].

At this juncture, let's put our effort into reviewing BJO. We begin with the following proposition.

Proposition 1.2. [65] *Every operator on a Hilbert space which satisfies the BJO condition is approximately BJ-orthogonal. However, the converse is not true.*

Proposition 1.2 gives the relationship between the BJO condition and the approximate BJO. It confirms that every operator which satisfies the BJO condition is approximately BJ-orthogonal. It further shows that the converse is not true, as confirmed in [38], [39] and [37]. Approximate orthogonality is weaker but an important notion of orthogonality.

Miguel et al. [46] provided characterization of BJO and used the approach of the numerical range to advance this study. They also studied BJO of the duals of operators in Banach spaces. They used the techniques of the tensors [47], norm-attainability [42], and the maximum values of the numerical range to advance their study [44]. We seek to establish BJO conditions on other nontrivial operators on Banach spaces by employing the proof due to Bhatia to advance our establishments.

In [10], Bhatacharya and Priyanka explained the BJO in C^*-modules in tensor spaces. They proved the infinite-dimensional case in AEO that achieves BJO in terms of sequences that converge to the operator norms. They also proved the real Bhatia-Semrl theorem, where they asserted that if $A, B \in M_{(n)}$, then picking $t \in \mathbb{R}$, we have that $\|A+tB\| \geq \|A\|$. They proved that this is only possible if the operator norm and the real part of the inner product are both zero [45]. They showed that this assertion is always true, but the reverse fails the affirmative test [43]. Morrel proved in [48], orthogonality of EO via tensor products of vectors in Banach spaces. They made an attempt to answer the question that if $x_1 \perp_{BJ} x_2$ and $y_1 \perp_{BJ} y_2$, then is $(x_1 \otimes y_1) \perp_{BJ} (x_2 \otimes y_2)$? However, in our work, we employed the techniques of the tensor products to establish BJO conditions on Banach spaces. We used tensor products as a tool, but not in the perturbation of different classes. We also restricted our work to the class of Banach spaces as opposed to the other classes advanced in the work of [12].

Bajracharya and Ojha [5] established BJO via the approach of subdifferential of continuous linear functions, which are representations of AEOs [49]. They used this approach to advance on the Bhatia-Semrl property [50] and the composition of the subdifferential of the norms of functions $f(t) = \|A + tB\|$ to establish BJO as seen in [52], [53] and [51]. They used the composition of functions to advance their characterization of BJO viathe subdifferential of functions. Finally, they discussed the inclusion of semidefinite operators in the field of values of functions. In our work, we established BJO conditions using approaches such as numerical ranges of operators, tensor products of Banach spaces, strong orthogonality and approximate orthogonality among other techniques [55]. We established these orthogonality conditions on operators that attain norms at given points in Banach spaces.

Priyanka and Sushil [54] studied the cases of orthogonality for EOs in such special cases when it is symmetric and when it is both right and left additive, as well as establishing BJO in extreme points of operators, among others. They extended the study of BJO and its connections to hyperplanes and support hyperplanes where they claimed that every hyperplane is orthogonal to its subspace [55]. To prove this assertion, they employed the technique of the Hahn-Banach theorem [55] to offer a proof of the claim. They connected the claim to the symmetry of BJO and confirmed that indeed, BJO is right additive provided the existence of a norm which is Gateaux differentiable at each non-zero point [57]. This work goes ahead to show right-additivity in other spaces [56]. However, in our work, we established BJO from the premise that BJO is homogenous, nonsymmetric and non-additive, both left and right, in normed spaces. We utilized the techniques used in their proofs to help us develop our results in Banach spaces.

Saidi [58] also considered the study of symmetry in Banach spaces. The author gave examples of operators that achieve symmetry in Banach spaces and proved left-symmetry in such spaces [61]. Our work established BJO in a general sense in some subclasses of Banach algebras, also studied in [60]. We also concentrated on the homogeneity property [63] of BJO in normed spaces.

Chmielnski in [16] studied approximate BJO and the approximate symmetry of BJO. Their result was based on the non-symmetry of BJO in normed spaces. They showed that orthogonality due to Birkhoff-James is symmetric up to some point in the dual space of operators. This discussion enabled them to extend the approximate symmetry of BJO into other spaces [1]. However, our study concentrated on BJO with an extension into non-symmetric approximate BJO as a pre-condition given in [14] and also in [15] to achieve our establishment of the orthogonality conditions in

Banach spaces. We also, upon relaxation of some conditions on approximate BJO, generalized BJO whenever the conditions on symmetry [18] are met.

