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# Spectral theorems associated with the generalized Dunkl-Wigner localization operators

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Abstract. The main crux of this paper is to introduce the Wigner transform associated with the generalized Dunkl operator and to give some new results related to this transform. Next, we introduce a new class of pseudo-differential operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ called localization operator which depends on a symbol $\sigma$ and two admissible functions $\psi_1$ and $\psi_2$, we give a criteria in terms of the symbol $\sigma$ for its boundedness and compactness, we also show that these operators belong to the Schatten-Von Neumann class $S^p$ for all $p \in [1, +\infty]$ and we give a trace formula.

**Keywords:** Fourier-Wigner transform, Localization operators, Dunkl operator, Schatten-von Neumann classes

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## 1. Introduction

The Wigner transform has a long story, which started in 1932 with Eugene Wigner's work as a probability quasi-distribution which allows the expression of quantum mechanical expectation values in the same form as the averages of classical statistical mechanics. It is also used in signal processing as a transform in time-frequency analysis; for more information, one can see [8, 20].


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A lot of attention has been given to various generalization of the classical Fourier transform. This paper focuses on the Fourier transform associated with the generalized Dunkl operator. More precisely, we consider the following first-order differential-difference operator defined by

$$\Delta_A(u) = \frac{\partial u}{\partial x} + \frac{A'(x)}{A(x)}\left(\frac{u(x)-u(-x)}{2}\right), \tag{1}$$

where $A$ is a nonnegative function satisfying certain conditions. The operator (1) plays an important role in analysis, and he generalized the Dunkl operator [**9, 13**], the Jacobi-Dunkl operator [**1**].

The eigenfunctions of the operator $\Delta_A$ satisfy a product formula which permits to develop a new harmonic analysis on the generalized Dunkl hypergroup denoted by $(\mathbb{R}, *_A)$, this hypergroup is commutative, with neutral element 0 and the identity map is the involution, the Haar $\mu_A$ measure on $(\mathbb{R}, *_A)$ is given by

$$d\mu_A(x) := A(x)dx. \tag{2}$$

For more information about the generalized Dunkl operator (1), one can see [**13**]. One of the aims of the Fourier transform is the study of the theory of localization operators called also Gabor multipliers, Toeplitz operators or Anti-Wick operators, this theory was initiated by Daubechies in [**7**], developed and detailed in the book [**22**] by Wong. Wong was the first one who defined the localization operators on the Weyl Heisenberg group in [**21**], next Boggiatto and Wong have extended this results on $L^p\left(\mathbb{R}^d\right)$ in [**2**]. The theory of localization operators associated with the Fourier-Wigner transform on hypergroups has been studied and known remarkable development for example in the spherical mean hypergroups [**16**], in the Hechman-Opdam hypergoups [**12**], in the Laguerre hypergoup [**15**], in the Dunkl hypergroup [**14**], in the Sturm-Liouville hypergroup [**4, 6, 17, 19**], we have also generalized this theory in the Laguerre-Bessel hypergroups [**3**]. However, to our knowledge, the localization operators for Wigner transform have not been studied on the generalized Dunkl hypergroup $(\mathbb{R}, *_A)$. The main purpose of this paper is twofold on the one hand we introduce the Fourier-Wigner transform associated with the generalized Dunkl operator (1) and we give some new results related to this transform on the other hand we introduce the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ associated with this transform and we give a criteria in terms of the symbol $\sigma$ for its boundedness and compactness, we also show that these operators belongs to the Schatten-Von Neumann classes $S^p$ for all $p \in [1; +\infty]$ and we give a trace formula.

The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the generalized Dunkl operator and Schatten-Von Neumann classes, in section 3 we will study the boundedness, compactness and the Schatten properties of the localization operator associated with the generalized Dunkl-Wigner transform.

## 2. Harmonic Analysis Associated with the Generalized Dunkl operator

In this section, we set some notations and we recall some results in harmonic analysis associated with the generalized Dunkl operator and the Schatten-Von Neumann classes. For more details, we refer the reader to [13, 22]. In the following, we denote by

- $\mathcal{D}_*(\mathbb{R})$ the space of even, differentiable functions on $\mathbb{R}$, with compact support.
- $L^p_A(\mathbb{R}), p \geq 1$, the space of measurable functions $f$ on $\mathbb{R}$ such that

$$\|f\|_{p,A} = \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p d\mu_A(x)\right)^{1/p} < +\infty & \text{si } 1 \leq p < +\infty, \\ \text{ess sup}_{x\in\mathbb{R}} |f(x)| < +\infty & \text{si } p = +\infty, \end{cases}$$

where $\mu_A$ is the measure given by the relation (2) and A is a nonnegative function defined on $\mathbb{R}$ and called the Chébli-Trimèche function, satisfying the following conditions see [13].

(i) There exists a positive even infinitely differentiable function $B$ on $\mathbb{R}$, with $B(x) \geq 1$ $x \in \mathbb{R}$, such that $A(x) = x^{2\alpha+1}B(x), \alpha > \frac{-1}{2}$.

(ii) $A$ is increasing on $\mathbb{R}$and $\lim_{x\to\infty} A(x) = \infty$.

(iii) $\frac{A'}{A}$ is decreasing on $(0, \infty)$, and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho$.

