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On fixed point results in S_b-metric spaces

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Abstract. In this paper, we present several fixed-point theorems for surjective mappings in complete S_b-metric spaces. We explore various conditions under which these fixed points exist.

Keywords: Fixed point, S-metric space; b-metric space; S_b-metric space

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1. Introduction

Fréchet [6] introduced the concept of metric spaces, establishing the foundation for a significant branch of mathematical research. Banach [2] made significant contributions to this field by formulating the fixed point theorem, widely recognized as one of the most important results in analysis and the cornerstone of metric fixed point theory. This theorem has been extended in numerous directions, showcasing its broad applicability. Since then, metric spaces have become fundamental to various areas of mathematics, such as functional analysis, nonlinear analysis, and topology. Many researchers have extensively generalized the structure of metric spaces.

The concept of the b-metric space was introduced by Czerwik [4], while Sedghi et al. [14] developed the notion of the S-metric space. Numerous fixed-point theorems for various types of contractive mappings have been developed within these spaces. Building on this foundation, Souayah et al. [15] introduced a novel structure

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known as the S_b-metric space, which is defined using b-metric and S-metric spaces. Dhanraj et al. [5] establisheda fixed-point theorem by employing an orthogonal Geraghty-type α-admissible contraction mapping in orthogonal complete Branciari b-metric spaces and also utilized this to derive the existence and uniqueness of the solution of the Volterra integral equation. Gnanaprakasam et al. [8] introduced orthogonal α-almost Istrațescu contractions and proved fixed point results in b-metric spaces. Gholidahneh et al. [7] extended modular b-metric spaces and obtained fixed point results for $\alpha\hat{\nu}$-Meir-Keeler contractions. Iqbal et al. [9] proposed a generalized multivalued (α, L)-almost contraction in b-metric spaces and established the existence and uniqueness of a fixed point. Iqbal et al. [10] introduced generalized weak contractions and established fixed point results in b-metric spaces. Mani et al. [11] derived fixed point results in bicomplex valued b-metric spaces. Prakasam et al. [12] established fixed point theorems for O-generalized contractions, generalizing known results and demonstrating the existence of solutions to integral equations. Branciari [3] established a fixed point theorem connected to the contraction mapping principle of Banach and Caccioppoli Aage and Salunke [1] proved some fixed point theorems for expansive self-maps in complete cone metric spaces.

In this paper, we present some fixed-point theorems for surjective mappings in complete S_b-metric spaces, examining various conditions that guarantee the existence of these fixed points.

2. Preliminaries

Czerwik [4] defined b-metric space as follows;

Definition 2.1. [4] Let X be a non empty set and $d : X \times X \to [0, \infty)$ be a mapping satisfying following properties:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, z) + d(z, y)] \ ,$$

for all $x, y, z \in X$.

Then d is called a b-metric on X and the ordered pair (X, d) is called b-metric space with coefficient s.

Sedghi et al. [14] introduced the notion of an S-metric space which is defined as follows;

Definition 2.2. [14] Let X be a non empty set and $S : X \times X \times X \to [0, \infty)$ be a mapping satisfying following properties:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$, for all $a,x,y,z \in X$ (rectangle inequality).

Then (X,S) is called a S-metric space.

Saji et al. [13] have proved some fixed point theorems for surjection satisfying various expansion type condition in S-metric space which are as follows;

Theorem 2.1. [13] *Let (X,S) be a complete S-metric space. Let T be a surjective map from X into X such that*

$$S(Tx,Tx,Ty)+mS(Tx,Tx,y) \geq aS(x,x,y)+b\max\{S(Tx,Tx,x),S(Ty,Ty,x)\},$$

for all $x,y \in X$, where $a,b,m > 0$, $a > m$, and $a+b-3m > 1$. Then T has a unique fixed point.

Theorem 2.2. [13] *Let (X,S) be a complete S-metric space. Let T be a surjective map from X into X such that*

$$S(Tx,Tx,Ty)+mS(Tx,Tx,y) \geq aS(x,x,y)+bS(Tx,Tx,x)+cS(Ty,Ty,y),$$

for all $x,y \in X$, where $a,b,c > 0$, $a-m > 1$, and $a+b+c-3m > 1$. Then T has a unique fixed point.