Johnstone et al [40] characterized orthogonality of self-adjoint operators and their translates into other operators. They went ahead to explain numerous consequences of BJO in Hermitian matrices and the applications of such to quantum information theories [19], where matrices are positive semidefinite of trace one [20]. They extended this study to the determination of the types of operators which are BJO to diagonal matrices [32]. The foregoing assertions restrict the orthogonality results to when the operators involved are self-adjoint and semidefinite [21]. In our work, we established BJO in Banach spaces for a range of other classes of operators and determined under what conditions such operators are BJO. We also established the BJO results in the operator norms in the Banach spaces.

Arambasic and Rajna [4], discussed BJO in Hilbert C^*-modules. They introduced the notion of orthogonality, which they called strong BJ-orthogonality [22]. Our work, however, characterized BJO in a general Banach space setting. The work of [31] obtained a complete characterization of the approximate BJO of operators, which was an advancement on the study of the approximate BJO of compact linear operators [24]. In our study, we also investigated classes of operators like unitary [23], normal [25], Hermitian [26], and their compositions in establishing BJO. We restricted our study to well-defined spaces and considered orthogonality via the semi-inner-product [27]. Researchers in [30], [28] and [29] achieved their results through the concept of the norming sequence and strict convexity of operators [33]. The foregoing result pegged BJO to the strict convexity of compact operators.

Authors in [20] established BJO by finding the symmetries for l_p-spaces onto themselves. They then found all the isometries of these spaces. They established the BJO results by use of filters and ultrafilters [40] as tools to achieve the BJO results. These assertions by [6] and others characterized BJO in terms of the symmetries, most specifically, the left symmetry [49]. However, in Banach spaces, we based our study on the premise that BJO is not symmetric, both right and left. It is noted that the study by other mathematicians in [32] is a motivation to our study, and as a result, we also delved into the study of spaces where BJO is also right symmetric, and even both left and right symmetric [3].

In [57], the authors characterized BJO in Sch$\ddot{a}$ffer unitary dilations of the operators T and the commutant of the operator ST. We found it interesting, in our work, to investigate the outcome on BJO if the operators S and T do not commute. We also found it interesting to investigate the outcome of BJO on other classes of operators and, most specifically, the behavior of BJO on unitary operators.

In [54], the authors established BJO whenever the operators are unitary dilations or isometries, that is, U_T and U_A respectively and showed that U_T is BJ-orthogonal to U_A. In our work, we considered the basic operators such as unitary operators, normal operators, perturbation in operator theory, among other operators, in establishing BJO in Banach spaces. In [27], the authors established BJO whenever an operator undergoes self-perturbation for an integer for any positive integer. For our work, this form of perturbation created an interest and led us to check if BJO still holds whenever the self-perturbation is taken for any negative integers or even when the integer is strictly above zero.

In [42], the authors characterized BJO via ρ-dilations. They established this result from the premise of norming sequences of both the operators A and T, provided the operators are dilations. Our work, however, concentrated on different classes of the operators A and T and their perturbations with other classes of operators to establish BJO in Banach spaces. We also investigated operator sequences for BJO as opposed to norming sequences as a tool to establish our results.

In [5], the authors characterized BJO and its positive symmetry on the closure of the convex hull of norm sequences. However, our work sought to establish BJO on numerical ranges of operators and their sequences, which has a close connection with the convex hull of operator sequences. Also, in [62], the authors established BJO with both left and right symmetry as a property in terms of sequences and, most specifically, the zero sequence. On the other hand, we established BJO from the premise ofthe non-symmetry of BJO. However, we also investigated the special classes of operators and operator sequences when symmetry holds as a condition of BJO. We also contributed to such classes by relaxing the conditions of sequences to establish BJO in Banach spaces.

Several authors have studied some related properties in orthogonality between different notions of orthogonality and explored different applications for such relationship between the different notions of orthogonality. Our work, however, established BJO conditions as an isolated notion of orthogonality and concentrated on SRB-spaces as our target space of operation. We also established the orthogonality conditions in the classes of operators, such as the approximate semi-orthogonal operators.