(iv) There exists a constant $\sigma > 0$, such that for all $x \in [x_0, \infty)\,, x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\sigma x}F(x), \text{ if } \rho > 0 \\ \frac{2\alpha+1}{x} + e^{-\sigma x}F(x), \text{ if } \rho = 0, \end{cases} \tag{3}$$

where $F$ is $C^\infty$ on $(0, \infty)$, bounded together with its derivatives.

- $L^p_\sigma(\mathbb{R}), p \geq 1$, the space of measurable functions $f$ on $\mathbb{R}$ such that

$$\|f\|_{p,\sigma} = \begin{cases} \left(\int_{\mathbb{R}} |f(\lambda)|^p d\sigma(\lambda)\right)^{1/p} < +\infty & \text{if } 1 \leq p < +\infty, \\ \underset{\lambda\in\mathbb{R}}{\text{ess sup}}|f(\lambda)| < +\infty & \text{if } p = +\infty. \end{cases}$$

where $d\sigma$ is the spectral measure defined on $\mathbb{R}$ by

$$d\sigma(\lambda) = \frac{|\lambda|}{4\sqrt{\lambda^2 - \rho^2}\left|c\left(\sqrt{\lambda^2 - \rho^2}\right)\right|^2} 1_{\mathbb{R}\setminus(-\rho,\rho)} d\lambda,$$

with $1_{\mathbb{R}\setminus(-\rho,\rho)}$ is the characteristic function of $\mathbb{R}\setminus(-\rho, \rho)$ and $C(\lambda)$ is the Harish-Chandra function given explicitely in [5, 11].

**2.1. The Eigenfunctions of the Generalized Dunkl operator.** For every $\lambda \in \mathbb{C}$, we consider the following Cauchy problem

$$\begin{cases} \Delta_A u(x) = i\lambda u(x) \\ u(0) = 1, \quad u'(0) = 0 \end{cases}$$

From [13], this system admits a unique solution denote by $\varphi_\lambda$ called the generalized Dunkl kernel, this function is infinitely differentiable on $\mathbb{R}$, even and satisfy the following important result:

$$|\varphi_\lambda(x)| \leq 1, \tag{4}$$

for all $\lambda, x \in \mathbb{R}$.

### 2.2. The Generalized Dunkl transform.

**Definition 2.1.** ([13]) The generalized Dunkl transform $\mathcal{F}_A$ defined on $L^1_A(\mathbb{R})$ by

$$\mathcal{F}_A(f)(\lambda) = \int_{\mathbb{R}} \varphi_\lambda(x) f(x) d\mu_A(x), \quad \text{for } \lambda \in \mathbb{R}. \tag{5}$$

**Particular cases**

- If the function A is of the form $A(x) = |x|^{2\alpha+1}$ with $\alpha \geq \frac{-1}{2}$, then $\Delta_A$ is the classical Dunkl operator and $\mathcal{F}_A$ coincide with the Dunkl transform see [9, 14].
- If $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x), \alpha \geq \beta \geq -\frac{1}{2}, \alpha \neq -\frac{1}{2}$, we regain the Jacobi-Dunkl operator given by

$$\Delta_A(u)(x) = \Delta_{\alpha,\beta} u(x) := \frac{\partial}{\partial x} u(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \frac{u(x) - u(-x)}{2},$$

in this case $\mathcal{F}_A$ coincide with the Jacobi-Dunkl transform see [1].

As we can see the generalized Dunkl transform (5) generalize many transforms in the literature and some basic properties of this transform are as follows, for the proofs, we refer the reader to [13].

**Proposition 2.1.**
*(1) (Riemann-Lebesgue) For all $f \in L^1_A(\mathbb{R})$, the function $\mathcal{F}_A(f)$ is continuous and we have*

$$\|\mathcal{F}_A(f)\|_{\infty,\sigma} \leq \|f\|_{1,A}. \tag{6}$$

*(2) (Inversion Formula) For all $f \in L^1_A(\mathbb{R})$ such that $\mathcal{F}_A(f) \in L^1_\sigma(\mathbb{R})$ we have*

$$f(x) = \int_{\mathbb{R}} \varphi_\lambda(x) \mathcal{F}_A(f)(\lambda) d\sigma(\lambda), \qquad a.e \quad x \in \mathbb{R}. \tag{7}$$

*(3) (Plancherel Theorem) The generalized Fourier transform extends uniquely to a unitary isomorphism from $L^2_A(\mathbb{R}_+)$ onto $L^2_\sigma(\mathbb{R})$ and for all $f \in L^2_A(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} |f(x)|^2 \, d\mu_A(x) = \int_{\mathbb{R}} |\mathcal{F}_A(f)(\lambda)|^2 \, d\sigma(\lambda). \tag{8}$$

**2.3. Generalized Dunkl translation operator.** *From [13], the character $\varphi_\lambda$ is multiplicative on $\mathbb{R}$ in the sense*

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}} \varphi_\lambda(z)K(x,y,z)A(z)dz, \tag{9}$$

*where $K(x,y,.)$ is a measurable positive function given explicitly in [13]. The product formula (9) permits to define a translation operator, a convolution product and to develop a new harmonic analysis on the generalized Dunkl setting .*

**Definition 2.2.** Let $x,y,z \in \mathbb{R}$ and $f$ be a measurable function on $\mathbb{R}$ the translation operator associated with the generalized Dunkl operator (1) is defined by:

$$\mathcal{T}_A^x(f)(y) = \int_{\mathbb{R}} f(z)K(x,y,z)A(z)dz.$$

The following proposition summarizes some properties of the generalized Dunkl translation operator, for the proofs we refer the reader to [13].