Souayah et al. [15] combined the concept of b-metric space and S-metric space and introduced a new metric space called S_b-metric space as follows;

Definition 2.3. [15] Let X be a non empty set and $s \geq 1$ be a given real number. Then a mapping $S_b : X \times X \times X \to [0,\infty)$ is said to be S_b-metric on X, if following properties are satisfied;

(i) $S_b(x,y,z) = 0$ if and only if $x = y = z$;
(ii) $S_b(x,x,y) = S_b(y,y,x)$;
(iii) $S_b(x,y,z) \leq s\big[\, S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a)\big], \quad \forall\, x,y,z,a \in X.$

Then (X,S_b) is called a S_b-metric space.

Example 2.4. [15] Let X be a nonempty set with card$(X) \geq 5$. Suppose $X = X_1 \cup X_2$ is a partition of X such that card$(X_1) \geq 4$. Let $s \geq 1$. Then

$$S_b(x,y,z) = \begin{cases} 0 & \text{if } x=y=z=0, \\ 3s & \text{if } (x,y,z) \in X_1^3, \\ 1 & \text{if } (x,y,z) \notin X_1^3, \end{cases}$$

for all $x,y,z \in X$, is a S_b-metric on X with coefficient $s \geq 1$.

Example 2.5. Let $X = \{a, b, c\}$ be a nonempty set and $s \geq 1$ be a real number. Define the mapping $S_b : X \times X \times X \to [0, \infty)$ as follows:

$$S_b(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \dfrac{1}{2} & \text{if exactly two of } x, y, z \text{ are equal,} \\ 1 & \text{if all three elements } x, y, z \text{ are distinct.} \end{cases}$$

Then

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$.
  - If $x = y = z$, then $S_b(x, y, z) = 0$ by definition.
  - Conversely, if $S_b(x, y, z) = 0$, the only case that satisfies this is $x = y = z$.

(ii) $S_b(x, x, y) = S_b(y, y, x)$.
  - Compute $S_b(x, x, y)$: $S_b(x, x, y) = \frac{1}{2}$, since exactly two elements are equal.
  - Compute $S_b(y, y, x)$: $S_b(y, y, x) = \frac{1}{2}$, since exactly two elements are equal.

  Hence, $S_b(x, x, y) = S_b(y, y, x)$.

(iii) $S_b(x, y, z) \leq s\big[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)\big]$ for all $x, y, z, t \in X$.
  - Case 1: $x = y = z$. Then $S_b(x, y, z) = 0$, and the inequality holds trivially.
  - Case 2: Exactly two of x, y, z are equal. Without loss of generality suppose that $x = y = a, z = b$ and $t = c$. Then

$$S_b(x, y, z) = S_b(a, a, b) = \frac{1}{2}$$

  and

$$S_b(x, x, t) = S_b(a, a, c) = \frac{1}{2},$$
$$S_b(y, y, t) = S_b(b, b, c) = \frac{1}{2},$$
$$S_b(z, z, t) = S_b(b, b, c) = \frac{1}{2}.$$

  Thus,

$$s\big[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)\big] = s\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{3}{2}s \geq 1 = S_b(x, y, z).$$

  - Case 3: x, y, z are distinct. Without loss of generality, let $x = a, y = b, z = c$ and $t = a$. Then

$$S_b(x, y, z) = S_b(a, b, c) = 1$$

  and

$$S_b(x, x, t) = S_b(a, a, a) = 0,$$

$$\begin{aligned}S_b(y,y,t)&=S_b(b,b,a)=\frac{1}{2},\\ S_b(z,z,t)&=S_b(c,c,t)=\frac{1}{2}.\end{aligned}$$

Therefore,

$$s\left(S_b(x,x,t)+S_b(y,y,t)+S_b(z,z,t)\right)=s\left(0+\frac{1}{2}+\frac{1}{2}\right)=s\geq 1=S_b(x,y,z).$$

Thus, all properties of the S_b-metric are satisfied. Therefore, S_b is S_b-metric on X.

3. Main Results

In this section, we propose some following results in complete S_b-metric space.

Theorem 3.1. *Let (X,S_b) be a complete S_b-metric space and T be a surjective mapping from X into X such that*

$$S_b(Tx,Tx,Ty)+\alpha S_b(Tx,Tx,y)\geq\beta S_b(x,x,y)+\gamma\max\left\{S_b(Tx,Tx,x),S_b(Ty,Ty,x)\right\}, \tag{1}$$

for all $x,y\in X$, where $\alpha,\beta,\gamma>0$ and $s\geq 1$ is a real number such that $\beta>\alpha s,\beta+\gamma-3\alpha s>1,\beta-s\big(\alpha-\gamma+2\alpha s+1\big)>0$. Then T has unique fixed point.