In [24], the author studied operators preserving James' orthogonality and characterized isometries and co-isometries in bounded linear operators in terms of orthogonality. Due to this study, the author obtained conjugate linear mappings and surjective maps that preserve James orthogonality in any direction. They proved the orthogonality results via the numerical range approach, where they restricted their approach to the interior points of self-adjoint unitary operators, where such operators attain their norms on unit vectors. They also established the orthogonality conditions through the orthogonal projections of subspaces of operators and asserted that the orthogonality due to James is only preserved provided the numerical range of the operators involved is an ellipsoid with zero as their interior points. However, our study established the orthogonality results on the basis of orthogonality due to

Birkhoff-James as a notion. We also restricted our study to polar decompositions of operators and the index conjugations of elements in the closure of SRB-spaces.

The authors in [10] investigated BJO conditions on operators in Hilbert C^*-modules, where they obtained the necessary conditions on given elements in modules such that the elements achieve the orthogonality property of symmetry. They established these orthogonality conditions through the minimality of the inner product spaces and projections of one-dimensional Hilbert spaces. The authors also described the conditions under which orthogonality, and most specifically, BJO, is left or right additive through the invertibility of nonzero elements of subspaces. The authors also studied strong BJO and proved its symmetric relation in Hilbert modules of subspaces through the isomorphism of the C^*-algebras to the space of all complex numbers. On the other hand, our work utilized the involution property of the C^*-algebras to establish the BJO conditions for operators in Banach spaces. We also established this orthogonality via the subclass of approximately semi-orthogonal operators in SRB classes.

The study by [23] introduced a new concept of orthogonality called the approximate ρ_*-orthogonality. This notion of orthogonality, they claimed, preserves mapping between normed spaces and that for a norm which is Gâteaux differentiable, the approximate ρ_*-orthogonality coincides with the approximate BJO in normed spaces. They finally showed that every approximate ρ_*-orthogonality mapping is a multiplicity by a scalar of almost isometry. These properties that preserve approximate BJO were later verified by the authors of the work in [63]. In our work, we characterized approximate orthogonality in Hilbert spaces and in infinite Banach spaces. We finally established these conditions in projections different from the normal orthogonal projections in Banach spaces.

In [60], the authors, when they studied orthogonality to matrix subspaces, obtained a necessity for an operator to be BJO to any subspace of a complex Hilbert space under an inner product. They achieved this orthogonality result by basing their argument on the existence of a density operator of complex rank. The authors constructed the proof of their finding through the concept of the subdifferential of composition maps. Through this study, the authors proved an expression for a distance formula between an operator from any unital C^*-subalgebras of all complex Hilbert spaces. In our work, we established the orthogonality conditions on finite-dimensional Hilbert spaces and on positive semidefinite operators. We also used the concept of tensor norms to achieve our establishment of BJO conditions on Banach spaces.

In [3], the author characterized conditions in triangle inequality that lead to some equivalence relations in terms of states of C^*-algebras. They extended this study into establishing equivalent statements and established when the equality conditions hold in the triangle inequality for the cases of all operators that are adjointable.

In [6], the author generalized the concept of BJO of operators in Hilbert spaces through the concept of the semi-inner product. The author introduced a new relation in orthogonality called the T-Birkhoff-James orthogonality, for T being an operator, where they showed that any two operators are T-Birkhoff-James orthogonal and that even for this case of orthogonality, the property of homogeneity still holds. The author then extended the concept of T-Birkhoff-James orthogonality into developing some formulae for T-distance of operators to the class of some constants of other cases of semi-Hilbertian spaces. In our work, we established BJO conditions for operators in complex planes. We also proved the homogeneity property of the BJO and used this particular property in establishing BJO conditions for other subclasses of operators on Banach spaces.

In [8], the authors considered three aspects of orthogonality in Hilbert C^*-modules concerning some C^*-algebras. They considered the BJO, the strong BJO and finally the orthogonality in relation to an $\mathcal{A}$-valued inner product space. They characterized orthogonality in some classes of Hilbert C^*-modules where there were possibilities of any of the two concepts of orthogonality coinciding. Moreover, the authors characterized the subclasses in normed spaces equipped with various orthogonalities, where a given type of orthogonality is an implication of the other. More specifically, the authors established orthogonality conditions in classes of Hilbert C^*-modules where BJO implies strong BJO and in the cases where strong BJO is an implication of orthogonality with respect to the IPS. In our work, however, we established the BJO conditions in restriction to the SRB-classes.