**Proposition 2.2.** *For all $x,y,z \in \mathbb{R}, f$ a measurable function on $\mathbb{R}$ we have*
*(1)*

$$\mathcal{T}_A^x(f)(y) = \mathcal{T}_A^y(f)(x). \tag{10}$$

*(2)*

$$\int_{\mathbb{R}} \mathcal{T}_A^x(f)(y)d\mu_A(x) = \int_{\mathbb{R}} f(y)d\mu_A(x). \tag{11}$$

*(4) for $f \in L_A^p(\mathbb{R})$ with $p \in [1,+\infty]$, $\mathcal{T}_A^x(f) \in L_A^p(\mathbb{R})$ and we have*

$$\|\mathcal{T}_A^x(f)\|_{p,A} \leq \|f\|_{p,A}. \tag{12}$$

*By using the generalized translation, we define the generalized convolution product of $f,g \in L_A^p(\mathbb{R})$ and $x \in \mathbb{R}$ by*

$$(f *_A g)(x) = \int_{\mathbb{R}} \mathcal{T}_A^x(f)(y)g(y)d\mu_A(y).$$

*This convolution is commutative, associative and its satisfies the following properties, for the proofs we refer the reader to [13].*

**Proposition 2.3.**
*(1) (Young's inequality) for all $p,q,r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$ and for all $f \in L_A^p(\mathbb{R}), g \in L_A^q(\mathbb{R})$ the function $f *_A g$ belongs to the space $L_A^r(\mathbb{R})$ and we have*

$$\|f *_A g\|_{r,A} \leq \|f\|_{p,A}\|g\|_{q,A}. \tag{13}$$

*(2) For $f,g \in L_A^2(\mathbb{R})$ the function $f *_A g$ belongs to $L_A^2(\mathbb{R})$ if and only if the function $\mathcal{F}_A(f)\mathcal{F}_A(g)$ belongs to $L_\sigma^2(\mathbb{R})$ and in this case we have*

$$\mathcal{F}_A(f *_A g) = \mathcal{F}_A(f)\mathcal{F}_A(g). \tag{14}$$

*and*

$$\int_{\mathbb{R}} |f *_A g(x)|^2 d\mu_A(x) = \int_{\mathbb{R}} |\mathcal{F}_A(f)(\lambda)|^2 |\mathcal{F}_A(g)(\lambda)|^2 d\sigma(\lambda). \tag{15}$$

**2.4. The Schatten-Von Neumann classes.** *Notation: we denote by*

• $l^p(\mathbb{N}), 1 \leq p \leq \infty$, *the set of all infinite sequences of real (or complex) numbers* $u := (u_j)_{j\in\mathbb{N}}$, *such that*

$$\|u\|_p := \begin{cases} \left(\sum_{j=1}^{\infty} |u_j|^p\right)^{\frac{1}{p}} < \infty, & \text{if} \quad 1 \leq p < \infty, \\ sup_{j\in\mathbb{N}} |u_j| < \infty, & \text{if} \quad p = +\infty. \end{cases}$$

• $B\left(L^p_A(\mathbb{R})\right), 1 \leq p \leq \infty$, *the space of bounded operators from* $L^p_A(\mathbb{R})$ *into itself. For* $p = 2$, *we define the space* $S_\infty := B\left(L^2_A(\mathbb{R})\right)$, *equipped with the norm,*

$$\|A\|_{S_\infty} := \sup_{v\in L^2_A(\mathbb{R}):\|v\|_{2,\alpha,\beta}=1} \|Av\|_{2,A}. \tag{16}$$

**Definition 2.3.**
(1) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator $A$ in $B\left(L^2_A(\mathbb{R})\right)$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.
(2) For $1 \leq p < \infty$, the Schatten class $S_p$ is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space $S_p$ is equipped with the norm

$$\|A\|_{S_p} := \left(\sum_{n=1}^{\infty} (s_n(A))^p\right)^{\frac{1}{p}}.$$

**Remark 2.4.** We note that the space $S_2$ is the space of Hilbert-Schmidt operators, and $S_1$ is the space of trace class operators.

**Definition 2.5.** The trace of an operator $A$ in $S_1$ is defined by

$$\text{tr}(A) = \sum_{n=1}^{\infty} \langle A\phi_n, \phi_n\rangle_{\mu_A}, \tag{17}$$

where $(\phi_n)_n$ is any orthonormal basis of $L^2_A(\mathbb{R})$.

**Remark 2.6.** If $A$ is positive, then

$$\text{tr}(A) = \|A\|_{S_1}. \tag{18}$$

Moreover, a compact operator $A$ on the Hilbert space $L^2_A(\mathbb{R})$ is Hilbert-Schmidt, if the positive operator $A^*A$ is in the space of trace class $S_1$. Then

$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = \text{tr}\left(A^*A\right) = \sum_{n=1}^{\infty} \|A\phi_n\|^2_{2,A}, \tag{19}$$

for any orthonormal basis $(\phi_n)_n$ of $L^2_A(\mathbb{R})$. For more information about the Schatten-Von Neumann classes one can see [22].