Proof. Suppose T satisfies (1). Let $x_0\in X$ be an arbitrary point in X. We define a sequence $\{x_n\}$ by $Tx_n=x_{n-1}$, for $n=1,2,3,\cdots$. Put $x=x_n$ and $y=x_{n+1}$ in (1), we get,

$$\begin{aligned}&S_b(Tx_n,Tx_n,Tx_{n+1})+\alpha S_b(Tx_n,Tx_n,x_{n+1})\\ \geq&\beta S_b(x_n,x_n,x_{n+1})+\gamma\max\left\{S_b(Tx_n,Tx_n,x_n),S_b(Tx_{n+1},T_{n+1},x_n)\right\}\end{aligned}$$

$$\begin{aligned}&S_b(x_{n-1},x_{n-1},x_n)+\alpha S_b(x_{n-1},x_{n-1},x_{n+1})\\ \geq&\beta S_b(x_n,x_n,x_{n+1})+\gamma\max\left\{S_b(x_{n-1},x_{n-1},x_n),S_b(x_n,x_n,x_n)\right\}\\ =&\beta S_b(x_n,x_n,x_{n+1})+\gamma S_b(x_{n-1},x_{n-1},x_n).\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\gamma)S_b(x_{n-1},x_{n-1},x_n)\geq&\beta S_b(x_n,x_n,x_{n+1})-\alpha S_b(s_{n-1},x_{n-1},x_{n+1})\\ \geq&\beta S_b(x_n,x_n,x_{n+1})-\alpha s\big[2S_b(x_{n-1},x_{n-1},x_n)+S_b(x_{n+1},x_{n+1},x_n)\big]\\ =&\beta S_b(x_n,x_n,x_{n+1})-\alpha s\big[2S_b(x_{n-1},x_{n-1},x_n)+S_b(x_n,x_n,x_{n+1})\big].\end{aligned}$$

Therefore

$$(1-\gamma+2\alpha s)(S_b(x_{n-1},x_{n-1},x_n)\geq(\beta-\alpha s)S_b(x_n,x_n,x_{n+1}).$$

It implies that

$$S_b(x_n,x_n,x_{n+1})\leq\frac{1-\gamma+2\alpha s}{\beta-\alpha s}S_b(x_{n-1},x_{n-1},x_n)=kS_b(x_{n-1},x_{n-1},x_n),$$

where $k = \frac{1-\gamma+2\alpha s}{\beta-\alpha s} < 1$ as $\beta + \gamma - 3\alpha s > 1$. Similarly, we can obtain $S_b(x_n, x_n, x_{n+1}) \leq k^2 S_b(x_{n-2}, x_{n-2}, x_{n-1})$. Continuing this process, we obtain,

$$S_b(x_n, x_n, x_{n+1}) \leq k^n S_b(x_0, x_0, x_1).$$

Setting $S_n = S_b(x_n, x_n, x_{n+1})$, we get,

$$S_n \leq k^n S_0, \quad \forall n \in \mathbb{N}.$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in X. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in the Definition 2.3, we get,

$$\begin{aligned}
S_b(x_n, x_n, x_m) &\leq s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1})\right] \\
&= s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_m)\right] \\
&\leq 2sS_b(x_n, x_n, x_{n+1}) + s\left[2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) + S_b(x_m, x_m, x_{n+2})\right] \\
&= 2sS_n + 2s^2 S_{n+1} + sS_b(x_{n+2}, x_{n+2}, x_m) \\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-1} S_{m-1} \\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots \\
&\leq 2sk^n S_0 + 2s^2 k^{n+1} S_0 + 2s^3 k^{n+2} S_0 + \cdots \\
&= 2sk^n \left[1 + (sk) + (sk)^2 + (sk)^3 + \cdots\right] S_0 \\
&= 2sk^n \left(\frac{1}{1-sk}\right) S_0.
\end{aligned}$$