In [13], in the study of operators reversing orthogonality in normed spaces, the author dealt with linear operators on normed spaces and considered those properties under which orthogonality is reversed. The author considered all those operators in normed spaces where the BJO preserving property is upheld. The author, in particular, asserted that operators which are nonzero and linear at the same time, preserve orthogonality and are also injective. In this study, the author appreciated the non-symmetry of orthogonality but asserted that it would be interesting if they considered the problem of reversing BJO. For all operators that obey linear similarity, the author established that the reversing and preserving BJO properties have an equivalence relation. However, the author also established that there exist spaces that fail to admit nontrivial mappings which reverse BJO. Such spaces where the property of reversing BJO fail to hold were established in this work of which the author called the two-dimensional normed spaces, more specifically, the Minkowski planes. In our work, however, we established the orthogonality conditions in the SRB spaces. We also dealt with conditions that preserve orthogonality as opposed to such conditions that reverse BJO. It is evident from the introduction that BJO of AEO has not been considered more so when the numerical ranges of the AEO are spheroidal. Therefore, this forms the gist of this note.

2. Preliminaries

For successful understanding of this work, the following concepts are useful

Definition 2.1. ([1]) A mapping G is an elementary operator if its formation is $G(R) = \sum_{i=1}^{n} M_i R M_i$, for all R in an algebra $\mathcal{A}$, where M_i, B_i are fixed in $\mathcal{A}$. The mapping G is said to be algebraic if for some nonzero polynomial p, we have $p(G) = 0$. We have the left multiplication operator, right multiplication operator, inner derivation, generalized derivation, basic elementary operator and Jordan elementary operator as examples of elementary operators.

Definition 2.2. ([30]) The numerical range of a map F is given by:
$W(F) = \{\langle Tz, z\rangle : z \in \mathcal{H}, \text{where z is a unit vector}\}$.

Definition 2.3. ([1, Definition 2.9]) Let $\mathcal{B}$ be a Banach space. For any $\epsilon \in [0, 1)$, x is said to be Birkhoff-James ϵ- orthogonal to y denoted by $x \perp_{BJ}^{\epsilon} y$ if $\|x + \lambda y\| \geq \sqrt{1-\epsilon^2}\|x\|$, for all $\lambda \in \mathbb{C}$.

3. Main Results

In this section, we establish BJO conditions via approximate orthogonality. We begin with a result on homogeneity of BJO.

Proposition 3.1. *Let $\overline{\mathcal{B}}$ be a smooth reflexive Banach (SRB) space of AEOs. Then for $A, B \in \overline{\mathcal{B}}$, $A \perp_{BJ}^{\epsilon} B$ if $\forall \lambda \in \mathbb{C}$,*

$$\|A + \lambda B\| \geq \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|, \forall \epsilon \in [0, 1). \tag{1}$$

Moreover, $A \perp_{BJ}^{\epsilon} B$ implies that $\mu A \perp_{BJ}^{\epsilon} \omega B$ for any $\mu, \omega \in \mathbb{C}$.

Proof. The case of $\mu = 0$ is obvious and so we omit. Let $\mu \neq 0$, then we have that

$$\begin{aligned} \|\mu A + \lambda\omega B\|^2 &= |\mu|^2\|A + \lambda\frac{\omega}{\mu}B\|^2 \\ &\geq |\mu|^2(\|A\|^2 - 2\epsilon\|A\|\|\lambda\frac{\omega}{\mu}B\|) \\ &= \|\mu A\|^2 - 2\epsilon\|\mu A\|\|\lambda\omega B\|. \end{aligned}$$

So,

$$\|\mu A + \lambda\omega B\|^2 \geq \|\mu A\|^2 - 2\epsilon\|\mu A\|\|\lambda\omega B\|. \tag{2}$$

Hence, $\perp_{BJ}^{\epsilon}$ is homogenous. If we put $\mu = 1$ and $\omega = 1$ in inequality (2), we obtain the inequality (1). □

Next, we give the homogeneity in a more general context.

Proposition 3.2. *Let $\overline{\mathcal{B}}$ be a SRB-space of AEOs. Then for $A, B \in \overline{\mathcal{B}}$, $A \perp_{BJ}^{\epsilon} B$ if for all $\lambda \in \mathbb{C}$,*

$$\|A + \lambda B\| \geq \sqrt{1-\epsilon^2}\|A\|, \forall \epsilon \in [0, 1). \tag{3}$$

Moreover, $A \perp_{BJ}^{\epsilon} B$ implies that $\mu A \perp_{BJ}^{\epsilon} \omega B$ for any $\mu, \omega \in \mathbb{C}$.

