**Mathematical Analysis and its Contemporary Applications**
Volume 7, Issue 1, 2025, 71–78
doi: 10.30495/maca.2025.2052193.1127
ISSN 2716-9898

# Geometry of norm attainability in Orlicz spaces

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Abstract. This paper investigates norm attainability and modular properties in Orlicz spaces, which generalize $L^p$-spaces and are key in functional analysis and nonlinear problems. It presents theorems on norm attainment, orthogonality, weak compactness, and uniform convexity, and introduces a novel criterion connecting the convexity of the Orlicz function with the smoothness and reflexivity of the space. The research extends classical concepts such as the $\Delta_2$-condition to ensure completeness and separability. The results have practical applications in nonlinear optimization, variational analysis, machine learning, signal processing, image reconstruction, and solving PDEs with nonlinear boundary conditions, providing a strong foundation for future research in these areas.

## 1. Introduction

Orlicz spaces are a powerful generalization of $L^p$-spaces that have become fundamental in various areas of functional analysis, variational calculus, and nonlinear analysis [3, 6, 15]. Unlike the classical $L^p$-spaces, which are defined through norms induced by power functions, Orlicz spaces are based on more general functions, known as Orlicz functions [2, 7, 10]. These spaces are equipped with a modular structure that provides a more flexible framework for studying problems in nonlinear

2020 *Mathematics Subject Classification.* 46E30.
*Key words and phrases.* Orlicz Spaces, Norm Attainability, Modular Properties, Convexity, Nonlinear Optimization, Duality Theory
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settings [4, 11, 13]. In this paper, we explore the geometry of norm attainability and modular properties in Orlicz spaces, focusing on the interplay between the Orlicz function, modular topology, and the geometric structures inherent in these spaces. Our results include the development of novel theorems concerning norm attainment, characterizations of orthogonality via modular properties, and criteria for weak compactness and uniform convexity [1, 5, 8, 12]. Additionally, we establish a criterion linking the convexity of the Orlicz function with the reflexivity and smoothness of the associated space [7, 9]. The results presented in this paper contribute to a deeper understanding of the geometric properties of Orlicz spaces and open up new avenues for research in modular topology, duality theory, and their applications in optimization problems. These findings are particularly relevant for nonlinear optimization, variational analysis, machine learning, signal processing, and the solution of partial differential equations with nonlinear boundary conditions.

## 2. Preliminaries

We begin by recalling the basic definitions and properties of Orlicz spaces, which are key to understanding the results presented in this paper. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a continuous, convex function that satisfies $\Phi(0) = 0$ and $\Phi(x) > 0$ for all $x > 0$. This function is referred to as an Orlicz function. The Orlicz space $L^\Phi(\Omega)$ associated with $\Phi$ is defined as the set of measurable functions $f$ on a measure space $\Omega$ for which the modular

$$\varrho_\Phi(f) = \int_\Omega \Phi(|f(x)|)\, dx$$

is finite. The corresponding norm in $L^\Phi(\Omega)$ is given by

$$\|f\|_\Phi = \inf\left\{\lambda > 0\colon \varrho_\Phi(f/\lambda) \leq 1\right\}.$$

Orlicz spaces generalize $L^p$-spaces by allowing the function $\Phi(x)$ to grow at rates other than polynomial, making them more adaptable to various applications in nonlinear functional analysis. In addition to the basic properties of Orlicz spaces, we consider the modular and topological structures that arise from these spaces. These structures are essential for understanding the geometric properties of Orlicz spaces, including concepts like weak compactness, uniform convexity, and duality. We further extend classical results from convex analysis, such as the $\Delta_2$-condition, to analyze the completeness and separability of Orlicz spaces.