Taking limit as $n, m \to \infty$, we get,

$$\lim_{n,m\to\infty} S_b(s_n, x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X, S_b) is complete S_b-metric space, the sequence $\{x_n\}$ converges to a point say x in X. As T is subjective, there exists a point $y \in X$ such that $Ty = x$. Now, consider,

$$\begin{aligned}
S_b(x_n, x_n, x) &= S_b(Tx_{n+1}, Tx_{n+1}, Ty) \\
&\geq -\alpha S_b(Tx_{n+1}, Tx_{n+1}, y) + \beta S_b(x_{n+1}, x_{n+1}, y) \\
&\quad + \gamma \max\left\{S_b(Tx_{n+1}, Tx_{n+1}, x_{n+1}), S_b(Ty, Ty, x_{n+1})\right\} \\
&= -\alpha S_b(x_n, x_n, y) + \beta S_b(x_{n+1}, x_{n+1}, y) \\
&\quad + \gamma \max\left\{S_b(x_n, x_n, x_{n+1}), S_b(x, x, x_{n+1})\right\}.
\end{aligned}$$

Letting $n \to \infty$, we get,

$$S_b(x, x, x) \geq -\alpha S_b(x, x, y) + \beta S_b(x, x, y) + \gamma \max\left\{S_b(x, x, x), S_b(x, x, x)\right\}.$$

It implies that $(\beta - \alpha) S_b(x, x, y) \leq 0$. This is possible only if $S_b(x, x, y) = 0$, as $\beta > \alpha s > \alpha$. Therefore, we have $x = y$. Hence x is a fixed point of T. Now, to prove uniqueness, suppose $z \neq x$ be another fixed point of T in X. Then,

$$\begin{aligned} S_b(x,x,z) &= S_b(Tx,Tx,Tz) \\ &\geq -\alpha S_b(Tx,Tx,z) + \beta S_b(x,x,z) + \gamma \max\{S_b(Tx,Tx,x), S_b(Tz,Tz,x)\} \\ &= -\alpha S_b(x,x,z) + \beta S_b(x,x,z) + \gamma \max\{S_b(x,x,x), S_b(z,z,x)\} \\ &= -\alpha S_b(x,x,z) + \beta S_b(x,x,z) + \gamma S_b(x,x,z) \\ &\geq (\beta+\gamma-\alpha) S_b(x,x,z). \end{aligned}$$

It implies that $S_b(x,x,z) > S_b(x,x,z)$, because $\beta+\gamma-\alpha > 0$, which is a contradiction. Therefore, we must have $x = z$. This proves uniqueness. □

Theorem 3.2. *Let (X, S_b) be a complete S_b-metric space and T be a surjective mapping from X into X such that*

$$S_b(Tx,Tx,Ty) + \alpha S_b(Tx,Tx,y) \geq \beta S_b(x,x,y) + \gamma S_b(Tx,Tx,x) + \delta S_b(Ty,Ty,y), \tag{2}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta > 0$ with $\beta > \alpha$ and $s \geq 1$ is a real number such that $\beta+\gamma+\delta-3\alpha s > 1$ and $\beta+\delta+s(\gamma-2\alpha s-\alpha-1) > 0$. Then T has unique fixed point.

Proof. Suppose T satisfies equality (2). Let $x_0 \in X$ be an arbitrary point in X. We define a sequence $\{x_n\}$ by $Tx_n = x_{n-1}$, for $n = 1, 2, 3, \cdots$. Put $x = x_n$ and $y = x_{n+1}$ in (2), we get,

$$\begin{aligned} S_b(Tx_n,Tx_n,Tx_{n+1}) + \alpha S_b(Tx_n,Tx_n,x_{n+1}) \geq & \beta S_b(S_n,x_n,x_{n+1}) + \gamma S_b(Tx_n,Tx_n,x_n) \\ & + \delta S_b(Tx_{n+1},Tx_{n+1},x_{n+1}). \end{aligned}$$

Hence,

$$\begin{aligned} S_b(x_{n-1},x_{n-1},x_n) + \alpha S_b(x_{n-1},x_{n-1},x_{n+1}) \geq & \beta S_b(x_n,x_n,x_{n+1}) + \gamma S_b(x_{n-1},X_{n-1},x_n) \\ & + \delta S_b(x_n,x_n,x_{n+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} (1-\gamma) S_b(x_{n-1},x_{n-1},x_n) &\geq (\beta+\delta) S_b(x_n,x_n,x_{n+1}) - \alpha S_b(x_{n-1},x_{n-1},x_{n+1}) \\ &\geq (\beta+\delta) S_b(x_n,x_n,x_{n+1}) \\ &\quad - \alpha s\big[2S_b(x_{n-1},x_{n-1},x_n) + S_b(x_{n+1},x_{n+1},x_n)\big]. \end{aligned}$$