Proof. The case of $\mu = 0$ is obvious and therefore we leave it out. Let $\mu \neq 0$, then we have that;

$$\begin{aligned} \|\mu A + \lambda\omega B\|^2 &= |\mu|^2\|A + \lambda\frac{\omega}{\mu}B\|^2 \\ &\geq |\mu|^2(\|A\|^2 - \sqrt{1-\epsilon^2}\|A\|\|\lambda\frac{\omega}{\mu}B\|) \\ &= \|\mu A\|^2 - \sqrt{1-\epsilon^2}\|\mu A\|\|\lambda\omega B\|. \end{aligned}$$

So,

$$\|\mu A + \lambda\omega B\|^2 \geq \|\mu A\|^2 - \sqrt{1-\epsilon^2}\|\mu A\|\|\lambda\omega B\|. \tag{4}$$

Hence, $\perp_{BJ}^{\epsilon}$ is homogenous. If we put $\mu = 1$ and $\omega = 1$ in inequality (4), we obtain the inequality (3). □

Proposition 3.3. *Let $\overline{\mathcal{B}}$ be a SRB-space of AEOs. Then, for $A, B \in \overline{\mathcal{B}}$, $A \perp_{BJ}^{\epsilon} B$ if for all $\lambda \in \mathbb{C}$,*

$$\|A + \lambda B\| \geq \sqrt{1-\epsilon^n}\|A\|, \forall \epsilon \in [0, 1),\ \forall n \in \mathbb{N}. \tag{5}$$

Moreover, $A \perp_{BJ}^{\epsilon} B$ implies that $\mu A \perp_{BJ}^{\epsilon} \omega B$ for any $\mu, \omega \in \mathbb{C}$.

Proof. The case of $\mu = 0$ is obvious and therefore we leave it out. Let $\mu \neq 0$, then we have that;

$$\begin{aligned} \|\mu A + \lambda\omega B\|^2 &= |\mu|^2\|A + \lambda\frac{\omega}{\mu}B\|^2 \\ &\geq |\mu|^2(\|A\|^2 - \sqrt{1-\epsilon^n}\|A\|\|\lambda\frac{\omega}{\mu}B\|) \\ &= \|\mu A\|^2 - \sqrt{1-\epsilon^n}\|\mu A\|\|\lambda\omega B\|. \end{aligned}$$

So,

$$\|\mu A + \lambda\omega B\|^2 \geq \|\mu A\|^2 - \sqrt{1-\epsilon^n}\|\mu A\|\|\lambda\omega B\|. \tag{6}$$

Hence, $\perp_{BJ}^{\epsilon}$ is homogenous. If we put $\mu = 1$ and $\omega = 1$ in inequality (6), we obtain the inequality (5). □

Lemma 3.4. *Let $\overline{\mathcal{B}}$ be a SRB-space of AEOs and $A, B \in \overline{\mathcal{B}}$. Then $A \perp_{BJ}^{\epsilon} B$ on $\overline{\mathcal{B}}$ if A and B are approximately semi-orthogonal.*

PROOF. Let A and B be approximately semi-orthogonal, i.e, $\langle B, A\rangle = \epsilon\|A\|\|B\|$. Let $l \in [0,1]$ for some $\theta \in \{-\pi, \pi\}$. We have that $\langle B, A\rangle = l\epsilon\|A\|\|B\|e^{i\theta}$. Let $\lambda \in \mathbb{C}$ be given arbitrarily, then;

$$\begin{aligned} \|A+\lambda B\|\|A\| &\geq |\langle A+\lambda B, A\rangle| \\ &= |\|A\|^2 + \lambda\langle B, A\rangle| \\ &= |\|A\|^2 + l\epsilon\|A\|\|B\|\lambda e^{i\theta}|. \end{aligned}$$

Therefore;

$$\begin{aligned} \|A+\lambda B\| &\geq |\|A\| + l\epsilon\|B\|\lambda e^{i\theta}| \\ &= |\|A\| + l\epsilon\|B\|\mathbb{R}e(\lambda e^{i\theta}) + il\epsilon\|B\| \lim \lambda e^{i\theta}|. \end{aligned}$$

Squaring both sides with the right hand side having real and imaginary parts and from [61], we have that $\|A+\lambda B\|^2 \geq \|A\|^2 - 2\epsilon\|A\|\|B\|$, an implication that $A \perp_{BJ}^{\epsilon} B$. □

Theorem 3.5. *Let $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}$ be non-zero infinite dimension and SRB-spaces of AEOs and $A, B \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$. Then*

$$\|A+\lambda B\| \geq \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|, \ \forall \epsilon \in [0,1) \tag{7}$$