**2.5. Generalized Dunkl-Wigner Transform.** The main purpose of this subsection is to introduce the generalized Fourier-Wigner transform associated with the generalized Dunkl operator (1) and to give some new results related to this transform.

**Notation:** We denote by

- $\mathcal{S}_*(\mathbb{R}^2)$ the Schwartz space defined on $\mathbb{R}^2$ equipped with it s usual topology.
- $L^p_{\theta_A}(\mathbb{R}^2)$,$1 \le p \le +\infty$ the space of measurable functions on $\mathbb{R}^2$ satisfying

$$\|f\|_{p,\theta_A} := \begin{cases} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x,\lambda)|^p \mathrm{d}\theta_A(x,\lambda)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[, \\ \underset{(x,\lambda)\in\mathbb{R}^2}{\text{ess sup}} |f(x,\lambda)|, & \text{if } p = +\infty. \end{cases}$$

where $\theta_A$ is the measure defined on $\mathbb{R}^2$ by

$$\mathrm{d}\theta_A(x,\lambda) := d\sigma(\lambda) \otimes d\mu_A(x)$$

**Definition 2.7.** The Fourier-Wigner transform associated with the operator $\Delta_A$ is defined on $\mathcal{D}_*(\mathbb{R}) \times \mathcal{D}_*(\mathbb{R})$ by

$$\mathcal{W}(f,g)(x,\lambda) := \int_{\mathbb{R}} f(y)\mathcal{T}^x_A(g)(y)\varphi_\lambda(y)d\mu_A(y). \tag{20}$$

**Particular case**:

- If the function A is of the form $A(x) = |x|^{2\alpha+1}$ with $\alpha \ge \frac{-1}{2}$ then $\mathcal{W}$ coincide with the classical Dunkl-Wigner transform see [14].

**Remark 2.8.** the transform $\mathcal{W}$ is a bilinear mapping from $\mathcal{D}_*(\mathbb{R}) \times \mathcal{D}_*(\mathbb{R})$ into $\mathcal{S}(\mathbb{R}^2)$ and can be written as

$$\mathcal{W}(f,g)(x,\lambda) = \mathcal{F}_A(f\mathcal{T}^x_A(g))(\lambda) \tag{21}$$

$$= (g *_A f\varphi_\lambda)(x). \tag{22}$$

We have the following results.

**Proposition 2.4.** *Let $f,g \in L^2_A(\mathbb{R})$ then $\mathcal{W}(f,g)$ is well defined and belongs to $L^2_{\theta_A}(\mathbb{R}^2) \cap L^\infty_{\theta_A}(\mathbb{R}^2)$ and we have*

$$\|\mathcal{W}(f,g)\|_{2,\theta_A} \le \|f\|_{2,A}\|g\|_{2,A}, \tag{23}$$

*and*

$$\|\mathcal{W}(f,g)\|_{\infty,\theta_A} \le \|f\|_{2,\theta_A}\|g\|_{2,A}. \tag{24}$$

Proof. Is a consequence of the Hölder's inequality and the relations (12) and (20). □

## 3. Localization operators Associated with the Generalized Dunkl-Wigner transform

**3.1. Introduction.** *In this section, we introduce and give sufficient conditions for the boundedness, compactness and Schatten class properties of localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ associated with the generalized Dunkl-Wigner transform in terms of properties of the symbol $\sigma$ and the functions $\psi_1$ and $\psi_2$.*

**Definition 3.1.** Let $\psi_1$ and $\psi_2$ be measurable functions on $\mathbb{R}$, $\sigma$ be a measurable function on the set $\mathbb{R}^2$, we define the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ associated with the generalized Dunkl-Wigner transform by

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)(y) := \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\mathcal{W}(f,\psi_1)(x,\lambda)\varphi_\lambda(y)\overline{\mathcal{T}_A^x(\psi_2)(y)}\mathrm{d}\theta_A(x,\lambda). \tag{25}$$

**Remark 3.2.** In accordance with the different choices of the symbol $\sigma$ and the different continuities required, we need to impose different conditions on $\psi_1$ and $\psi_2$, and then we obtain an operator on $L^p_A(\mathbb{R})$ for all $1 \leq p \leq +\infty$.

It is more convenient to interpret the definition of $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ in a weak sense, that is for all $f \in L^p_A(\mathbb{R}), g \in L^q_A(\mathbb{R})$ we have

$$\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_A} = \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\mathcal{W}(f,\psi_1)(x,\lambda)\overline{\mathcal{W}(g,\psi_2)(x,\lambda)}\mathrm{d}\theta_A(x,\lambda). \tag{26}$$

We have the following result.