It implies that

$$S_b(x_n,x_n,x_{n+1} \leq \frac{1-\gamma+2\alpha s}{\beta+\delta-\alpha s} S_b(x_{n-1},x_{n-1},x_n).$$

Therefore, $S_b(x_n,x_n,x_{n+1}) \leq k S_b(x_{n-1},x_{n-1},x_n)$, where $k = \dfrac{1-\gamma+2\alpha s}{\beta+\delta-\alpha s} < 1$. Similarly we can obtain that $S_b(x_n,x_n,x_{n+1}) \leq k^2 S_b(x_{n-2},x_{n-2},x_{n-1})$. Continuing the same process and setting $S_n = S_b(s_n,x_n,x_{n+1})$, we obtain,

$$S_n \leq k^n S_0, \qquad \forall n \in \mathbb{N}.$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in X. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in the Definition 2.3, we get,

$$\begin{aligned}
S_b(x_n, x_n, x_m) &\leq s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1})\right] \\
&= s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_m)\right] \\
&\leq 2sS_b(x_n, x_n, x_{n+1}) + s\left[2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) + S_b(x_m, x_m, x_{n+2}\right] \\
&= 2sS_n + 2s^2 S_{n+1} + sS_b(x_{n+2}, x_{n+2}, x_m) \\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-1} S_{m-1} \\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots \\
&\leq 2sk^n S_0 + 2s^2 k^{n+1} S_0 + 2s^3 k^{n+2} S_0 + \cdots \\
&= 2sk^n \left[1 + (sk) + (sk)^2 + (sk)^3 + \cdots\right] S_0 \\
&= 2sk^n \left(\frac{1}{1 - sk}\right) S_0.
\end{aligned}$$

Taking limit as $n, m \to \infty$, we get,

$$\lim_{n,m\to\infty} S_b(s_n, x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X, S_b) is complete S_b metric space, the sequence $\{x_n\}$ converges to a point say x in X. As T is subjective, there exists a point $y \in X$ such that $Ty = x$. Now,

$$\begin{aligned}
S_b(x_n, x_n, x) &= = S_b(Tx_{n+1}, Tx_{n+1}, Ty) \\
&\geq -\alpha S_b(Tx_{n+1}, T_{n+1}, y) + \beta S_b(x_{n+1}, x_{n+1}, y) + \gamma S_b(Tx_{n+1}, Tx_{n+1}, x_{n+1}) \\
&\quad + \delta S_b(Ty, Ty, y) \\
&= -\alpha S_b(x_n, x_n, y) + \beta S_b(x_{n+1}, x_{n+1}, y) + \gamma S_b(x_n, x_n, x_{n+1}) + \delta S_b(x, x, y).
\end{aligned}$$

Letting $n \to \infty$, we obtain,

$$S_b(x, x, y) \geq -\alpha S_b(x, x, y) + \beta S_b(x, x, y) + \gamma S_b(x, x, x) + \delta S_b(x, x, y).$$

Therefore, $(-\alpha + \beta + \delta) S_b(x, x, y) \leq 0$, which implies that $S_b(x, x, y) = 0$ (as $\beta > \alpha$ implies that $-\alpha + \beta + \delta > 0$) and hence $x = y$, which shows that x is a fixed point of T. Now, to prove uniqueness, suppose $z \neq x$ be another fixed point of T in X. Then,

$$\begin{aligned}
S_b(x, x, z) &= S_b(Tx, Tx, Tz) \\
&\geq -\alpha S_b(Tx, Tx, z) + \beta S_b(x, x, z) + \gamma S_b(Tx, Tx, x) + \delta S_b(Tz, Tz, z) \\
&= -\alpha S_b(x, x, z) + \beta S_b(x, x, z) + \gamma S_b(x, x, x) + \delta S_b(z, z, z) \\
&\geq (\beta - \alpha) S_b(x, x, z).
\end{aligned}$$

Therefore, $S_b(x, x, z) > S_b(x, x, z)$, because $\beta > \alpha$, a contradiction. Hence, we must have $x = z$. This proves uniqueness. □

Theorem 3.3. *Let (X, S_b) be a complete S_b-metric space and T be a surjective mapping from X into X such that*

$$S_b(Tx, Tx, Ty) \geq kS_b(x, x, y) + lS_b(x, x, Ty), \tag{3}$$

for all $x, y \in X$, where $k, l > 0$ with $k + l > 0$ and $s \geq 1$ is a real number such that $k > s$. Then T has unique fixed point.