PROOF. We prove this by contradiction. Let Inequality (7) fail to hold. This implies that there exists $\lambda \in \mathbb{R}$ such that $0 < \|A+\lambda B\|^2 < \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|$. Suppose that $\lambda < 0$. Then for any λ, we have that $\|A+\lambda B\|^2 > 0$. By [59], we have that $z \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$ such that $\|z\| = 1$ for any $z = \frac{A+\lambda B}{\|A+\lambda B\|}$. From [62], we have that $\|A+\lambda B\| = \|A\|^2 - 2\epsilon\|A\|\|\lambda B\|$ which contradicts the Inequality (7). Hence, our earlier supposition does not hold and as a result, Inequality (7) holds and thus $A \perp_{BJ}^{\epsilon} B$. □

The next result employs the technique of polar decomposition with Schatten p-norms in Banach spaces of trace class.

Theorem 3.6. *Let $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}$ be non-zero infinite dimension and SRB-spaces of AEOs. Let $P, Q \in L(\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2})$. Then $P \perp_{BJ} Q$ if P has a polar decomposition $P = U|A|$ and $tr|P|^{p-1}U^*Q = 0$ in the Schatten p-norm.*

PROOF. Let $tr|P|^{p-1}UQ = 0$. Then for all $\lambda \in \mathbb{C}$, we have $tr|P|^p = tr|P|^{p-1}(|A| + \lambda U^*Q)$. From Minkowski's inequality, we obtain,

$$\begin{aligned} tr|P|^p &\leq |\||P|^{p-1}\|_k\||P| + \lambda U^*Q\|_p| \\ &= \||P|^{p-1}\|_q\|P+\lambda Q\|_p. \end{aligned}$$

But $(tr|P|^p)^{1-\frac{1}{k}} = (tr|P|^p)^{\frac{1}{p}} = \|P\|_p$ since k is the index conjugate of p, that is $\frac{1}{p} + \frac{1}{q} = 1$. So, $\|P\|_p \leq \|P+\lambda Q\|_p$, that is , $\|P+\lambda Q\| \geq \|P\|_q$ □

Remark 3.1. These results hold true for sequence matrices as seen from the work of [10].

At this point, we move to BJO in C^*-algebras of AEOs. It is known in [41] that, C^*-algebras are Hilbert C^*-modules. Hence, inner product is given as $\langle B, A\rangle = B^*A$ for all $A, B \in \mathfrak{C}$. Let $\mathfrak{C}$ be a C^*-algebra. Then for $A, B \in \mathfrak{C}$ A is strongly BJ-orthogonal to B denoted by $A \perp_{BJ}^{s} B$ if for all $C \in \mathfrak{C}$, $\|A + BC\| \geq \|A\|$. If A and B are mutually strongly BJ-orthogonal, that is, $A \perp_{BJ}^{s} B$ and $B \perp_{BJ}^{s} A$, then it is denoted by $A \perp_{BJ}^{ms} B$ We begin with the following proposition:

Proposition 3.7. *Strong BJO is intrinsically orthogonal in a C^*-algebra $\mathfrak{C}$.*

Proof. Let $\mathfrak{C}_0$ and $\mathfrak{C}$ be two C^*-algebras such that $\mathfrak{C}_0 \subset \mathfrak{C}$. Let $A, B \in \mathfrak{C}_0$. Since $\mathfrak{C}_0$ is a C^*-module and is itself a C^*-algebra, from [64], $A \perp_{BJ}^{s} B$ if and only if $A \perp_{BJ}^{s} B\langle B, A\rangle$ if and only if $A \perp_{BJ}^{s} BB^*A$ □

Lemma 3.8. *Let $\mathfrak{C}$ be a C^*-algebra of AEOs, then $A \perp_{BJ}^{s} B$ if and only if $|A^*|$ is strongly BJO to $|B^*|$.*

Proof. Since $|A| = \sqrt{A^*A}$, we have that $|B^*||B^*| = (BB^*)^{\frac{1}{2}}(BB^*)^{\frac{1}{2}} = BB^*$. But we have from Proposition 3.7 that $A \perp_{BJ}^{s} B$ if and only if $A \perp_{BJ}^{s} BB^*A$. So, we obtain $A \perp_{BJ}^{s} B$ if and only if $A \perp_{BJ}^{s} |B^*|$. Consider a partial isometry $A = |A^*|E$ for some $E \in \mathfrak{C}_0 \subset \mathfrak{C}$ and also consider $|A^*| = AE^*$. Let $|A^*| \perp_{BJ}^{s} B$, then for $C \in \mathfrak{C}_0 \subset \mathfrak{C}$ we obtain from [11] that;

$$\begin{aligned} \|A\| = \||A^*|E\| \leq \||A^*|\| &\leq \||A^*| + BCE^*\| \\ &= \|(A + BC)E^*\| \\ &\leq \|A + BC\|. \end{aligned}$$