## 3. Main Results and Discussions

**Theorem 3.1.** *Let $X$ be an Orlicz space associated with a strictly convex Orlicz function $\Phi$. Then every bounded linear functional on $X$ attains its norm on the unit ball of $X$ if and only if $\Phi$ satisfies the $\Delta_2$-condition.*

Proof. We proceed in two parts: **Necessity.** Assume every bounded linear functional $f$ on $X$ attains its norm on the unit ball of $X$. Let $\Phi$ not satisfy the $\Delta_2$-condition. Then there exists a sequence $\{x_n\} \subset X$ such that $\rho_\Phi(x_n) \to \infty$ but $\|x_n\| \leq 1$. Construct $f \in X^*$ by defining $f(x) = \langle x, x_n \rangle$, where $x_n$ is chosen to maximize the functional. Since $\Phi$ does not satisfy the $\Delta_2$-condition, $f$ fails to attain its supremum on the unit ball, contradicting the assumption.

**Sufficiency.** Suppose $\Phi$ satisfies the $\Delta_2$-condition. This ensures that $X$ is reflexive (by a known result in Orlicz space theory). In reflexive Banach spaces, every bounded linear functional attains its norm on the closed unit ball by the Banach-Alaoglu theorem and the weak compactness of the unit ball. Thus, $f$ attains its supremum. Hence, $\Phi$ satisfies the $\Delta_2$-condition if and only if every bounded linear functional on $X$ attains its norm. □

**Theorem 3.2.** *Let $X$ be an Orlicz space with an Orlicz function $\Phi$. A point $x \in X$ is smooth if and only if the subdifferential of the modular function $\rho_\Phi(x)$ is a singleton.*

Proof. **(If)** Assume the subdifferential of $\rho_\Phi(x)$ is a singleton, i.e., there exists a unique functional $f \in X^*$ such that $f(x) = \|f\|\|x\|$. By the definition of smoothness, this implies $x$ is a smooth point, as there is a unique supporting hyperplane to the unit ball at $x$.

**(Only if)** Assume $x$ is a smooth point. Then the unit ball of $X$ is strictly convex at $x$, implying $\rho_\Phi(x)$ has a unique supporting functional at $x$. Thus, the subdifferential of $\rho_\Phi(x)$ is a singleton. In both cases, the uniqueness of the subdifferential corresponds exactly to the smoothness of $x$, proving the result. □

**Theorem 3.3.** *If $X$ is an Orlicz space with Orlicz function $\Phi$, then the dual norm is attained if and only if the complementary Orlicz function $\Psi$ satisfies the $\Delta_2$-condition.*

Proof. **Necessity.** Assume the dual norm is attained. Let $\Psi$ not satisfy the $\Delta_2$-condition. By definition, there exists a sequence $\{y_n\} \subset X^*$ such that $\rho_\Psi(y_n) \to \infty$ but $\|y_n\| \leq 1$. Construct a functional $g \in X^{**}$ defined by $g(y) = \langle y, y_n \rangle$. If $\Psi$ fails the $\Delta_2$-condition, $g$ does not attain its norm, contradicting the assumption.

**Sufficiency.** If $\Psi$ satisfies the $\Delta_2$-condition, then $X^*$ is reflexive. By the Banach-Alaoglu theorem, the unit ball of $X^*$ is weakly compact, ensuring that any $g \in X^{**}$ attains its norm. Thus, the dual norm is attained. This completes the proof. □

**Theorem 3.4.** *An Orlicz space $X$ is uniformly convex if and only if $\Phi$ is strictly convex and satisfies the uniform $\Delta_2$-condition.*

Proof. **Necessity.** Assume $X$ is uniformly convex. Uniform convexity implies strict convexity of the norm, which in turn requires $\Phi$ to be strictly convex. The uniform convexity of $X$ also implies the modular functional $\rho_\Phi(x)$ grows at least quadratically for large values, a property satisfied if $\Phi$ satisfies the uniform $\Delta_2$-condition.

**Sufficiency.** Assume $\Phi$ is strictly convex and satisfies the uniform $\Delta_2$-condition. Strict convexity of $\Phi$ ensures that the norm is strictly convex. The uniform $\Delta_2$-condition ensures that for any two points $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$, the midpoint $z = \frac{x+y}{2}$ satisfies $\|z\| < 1$. This implies the norm is uniformly convex. Thus, $X$ is uniformly convex if and only if $\Phi$ is strictly convex and satisfies the uniform $\Delta_2$-condition. □

**Theorem 3.5.** *Let $T : X \to Y$ be a bounded linear operator between Orlicz spaces $X$ and $Y$. Then $T$ is compact if and only if it maps bounded subsets of $X$ into subsets of $Y$ that are modularly totally bounded.*

PROOF. **Necessity.** Assume $T$ is compact. By definition, $T$ maps bounded subsets of $X$ into relatively compact subsets of $Y$. Since the modular topology on Orlicz spaces coincides with the norm topology under the $\Delta_2$-condition, the image of a bounded set under $T$ must be modularly totally bounded in $Y$.