Proof. Suppose T satisfies inequality (3). Let $x_0 \in X$ be an arbitrary point in X. We define a sequence $\{x_n\}$ by $Tx_n = x_{n-1}$, for $n = 1, 2, 3, \ldots$. Put $x = x_n$ and $y = x_{n+1}$ in (3), we get,

$$S_b(Tx_n, Tx_n, Tx_{n+1}) \geq kS_b(x_n, x_n, x_{n+1}) + lS_b(x_n, x_n, Tx_{n+1}).$$

Therefore

$$S_b(x_{n-1}, x_{n-1}, x_n) \geq kS_b(x_n, x_n, x_{n+1}) + lS_b(x_n, x_n, x_n).$$

It implies that $S_b(x_n, x_n, x_{n+1}) \leq \dfrac{1}{k} S_b(x_{n-1}, x_{n-1}, x_n)$. Taking $\alpha = \dfrac{1}{k} < 1$, we get,

$$S_b(x_n, x_n, x_{n+1}) \leq \alpha S_b(x_{n-1}, x_{n-1}, x_n).$$

Similarly, we can show that $S_b(x_n, x_n, x_{n+1}) \leq \alpha^2 S_b(x_{n-2}, x_{n-2}, x_{n-1})$. Continuing the same process, we obtain, $S_b(x_n, x_n, x_{n+1}) \leq \alpha^n S_b(x_0, x_0, x_1)$. Setting $S_n = S_b(x_n, x_n, x_{n+1})$, we get,

$$S_n \leq \alpha^n S_0, \quad \forall n \in \mathbb{N}.$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in X. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in the Definition 2.3, we get,

$$\begin{aligned}
S_b(x_n, x_n, x_m) &\leq s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1})\right] \\
&= s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_m)\right] \\
&\leq 2sS_b(x_n, x_n, x_{n+1}) + s\left[2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) + S_b(x_m, x_m, x_{n+2}\right] \\
&= 2sS_n + 2s^2 S_{n+1} + sS_b(x_{n+2}, x_{n+2}, x_m) \\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-1} S_{m-1} \\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots \\
&\leq 2s\alpha^n S_0 + 2s^2 k^{n+1} S_0 + 2s^3 \alpha^{n+2} S_0 + \cdots \\
&= 2s\alpha^n \left[1 + (s\alpha) + (s\alpha)^2 + (s\alpha)^3 + \cdots\right] S_0 \\
&= 2s\alpha^n \left(\frac{1}{1 - s\alpha}\right) S_0.
\end{aligned}$$

Taking limit as $n, m \to \infty$, we get,

$$\lim_{n,m\to\infty} S_b(s_n, x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X, S_b) is complete S_b metric space, the sequence $\{x_n\}$ converges to a point say x in X. As T is subjective, there exists a point $y \in X$ such that $Ty = x$. Now, we have,

$$\begin{aligned} S_b(x_n, x_n, x) &= S_b(Tx_{n+1}, Tx_{n+1}, Ty) \\ &\geq kS_b(x_{n+1}, x_{n+1}, y) + lS_b(x_{n+1}, x_{n+1}, Ty) \\ &= kS_b(x_{n+1}, x_{n+1}, y) + lS_b(x_{n+1}, x_{n+1}, x). \end{aligned}$$

Letting $n \to \infty$, we obtain, $S_b(x, x, x) \geq kS_b(x, x, y) + lS_b(x, x, x)$. Therefore, $kS_b(x, x, y) \leq 0$, which implies that $S_b(x, x, y) = 0 \quad (\text{as } k > 1)$. It implies that $x = y$. Therefore, x is a fixed point of T. Now, to prove uniqueness, suppose $z \neq x$ be another fixed point of T in X. Then,

$$\begin{aligned} S_b(x, x, z) &= S_b(Tx, Tx, Tz) \\ &\geq kS_b(x, x, z) + lS_b(x, x, Tz) \\ &= kS_b(x, x, z) + lS_b(x, x, z) \\ &= (k + l)S_b(x, x, z). \end{aligned}$$

It implies that $S_b(x, x, z) > S_b(x, x, z)$ as $k + l > 0$, which is a contradiction. Hence, we must have $x = z$. This proves the uniqueness. □

Theorem 3.4. *Let (X, S_b) be a complete S_b-metric space and T be a surjective mapping from X into X such that*

$$S_b(Tx, Ty, Tz) \geq \Omega\left(S_b(Tx, Tx, x) + S_b(Ty, Ty, y) + S_b(Tz, Tz, z)\right), \tag{4}$$

for all $x, y, z \in X$, where $0 < \Omega < \dfrac{1}{2}$ and $s \geq 1$ is a real number such that $s < \dfrac{1}{1 - 2\Omega}$. Then T has a fixed point.