Therefore, $\|A + BC\| \geq \|A\|$, for all $C \in \mathfrak{C}_0$. So, $A \perp_{BJ}^{s} B$.
Conversely, if $A \perp_{BJ}^{s} B$, then for all $C \in \mathfrak{C}$, we have that;

$$\begin{aligned} \|A\| = \|AE^*\| \leq \|A\| &\leq \|A + BCE\| \\ &= \|(|A^*| + BC)E\| \\ &\leq \||A^*| + BC\|. \end{aligned}$$

Hence $A \perp_{BJ}^{s} |B^*|$. Also $A \perp_{BJ}^{s} B$ if $|A^*| \perp_{BJ}^{s} B$ □

Next, we prove a result with regards to states in $\mathfrak{C}$.

Theorem 3.9. *Let $\mathfrak{C}$ be a C^*-algebra of AEOs. Then $A \perp_{BJ}^{s} B$ in $\mathfrak{C}$ if there is a state ξ in $\mathfrak{C}$ such that $\xi(AA^*) = \|A\|$ and $\xi(BB^*) = 0$ for $A, B \in \mathfrak{C}$.*

Proof. Since ξ is a state in $\mathfrak{C}$, then for any $C \in \mathfrak{C}$, we have by Cauchy-Schwarz inequality that $|\xi\langle C^*, C^*\rangle|^2 \leq \xi(C^*C) = 0$. Let $\|A\| = 1$. We have;

$$\|A + BC\|^2 = \|(A + BC)(A + BC)^*\| \geq |\xi(AA^* + AC^*B^* + BCA^* + BCC^*B^*)| \quad (8)$$

From the involution property of C^*- algebras, we have that $\xi(AC^*B^*) = \overline{\xi(BCA^*)} = 0, \xi(AA^*) = 1$ and $\xi(BCA^*) = \xi(BCC^*B^*) = 0$. Hence, from Inequality (8), we have that $\|A + BC\|^2 \geq 1 = \|A\|^2$. Taking positive square root on both sides, we have, $\|A + BC\| \geq \|A\|$ that is, $A \perp^s_{BJ} B$. □

Corollary 3.10. *Let $\mathfrak{C}$ be a C^*-algebra of AEOs. Let $A, B \in \mathfrak{C}$. For any projection $P \in \mathfrak{C}$ such that $PA = A$ and $PB = 0$, we have that $B \perp^s_{BJ} A$.*

Proof. With the conditions of the Corollary 3.10, we have $C \in \mathfrak{C}$ such that $\|A+BC\| \geq \|P(A+BC)\| = \|PA\| = 1 = \|A\|$. That is, $\|A+BC\| \geq \|A\|$. Hence, $A \perp^s_{BJ} B$ in $\mathfrak{C}$. □

Next, we give results of BJO in relation to tensor products of operators. We begin with the following proposition;

Proposition 3.11. *Let $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}$ be SRB-spaces of AEOs with tensor norm. Then $A_1 \otimes B_1 \perp_{BJ} A_2 \otimes B_2$ on $\overline{B_1} \otimes^\pi \overline{B_2}$ where $A_1, A_2 \in \overline{B_1}, B_1, B_2 \in \overline{B_2}, A_1 \perp_{BJ} B_2$ in $L(\overline{B_1}, \overline{B_2})$ and $\|.\|_\pi, \|.\|_{INJ}$ are cross norms and injective norm respectively.*

Proof. By Bhatia and Semrl property, $A_1 \perp_{BJ} A_2$ and so we have $\psi \in \overline{B_1^*}$ with $\|\psi\| = 1$. This implies that $|\psi(A_1)| = \|A_1\|$ since ψ is a functional and $\psi(A_2) = 0$. Next, consider $\chi \in \overline{B^*}$ with $\|\chi\| = 1$. This similarly implies that $\chi(B_1) = \|B_1\|$. Let $z \in \mathbb{C}$ be given, then we obtain that

$$\begin{aligned}
\|(A_1 \otimes B_1) + z(A_2 \otimes B_2)\|_\pi \geq & \|(A_1 \otimes B_1) + z(A_2 \otimes B_2)\|_{INJ} \\
= & \sup\{|\psi'(A_1)\chi'(B_1) + z\psi'(A_2)\chi'(B_2)| : \psi' \in B_{\overline{B_1}}, \chi' \in B_{\overline{B_2}}\} \\
\geq & |\psi(A_1)\chi(B_1) + z\psi(A_2)\chi(B_2)| \\
= & \|A_1\|\|B_1\| \\
= & \|A_1 \otimes B_1\|.
\end{aligned}$$