**Proposition 3.1.** *Let $1 \leq p \leq +\infty$, the adjoint of the linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^p_A(\mathbb{R}) \longrightarrow L^p_A(\mathbb{R})$$

*is the operator*

$$\mathcal{L}^*_{\psi_1,\psi_2}(\sigma)\ L^{p'}_A(\mathbb{R}) \longrightarrow L^{p'}_A(\mathbb{R})$$

*where*

$$\mathcal{L}^*_{\psi_1,\psi_2}(\sigma) = \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}). \tag{27}$$

Proof. Let $f \in L^p_A(\mathbb{R}), g \in L^{p'}_A(\mathbb{R})$ by using the relation (26), we have

$$\begin{aligned}\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_A} =& \overline{\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\sigma(x,\lambda)}\mathcal{W}(g,\psi_2)(x,\lambda)\overline{\mathcal{W}(f,\psi_1)(x,\lambda)}d\theta_A(x,\lambda)}\\ =&\overline{\langle \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}))(g) \mid f\rangle_{\mu_A}} = \langle f \mid \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}))(g)\rangle_{\mu_A},\end{aligned}$$

we get

$$\mathcal{L}^*_{\psi_1,\psi_2}(\sigma) = \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}).$$

□

*In the sequel of this section, $\psi_1$ and $\psi_2$ will be any functions in $L^2_A(\mathbb{R})$ such that $\|\psi_1\|_{2,A} = \|\psi_2\|_{2,A} = 1$. We note that this hypothesis is not essential and the result still true up some constant depending on $\|\psi_1\|_{2,A}$ and $\|\psi_2\|_{2,A}$.*

**3.2. Boundedness for $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ in $S_\infty$.** *The main purpose of this subsection is to prove that the linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R})$$

*is bounded for all symbol $\sigma \in L^p_{\theta_A}(\mathbb{R}^2)$ with $1 \leq p + \infty$. We consider first the problem for $\sigma \in L^1_{\theta_A}(\mathbb{R}^2)$, next $\sigma \in L^\infty_{\theta_A}(\mathbb{R}^2)$ and we conclude by using interpolation theory.*

**Proposition 3.2.** *Let $\sigma \in L^2_\theta(\mathbb{R}^2)$ then the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is in $S_\infty$ and we have*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{1,\theta_A}. \tag{28}$$

Proof. Let $f, g \in L^2_A(\mathbb{R})$ by using the relation (26), we have

$$|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_A}| \leq \|\mathcal{W}(f,\psi_1)\|_{\infty,\theta}\|\mathcal{W}(g,\psi_2)\|_{\infty,\theta}\|\sigma\|_{1,\theta_A},$$

by using the relation (24), we get

$$\left|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_A}\right| \leq \|f\|_{2,A}\|g\|_{2,A}\|\sigma\|_{1,\theta_A},$$

by (24) we find that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{1,\theta_A}$$

□

**Proposition 3.3.** *Let $\sigma \in L^\infty_{\theta_A}(\mathbb{R}^2)$, the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is in $S_\infty$ and we have*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta_A}. \tag{29}$$

Proof. Let $f, g \in L^2_A(\mathbb{R})$ by using the relation (26) and Hölder's inequality we find that

$$|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_A}| \leq \|\sigma\|_{\infty,\theta_A}\|\mathcal{W}(f,\psi_1)\|_{2,\theta}\|\mathcal{W}(g,\psi_2)\|_{2,\theta_A},$$

by using the relation (23) we get

$$|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_A}| \leq \|\sigma\|_{\infty,\theta_A}\|f\|_{2,A}\|g\|_{2,A},$$

thus

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta_A}.$$

□

*We can now associate a localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ to every symbol $\sigma$ in $L^p_{\theta_A}(\mathbb{R}^2)$, for all $1 \leq p \leq +\infty$, and prove that $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ belongs to $S_\infty$.*

**Theorem 3.4.** *Let $\sigma \in L^p_\theta(\mathbb{R}^2)$,$1 \leq p \leq +\infty$ then there exists a unique bounded linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R})$$

*such that*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{p,\theta_A}. \tag{30}$$

Proof. Let $\sigma \in L^p_\theta(\mathbb{R}^2), 1 \leq p \leq +\infty$ and $f \in L^2_A(\mathbb{R})$ we consider the following operator

$$T : L^1_{\theta_A}(\mathbb{R}^2) \cap L^\infty_{\theta_A}(\mathbb{R}^2) \longrightarrow L^2_A(\mathbb{R}),$$

given by

$$T(\sigma) = \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f),$$

then by using the relations (28) and (29), we have

$$\|T(\sigma)\|_{2,A} \leq \|f\|_{2,A}\|\sigma\|_{1,\theta_A} \tag{31}$$

and

$$\|T(\sigma)\|_{2,A} \leq \|f\|_{2,A}\|\sigma\|_{\infty,\theta_A}, \tag{32}$$

by using the relations (31), (32), and the Riesz-Thorin interpolation Theorem see [**18, 22**], the operator $T$ may be uniquely extended to a linear operator on $L^p_{\theta_A}(\mathbb{R}^2)$ for all $1 \leq p \leq +\infty$ and we have

$$\|T(\sigma)\|_{2,A} = \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)\|_{2,A} \leq \|f\|_{2,A}\|\sigma\|_{p,\theta_A}, \tag{33}$$

since (33) true for all $f \in L^2_A(\mathbb{R})$ which gives the desired result. □

**3.3. $L^p_A$-Boundedness of localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.** *Using Schur's technique* [**10**] *our main purpose of this subsection is to prove that the linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^p_A(\mathbb{R}) \longrightarrow L^p_A(\mathbb{R}),$$