Proof. Suppose T satisfies (4). Let $x_0 \in X$ be an arbitrary point in X. We define a sequence $\{x_n\}$ by $Tx_n = x_{n-1}$, for $n = 1, 2, 3, \ldots$. Put $x = y = x_n$ and $z = x_{n+1}$ in (4), we get,

$$\begin{aligned} S_b(Tx_n, Tx_n, Tx_{n+1}) \geq \Omega\left(S_b(Tx_n, Tx_n, x_n) + S_b(Tx_n, Tx_n, x_n)\right) \\ + S_b(Tx_{n+1}, Tx_{n+1}, x_{n+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} (1 - 2\Omega)S_b(Tx_n, Tx_n, x_n) &\geq S_b(Tx_{n+1}, Tx_{n+1}, x_{n+1}) \\ (1 - 2\Omega)S_b(x_{n-1}, x_{n-1}, x_n) &\geq S_b(x_n, x_n, x_{n+1}). \end{aligned}$$

It implies that $S_b(x_n, x_n, x_{n+1}) \leq (1 - 2\Omega)S_b(x_{n-1}, x_{n-1}, x_n)$. Let $k = 1 - 2\Omega < 1$. Therefore, $S_b(x_n, x_n, x_{n+1}) \leq kS_b(x_{n-1}, x_{n-1}, x_n)$. Similarly, we can obtain

$S_b(x_n, x_n, x_{n+1}) \leq k^2 S_b(x_{n-2}, x_{n-2}, x_{n-1})$. Continuing the same process and setting $S_n = S_b(x_n, x_n, x_{n+1})$, we obtain,

$$S_n \leq k^n S_0, \quad \forall n \in \mathbb{N} \ .$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in X. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in the Definition 2.3, we get,

$$\begin{aligned} S_b(x_n, x_n, x_m) &\leq s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_m, x_m, x_{n+1})\right] \\ &= s\left[2S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_m)\right] \\ &\leq 2sS_b(x_n, x_n, x_{n+1}) + s\left[2sS_b(x_{n+1}, x_{n+1}, x_{n+2}) + S_b(x_m, x_m, x_{n+2}\right] \\ &= 2sS_n + 2s^2 S_{n+1} + sS_b(x_{n+2}, x_{n+2}, x_m) \\ &\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-1} S_{m-1} \\ &\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots \\ &\leq 2sk^n S_0 + 2s^2 k^{n+1} S_0 + 2s^3 k^{n+2} S_0 + \cdots \\ &= 2sk^n \left[1 + (sk) + (sk)^2 + (sk)^3 + \cdots\right] S_0 \\ &= 2sk^n \left(\frac{1}{1 - sk}\right) S_0 \ . \end{aligned}$$

Taking limit as $n, m \to \infty$, we get,

$$\lim_{n,m\to\infty} S_b(s_n, x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X, S_b) is complete S_b metric space, the sequence $\{x_n\}$ converges to a point say x in X. As T is subjective, there exists a point $y \in X$ such that $Ty = x$. Now, consider,

$$\begin{aligned} S_b(x_n, x_n, x) &= S_b(Tx_{n+1}, Tx_{n+1}, Ty) \\ &\geq \Omega\left(S_b(Tx_{n+1}, Tx_{n+1}, x_{n+1}) + S_b(Tx_{n+1}, Tx_{n+1}, x_{n+1}) + S_b(Ty, Ty, y)\right) \\ &\geq \Omega\left(2S_b(x_n, x_n, x_{n+1}) + S_b(x, x, y)\right) \ . \end{aligned}$$

Letting $n \to \infty$, we obtain $S_b(x, x, x) \geq \Omega\left(2S_b(x, x, x) + S_b(x, x, y)\right)$. Therefore, $\Omega S_b(x, x, y) \leq 0$. It implies that $S_b(x, x, y) = 0$ because $0 < \Omega < \dfrac{1}{2}$. Thus, $x = y$, which shows that x is a fixed point of T. □

Example 3.1. Let $X = \{a, b, c\}$ and consider S_b-metric defined in Example 2.5. Define $T : X \to X$ as follows:

$$T(x) = \begin{cases} a & \text{if } x = a, \\ c & \text{if } x = b, \\ b & \text{if } x = c. \end{cases}$$