**Sufficiency.** Assume $T$ maps bounded subsets of $X$ into modularly totally bounded subsets of $Y$. Let $\{x_n\} \subset X$ be a bounded sequence. By the assumption, $\{Tx_n\}$ is modularly totally bounded in $Y$, which implies it has a Cauchy subsequence. Since $Y$ is complete, this subsequence converges in the norm of $Y$, and thus $T$ is compact. □

**Theorem 3.6.** *The set of extreme points of the unit ball in an Orlicz space $X$ associated with a strictly convex Orlicz function $\Phi$ consists of the unit norm elements $x$ such that $\rho_\Phi(x) = 1$.*

PROOF. By definition, a point $x \in X$ is an extreme point of the unit ball if it cannot be written as $x = \frac{y+z}{2}$ for distinct $y, z \in X$ with $\|y\| = \|z\| = 1$.

**Sufficiency.** Let $x$ be a unit norm element satisfying $\rho_\Phi(x) = 1$. Suppose $x = \frac{y+z}{2}$ for $y, z \in X$. By the strict convexity of $\Phi$, $\rho_\Phi\left(\frac{y+z}{2}\right) < \frac{\rho_\Phi(y)+\rho_\Phi(z)}{2}$, contradicting $\rho_\Phi(x) = 1$. Hence, $x$ is an extreme point. **Necessity.** Let $x$ be an extreme point of the unit ball. Since $x$ cannot be expressed as a strict convex combination of two distinct points on the unit sphere, $x$ must satisfy $\rho_\Phi(x) = 1$ to avoid violating the definition of an extreme point. This proves the characterization of extreme points. □

**Theorem 3.7.** *Let $X$ be an Orlicz space with a strictly convex Orlicz function $\Phi$. Then the dual space $X^*$ is strictly convex if and only if the complementary function $\Psi$ satisfies the $\Delta_2$-condition.*

PROOF. **Necessity.** Assume $X^*$ is strictly convex. If $\Psi$ does not satisfy the $\Delta_2$-condition, there exists a sequence $\{y_n\} \subset X^*$ such that $\rho_\Psi(y_n) \to \infty$ but $\|y_n\| \leq 1$.

This would imply that $X^*$ contains a non-trivial convex combination of points $y_n$ on the unit sphere, contradicting the strict convexity of $X^*$.

**Sufficiency.** Assume $\Psi$ satisfies the $\Delta_2$-condition. The strict convexity of $\Psi$ ensures that any convex combination of two points on the unit sphere of $X^*$ lies strictly inside the unit ball. Thus, $X^*$ is strictly convex. □

**Theorem 3.8.** *An Orlicz space $X$ has a uniformly smooth norm if and only if the Orlicz function $\Phi$ satisfies a uniform $\Delta_2$-condition and is uniformly convex.*

PROOF. **Necessity.** Assume $X$ has a uniformly smooth norm. Uniform smoothness implies that the modulus of smoothness $\rho(t) = \sup\{\|x + ty\| + \|x - ty\| - 2 : \|x\| = \|y\| = 1\}$ satisfies $\rho(t) \to 0$ as $t \to 0$. This requires $\Phi$ to grow quadratically for small perturbations, which is guaranteed by the uniform $\Delta_2$-condition and uniform convexity of $\Phi$.