*is bounded for all $1 \leq p \leq +\infty$, we have the following result.*

**Theorem 3.5.** *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2)$ and $\psi_1,\ \psi_2 \in L^1_A(\mathbb{R}) \cap L^\infty_A(\mathbb{R})$ then the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ extend to a unique bounded linear operator from $L^p_A(\mathbb{R})$ into itself for all $1 \leq p \leq +\infty$, furthermore we have*

$$\|\mathcal{L}_{u,v}(\sigma)\|_{B\left(L^p_A(\mathbb{R})\right)} \leq \max(\|\psi_1\|_{\infty,A}\|\psi_2\|_{1,A}\|\sigma\|_{1,\theta_A}, \|\psi_1\|_{1,A}\|\psi_2\|_{\infty,A}\|\sigma\|_{1,\theta_A}).$$

Proof. Let $F$ be the function defined on $\mathbb{R}^2$ by

$$F(y,s) = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \sigma(x,\lambda)\varphi_\lambda(y)\overline{\mathcal{T}^x_A(v)(y)}\varphi_\lambda(s)\mathcal{T}^x_A(u)(s)\mathrm{d}\theta_A(x,\lambda),$$

by using Fubini's theorem we find that

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)(y) = \int_{\mathbb{R}} F(y,s)f(s)d\mu_A(s),$$

furthermore by using the relation (10) and Fubini's theorem we find that

$$\int_{\mathbb{R}} |F(y,s)|d\mu_A(y) \leq \|\psi_1\|_{\infty,A}\|\psi_2\|_{1,A}\|\sigma\|_{1,\theta_A} \tag{34}$$

and

$$\int_{\mathbb{R}} |F(y,s)|d\mu_A(s) \leq \|\psi_1\|_{1,A}\|\psi_2\|_{\infty,A}\|\sigma\|_{1,\theta_A} \tag{35}$$

by using the relations (34), (35), and Schur's lemma [10] we can conclude that the linear operator

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^p_A(\mathbb{R}) \longrightarrow L^p_A(\mathbb{R}),$$

is bounded for all $1 \leq p \leq +\infty$ and we have

$$\|\mathcal{L}_{u,v}(\sigma)\|_{B\left(L^p_\alpha(\mathbb{R})\right)} \leq max(\|\psi_1\|_{\infty,A}\|\psi_2\|_{1,A}\|\sigma\|_{1,\theta_A}, \|\psi_1\|_{1,A}\|\psi_2\|_{\infty,A}\|\sigma\|_{1,\theta_A}).$$

□

**3.4. Trace of the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.** *The main result of this subsection is to prove that the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R}),$$

*is in the Schatten-Von Neumann class $S^p$ for all $1 \leq p \leq +\infty$, firstly we have the following result*

**Theorem 3.6.** *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2)$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R})$$

*is an Hilbert-Schmidt operator in particular it is compact and we have*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{HS} \leq 1 + \|\sigma\|^2_{1,\theta_A}.$$

Proof. Let $(\phi_k)_k$ be an orthonormal basis of $L^2_A(\mathbb{R})$, by using Fubini's theorem and the relations (21) and (26), we get

$$\begin{aligned} &\||\ \mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\phi_k\right)\|^2_{2,A} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\mathcal{F}_A\left(\phi_k \mathcal{T}^x_A(\psi_1)\right)(\lambda)\overline{\mathcal{F}_A\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\phi_k\right)\mathcal{T}^x_A(\psi_2)\right)(\lambda)}\mathrm{d}\theta_A(x,\lambda), \end{aligned}$$

by using the relation (27), we get

$$\mathcal{F}_A\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\phi_k\right)\mathcal{T}^x_{\alpha,\beta}(v)\right)(\lambda) = \left\langle \phi_k \mid \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma})\left(\overline{\mathcal{T}^x_A(\psi_2)}\right)\right\rangle_{\mu_A},$$

and by using Fubini's theorem, we find that

$$\begin{aligned} &\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|^2_{HS} \\ \leq &\frac{1}{2}\int_{\mathbb{R}^2}|\sigma(x,\lambda)|\left[\sum_{k=1}^{+\infty}\left|\langle\varphi_\lambda\mathcal{T}^x_A(\psi_1)\mid\phi_k\rangle_{\mu_A}\right|^2 + \left|\left\langle\mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma})\left(\overline{\mathcal{T}^x_A(\psi_2)\varphi_\lambda}\right)|\phi_k\right\rangle_{\mu_A}\right|^2\right]\mathrm{d}\theta_A(x,\lambda) \end{aligned}$$

By using Parseval's identity, the relations (4), (14), (28), and the fact that $\|\psi_1\|_{2,A} = \|\psi_2\|_{2,A} = 1$, we find that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|^2_{HS} \leq \frac{1}{2}\|\sigma\|_{1,\theta}\left(1 + \|\sigma\|^2_{1,\theta_A}\right) \leq \left(1 + \|\sigma\|^2_{1,\theta_A}\right)^2 < \infty$$

which proves that $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is an Hilbert-Schmidt operator so compact and we have

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{HS} \leq 1 + \|\sigma\|^2_{1,\theta_A}.$$

□

*In the following we prove that the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R})$$

*is compact for all $\sigma \in L^p_{\theta_A}(\mathbb{R}^2)$.*

**Proposition 3.7.** *Let $\sigma \in L^p_{\theta_A}(\mathbb{R}^2), 1 \leq p < +\infty$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R})$$