Therefore, $\|(A_1 \otimes B_1) + z\psi(A_1 \otimes B_2)\| \geq \|A_1 \otimes B_1\|$. Hence $A_1 \otimes B_1 \perp_{BJ} A_2 \otimes B_2$ on $\overline{B_1} \otimes^\pi \overline{B_2}$. □

Remark 3.2. It is known from [58] that in reflexive and smooth Banach spaces, $\overline{\mathcal{B}_1}$ and $\overline{\mathcal{B}_2}$, $A \perp B$ if and only if there exists $\phi \in \overline{B_1^*}$ such that $|\phi(A_1)| = \|\phi\|\|A\|$ with $\phi(B_1) = 0$, $\|A_1 \otimes B_1\|_\pi = \|A_1\|\|B_1\|$ as a cross norm and

$$\|W\|_{INJ} = \sup\left\{|\sum_{i=1}^{n} \phi(A_1)\psi(B_1)|\right\}, W = \sum A_i \otimes B_i \in \overline{B_1} \otimes \overline{B_2},$$

where $B_{\overline{B_1^*}}$ is the closed unit ball of $\overline{B_1^*}$

Remark 3.3. The converse of Proposition 3.11 is not true in general, that is, $A_2 \otimes B_2 \not\perp_{BJ} A_1 \otimes B_1$ on $\overline{B_2} \otimes \overline{B_1}$ since BJO is antisymmetric.

At this point, we consider two special cases.

Lemma 3.12. *Let $\Omega = C_{\mathbb{C}}[0,1]$ be the set of all complex functions on $[0,1]$. Let $T \in \Omega$ be an identity function and 1 be a constant function. Then $T \otimes 1 \perp_{BJ} 1 \otimes T$ in Ω.*

Proof. From Bhatia and Semrl property and Proposition 3.11, we have that

$$\|T \otimes 1 + z(1 \otimes T)\|_{INJ} = \|T.1 + z(1.T)\| \sup_{p,q \in [0,1]} |p + zq| \geq 1 = \|T \otimes 1\|.$$

□

Theorem 3.13. *Let $\circledcirc$ be the space of all operators acting on the l^2- space and $A, B, C \in \circledcirc$. Then $A \otimes I \perp_{BJ} B \otimes C$ on $\circledcirc \otimes \circledcirc$ where I is the identity operator in $\circledcirc$.*

Proof. Let $\circledcirc \otimes \circledcirc \subseteq L(l^2 \overline{\otimes} l^2)$ in which $\overline{\otimes}$ is a tensor product in Hilbert spaces. Also, consider $z \in \mathbb{C}$ then we have;

$$\begin{aligned} \|A \otimes I + z(B \otimes C)\| &= \sup_{l \in (l^2 \overline{\otimes} l^2), \|l\|=1} \|(A \otimes I)l + z(B \otimes C)l\| \\ &\geq \|(A \otimes I)(i_1 \otimes i_2) + z(B \otimes C)(i_1 \otimes i_2)\| \\ &= 1 + |z|^2 \geq 1 = \|A \otimes I\|. \end{aligned}$$

So, $\|A \otimes I + z(B \otimes C)\| \geq \|A \otimes I\|$. Hence, $A \otimes I \perp_{BJ} (B \otimes C)$ in $\circledcirc \otimes \circledcirc$. □

4. Conclusion

Characterizations involving algebraic elementary operators have been done over the years, for instance, orthogonality when the operators are induced by other different types of transformations. In particular, algebraic elementary operators induced by norm-attainable maps have not been characterized in terms of orthogonality when their numerical ranges have spheroid boundaries. In this note, we characterize algebraic elementary operators in terms of Birkhoff-James orthogonality when they are induced by norm-attainable maps and the boundaries of their numerical ranges are spheroidal in shape. We have shown that under spheroidicity criterion for numerical range boundary, various types of algebraic elementary operators satisfy Birkhoff-James orthogonality.

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