Then, clearly T is a surjective mapping. Without loss of generality, let $x = a, y = b, z = c$. Then

$$S_b(Tx, Ty, Tz) = S_b(Ta, Tb, Tc) = S_b(a, c, b) = 1,$$

$$S_b(Tx, Tx, x) = S_b(Ta, Ta, a) = S_b(a, a, a) = 0,$$

$$S_b(Tx, Tx, x) = S_b(Ta, Ta, a) = S_b(a, a, a) = 0,$$

$$S_b(Ty, Ty, y) = S_b(Tb, Tb, b) = S_b(c, c, b) = \frac{1}{2},$$

$$S_b(Tz, Tz, z) = S_b(Tc, Tc, c) = S_b(b, b, c) = \frac{1}{2}.$$

Therefore, by $0 < \Omega < \frac{1}{2}$, we have

$$\begin{aligned}\Omega\left(S_b(Tx, Tx, x) + S_b(Ty, Ty, y) + S_b(Tz, Tz, z)\right) =& \Omega\left(0 + \frac{1}{2} + \frac{1}{2}\right)\\ =&\Omega\\ <&\frac{1}{2}\\ <&1\\ =&S_b(Tx, Ty, Tz).\end{aligned}$$

That is,

$$S_b(Tx, Ty, Tz) > \Omega\left(S_b(Tx, Tx, x) + S_b(Ty, Ty, y) + S_b(Tz, Tz, z)\right).$$

Therefore, T satisfies all the conditions of the Theorem 3.4. Hence, T has a fixed point $x = a$.

4. Conclusions and Future Works

In this paper, we established several fixed-point theorems of surjective mappings in complete S_b-metric spaces. There is a scope to study the existence of common fixed point for two self mapping under certain conditions in complete S_b-metric spaces.

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References

[1] C. T. Aage, and J. N. Salunke, *Some fixed point theorems for expansion onto mappings on cone metric spaces*, Acta Math. Sin. (Engl. Ser.), **27**(6) (2011), 1101–1106.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[3] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57**(1-2) (2000), 31–37.

[4] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostraviensis, **01**(1) (1993), 5–11.

[5] M. Dhanraj, A. J. Gnanaprakasam, G. Mani, O. Ege, and M. De la Sen, *Solution to integral equation in an O-Complete Branciari b-metric spaces*, Axioms, **11**(12) (2022), 1–14.

[6] M. M. Frechet, *Sur quelques points du calcul fonctionnel*, Rend. Circ. Matem. Palermo **22** (1906), 1–72.

[7] A. Gholidahneh, S. Sedghi, O. Ege, Z. D. Mitrovic, and M. De la Sen, *The Meir-Keeler type contractions in extended modular b-metric spaces with an application*, AIMS Math., **6**(2) (2021), 1781–1799.

[8] A. J. Gnanaprakasam, G. Mani, O. Ege, A. Aloqaily, and N. Mlaiki, *New fixed point results in orthogonal b-metric spaces with related applications*, Math. Sigma, **11**(3) (2023), 1-18.

[9] M. Iqbal, A. Batool, O. Ege, and M. De la Sen, *Fixed point of almost contraction in b-metric spaces*, J. Math., **2020**(1) (2020), 1–6.

[10] M. Iqbal, A. Batool, O. Ege, and M. De la Sen, *Fixed point of generalized weak contraction in b-metric spaces*, J. Funct. Spaces, **2021.1** (2021), 2042162.

[11] G. Mani, A. J. Gnanaprakasam, O. Ege, N. Fatima, and N. Mlaiki, *Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued b-metric space*, SUT J. Math. Symmetry, **15**(2) (2023), 297, 1–15.

[12] S. K. Prakasam, A. J. Gnanaprakasam, O. Ege, G. Mani, S. Haque, and N. Mlaiki, *Fixed point for an OgF-c in O-complete b-metric-like spaces*, AIMS Math., **8**(1) (2023), 1022–1039.

[13] L. T. Saji, G. Uthaya Sankar, and A. Subramanian, *Fixed point theorems in S-metric spaces*, Malaya J. Math., **8** (2020), 2296–2298.

[14] S. Sedghi, N. Shobe, and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik, **64**(3) (2012), 258—266.

[15] N. Souayah and N. Mlaiki, *A fixed point theorem in S_b-metric spaces*, J. Math. Comput. Sci., **16** (2016), 131—139.

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