*is compact.*

Proof. Let $\sigma \in L^p_{\theta_A}(\mathbb{R}^2)$ with $1 \leq p < +\infty$ and let $(\sigma_n)_n$ be a sequence of functions in $L^1_{\theta_A}(\mathbb{R}^2) \cap L^\infty_{\theta_A}(\mathbb{R}^2)$ such that $\sigma_n \longrightarrow \sigma$ in $L^p_{\theta_A}(\mathbb{R}^2)$ as $n \longrightarrow \infty$ then by using the relation (30), we find that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma_n) - \mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma_n - \sigma\|_{p,\theta_A},$$

hence $\mathcal{L}_{\psi_1,\psi_2}(\sigma_n) \longrightarrow \mathcal{L}_{\psi_1,\psi_2}(\sigma)$ in $S_\infty$ as $n \longrightarrow \infty$ on the other hand by Theorem 3.6, we have $\mathcal{L}_{\psi_1,\psi_2}(\sigma_n)$ is in $S_2$ hence compact, it follows that $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is compact. □

*In the next theorem we obtain a $L^1_A$-compactness result for the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.*

**Theorem 3.8.** *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2)$, $\psi_1$ and $\psi_2$ in $L^1_A(\mathbb{R}) \cap L^\infty_A(\mathbb{R})$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^1_A(\mathbb{R}) \longrightarrow L^1_A(\mathbb{R})$$

*is compact.*

Proof. by using theorem 3.2 the linear operator

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^1_A(\mathbb{R}) \longrightarrow L^1_A(\mathbb{R})$$

is well defined, let $(f_n) \subset L^1_A(\mathbb{R}_+)$ such that $f_n \longrightarrow 0$ weakly in $L^1_A(\mathbb{R})$ as $n \longrightarrow \infty$, it is enough to prove that $\lim_{n\to+\infty} \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f_n)\|_{1,A} = 0$. By using the relation (25), we have

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f_n)\|_{1,A} \leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |\sigma(x,\lambda)\, \|\mathcal{W}(f_n,\psi_1)(x,\lambda)\|\, \mathcal{T}^x_A(\psi_2)(y)|\, \mathrm{d}\theta_A(x,\lambda)\right] d\mu_A(y). \tag{36}$$

Using the fact that $f_n \longrightarrow 0$ weakly in $L^1_A(\mathbb{R})$ as $n \longrightarrow \infty$, we deduce that

$$\lim_{n\to+\infty} |\mathcal{W}(f_n,\psi_1)(x,\lambda)\|\mathcal{T}^x_A(\psi_2)(y)| = 0, \tag{37}$$

for all $x, y, \lambda \in \mathbb{R}$, on the other hand as $f_n \longrightarrow 0$ weakly in $L^1_A(\mathbb{R})$ as $n \longrightarrow \infty$, there exists a positive conctant $c$ such that $\|f_n\|_{1,A} \leq c$, so we find that

$$|\, \mathcal{W}(f_n,\psi_1)\left((x,\lambda)\right\|\mathcal{T}^x_{\alpha,\beta}(\psi_2)(y)| \leq c|\sigma(x,\lambda)\|\, \psi_1\|_{\infty,\alpha,\beta}|\psi_2(y)|, \tag{38}$$

by using Fubuni's theorem we get

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|\sigma(x,\lambda)\,\|\mathcal{W}(f_n,\psi_1)(x,\lambda)\|\,\mathcal{T}_A^x(\psi_2)(y)|\,\mathrm{d}\theta_A(x,\lambda)\right]d\mu_A(y)<\infty. \tag{39}$$

Thus from the relations (36), (37), (38), (39), and the Lebesgue dominated convergence theorem we deduce that $\lim_{n\to+\infty}\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(f_n)\|_{1,A}=0$ and the proof is complete. □

*In the following we show that the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is in the trace class $S^1$.*

**Theorem 3.9.** *Let $\sigma\in L^1_{\theta_A}(\mathbb{R}^2)$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma):L^2_A(\mathbb{R})\longrightarrow L^2_A(\mathbb{R})$$

*is in the trace class operators $S_1$ and we have*

$$\|\tilde{\sigma}\|_{1,\theta_A}\le\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1}\le\|\sigma\|_{1,\theta_A} \tag{40}$$

*where $\tilde{\sigma}$ is the function given by*

$$\tilde{\sigma}(x,\lambda)=\left\langle\mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\varphi_\lambda\mathcal{T}^x_{\alpha,\beta}(\psi_1)\right)\mid\varphi_\lambda\mathcal{T}^x_A(\psi_2)\right\rangle_{\mu_A}.$$

Proof. Let $\sigma\in L^1_{\theta_A}(\mathbb{R}^2)$ by using Theorem 3.4, we have $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is a compact operator, using [22], there exists an orthonormal basis $\phi_j$ for $j=1,2,\ldots$, for the orthogonal complement of the kernel of the operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ consisting of eigenvectors of $|\mathcal{L}_{\psi_1,\psi_2}(\sigma)|$ and $(h_j)$, $j=1,2,\ldots$, an orthonormal set in $L^2_A(\mathbb{R})$ such that the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ can be diagonalized as

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)=\sum_{j=1}^{+\infty}s_j\,\langle f\mid\phi_j\rangle_{\mu_A}h_j, \tag{41}$$

where $s_j$ for $j=1,2,\ldots$, are the positive singular values of $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ corresponding to $\phi_j$, then we get

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1}=\sum_{j=1}^{+\infty}s_j=\sum_{j=1}^{+\infty}\langle\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(\phi_j)\mid h_j\rangle_{\mu_A},$$

by using the relations (25) and (26), we find that

$$\begin{aligned}&\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1}\\ \le&\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}|\sigma(x,\lambda)|\left[\sum_{j=1}^{+\infty}\left|\langle\varphi_\lambda\mathcal{T}^x_A(\psi_1)\mid\phi_j\rangle_{\mu_A}\right|^2|+\sum_{j=1}^{+\infty}\left|\langle\varphi_\lambda\mathcal{T}^x_A(\psi_2)\mid h_j\rangle_{\mu_A}\right|^2\right]\mathrm{d}\theta_A(x,\lambda)\end{aligned}$$

by using Parseval's identity, we get

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1}\le\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}|\sigma(x,\lambda)\|\varphi_\lambda\mathcal{T}^x_A(\psi_2)\,\|^2_{2,A}+\|\,\varphi_\lambda\mathcal{T}^x_A(\psi_2)\|^2_{2,A}\mathrm{d}\theta_A(x,\lambda).$$

By using the relations (4), (12), and the fact that $\|\psi_1\|_{2,A}=\|\psi_2\|_{2,A}=1$, we get

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1}\leq\|\sigma\|_{1,\theta_A}.$$

Now, we prove that $|\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ satisfies the first member of (40), it is easy to see that $\tilde{\sigma}\in L^1_{\theta_A}(\mathbb{R}^2)$ and by using the relation (41) and Fubini's theorem, we find that

$$\begin{aligned}\|\tilde{\sigma}\|_{1,\theta_A}&\leq\frac{1}{2}\sum_{j=1}^{+\infty}s_j\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\left|\langle\varphi_\lambda\mathcal{T}_A^x(\psi_1)\mid\phi_j\rangle_{\mu_A}\right|^2+\left|\langle h_j\mid\varphi_\lambda\mathcal{T}_A^x(\psi_2)\rangle_{\mu_A}\right|^2\right)\mathrm{d}\theta_A(x,\lambda)\right]\\&=\frac{1}{2}\sum_{j=1}^{+\infty}s_j\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{W}\left(\phi_j,\psi_1\right)(x,\lambda)\right|^2+\left|\mathcal{W}\left(h_j,\psi_2\right)(x,\lambda)\right|^2\right]\mathrm{d}\theta_A(x,\lambda),\end{aligned}$$

by using the relation (23) and the fact that $\|\psi_1\|_{2,A}=\|\psi_2\|_{2,A}=1$, we get

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}|\tilde{\sigma}(x,\lambda)|d\theta_A(x,\lambda)\leq&\frac{1}{2}\sum_{j=1}^{+\infty}s_j\left(\|\psi_1\|^2_{2,A}+\|\psi_2\|^2_{2,A}\right)\\=&\sum_{j=1}^{+\infty}s_j\\=&\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1},\end{aligned}$$

the proof is complete. □

*In the following we give a trace formula for the localization operators* $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.

**Theorem 3.10.** *Let* $\sigma\in L^1_{\theta_A}(\mathbb{R}^2)$ *we have the following trace formula*

$$\operatorname{Tr}\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\right)=\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(x,\lambda)\left\langle\varphi_\lambda\mathcal{T}_A^x(\psi_1)\mid\varphi_\lambda\mathcal{T}_A^x(\psi_2)\right\rangle_{\mu_A}\mathrm{d}\theta_A(x,\lambda)\tag{42}$$

Proof. Let $\{\phi_j, j=1,2,\ldots\}$ be an orthonormal basis for $L^2_A(\mathbb{R})$. From Theorem 3.9, the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ belongs to $S_1$, then by the definition of the trace given by the relation (18), Fubini's theorem and Parseval's identity, we get

$$\begin{aligned}\operatorname{Tr}\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\right)&=\sum_{j=1}^{\infty}\left\langle\mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\phi_j\right),\phi_j\right\rangle_{\mu_A}\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(x,\lambda)\sum_{j=1}^{\infty}\left\langle\phi_j,\overline{\varphi_\lambda\mathcal{T}_A^x(\psi_1)}\right\rangle_{\mu_A}\left\langle\overline{\varphi_\lambda\mathcal{T}_A^x(\psi_2)},\phi_j\right\rangle_{\mu_A}\mathrm{d}\theta_A(x,\lambda)\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(x,\lambda)\left\langle\varphi_\lambda\mathcal{T}_A^x(\psi_1)\mid\varphi_\lambda\mathcal{T}_A^x(\psi_2)\right\rangle_{\mu_A}\mathrm{d}\theta_A(x,\lambda),\end{aligned}$$

and the proof is complete. □

*In the following we give the main result of this section.*

**Corollary 3.11.** *Let $\sigma$ in $L^p_{\theta_A}(\mathbb{R}^2), 1 \leq p \leq +\infty$ then, the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}) \longrightarrow L^2_A(\mathbb{R})$$

*is in $S^p$ and we have*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_p} \leq \|\sigma\|_{p,\theta_A}.$$

Proof. The result follows from (29) and (40) and by interpolation theory see [18]. □

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