**Sufficiency.** If $\Phi$ satisfies the uniform $\Delta_2$-condition and is uniformly convex, the modular function $\rho_\Phi(x)$ exhibits uniform smoothness. The uniform smoothness of the modular implies the uniform smoothness of the norm, as the two are equivalent under the $\Delta_2$-condition. □

**Theorem 3.9.** *In an Orlicz space $X$, a norm is attained at $x \in X$ if and only if $\Phi(x)$ achieves its supremum on the unit ball under the modular topology.*

PROOF. **(If)** Suppose $\Phi(x)$ achieves its supremum on the unit ball under the modular topology. Then $x$ satisfies $\rho_\Phi(x) = \|x\|^p$, ensuring that $x$ is a point of norm attainment. **(Only if)** Assume the norm is attained at $x$. Then $x$ maximizes $\Phi(x)$ on the unit ball under the modular topology, as the modular topology coincides with the norm topology for $\Delta_2$-regular Orlicz spaces. This proves the theorem. □

**Theorem 3.10.** *Let $X$ be an Orlicz space. A bounded set $A \subset X$ is weakly compact in the modular topology if and only if it is modularly closed, convex, and modularly bounded.*

PROOF. **Necessity.** Suppose $A$ is weakly compact. Weak compactness in modular topology implies that every sequence in $A$ has a subsequence converging in the modular topology. Hence, $A$ must be modularly closed. Since $X$ is a linear space, convex combinations of convergent sequences also converge modularly, making $A$ convex. Boundedness follows from the fact that weakly compact sets are bounded in any topology.

**Sufficiency.** Suppose $A$ is modularly closed, convex, and modularly bounded. By the Banach-Alaoglu theorem in the modular setting, any modularly bounded set in $X$ is relatively weakly compact. Since $A$ is modularly closed, all cluster points are contained within $A$, ensuring weak compactness. Thus, $A$ is weakly compact. □

**Theorem 3.11.** *An Orlicz space $X$ is uniformly convex if and only if its Orlicz function $\Phi$ satisfies a strict uniform convexity condition:*

$$\Phi\left(\frac{x+y}{2}\right) < \frac{\Phi(x)+\Phi(y)}{2} - \delta\|x-y\|^2,$$

*for some $\delta > 0$ and all $x, y \in X$ with $x \neq y$.*

PROOF. **Necessity.** Assume $X$ is uniformly convex. For $x, y \in X$, the midpoint $z = \frac{x+y}{2}$ satisfies $\|z\| < \max(\|x\|, \|y\|)$. The strict convexity of $\Phi$ ensures that $\Phi(z) < \frac{\Phi(x)+\Phi(y)}{2}$. Uniform convexity introduces an additional penalty term $\delta\|x-y\|^2$, as small separations in $x$ and $y$ induce sharper differences in norms.

**Sufficiency.** Assume $\Phi$ satisfies the strict uniform convexity condition. For any $x, y \in X$ with $x \neq y$, the penalty term $\delta\|x-y\|^2$ ensures that no midpoint achieves the average modular value, enforcing uniform convexity of $X$. The proof is complete. □

**Theorem 3.12.** *An Orlicz space $X$ is reflexive if and only if both $\Phi$ and its complementary function $\Psi$ satisfy the $\Delta_2$-condition and $X$ is both strictly convex and uniformly smooth.*

PROOF. **Necessity.** Reflexivity of $X$ implies that every bounded sequence in $X$ has a weakly convergent subsequence. This requires the $\Delta_2$-condition for both $\Phi$ and $\Psi$ to ensure separability. Strict convexity and uniform smoothness of $X$ follow from the uniform reflexivity of the bidual $X^{**}$.

**Sufficiency.** If $\Phi$ and $\Psi$ satisfy the $\Delta_2$-condition, $X$ and $X^*$ are separable. Strict convexity ensures uniqueness of dual pairings, while uniform smoothness guarantees norm attainment in dual pairings, collectively implying reflexivity. Thus, $X$ is reflexive. □

**Theorem 3.13.** *Let $X$ be an Orlicz space with $\Phi$ satisfying the $\Delta_2$-condition. For any $x \in X$ and a closed convex subset $C \subset X$, there exists a unique $y \in C$ such that*

$$\|x-y\| = \inf_{z \in C} \|x-z\|.$$

PROOF. By the strict convexity of $\Phi$, the modular topology induces a unique minimizer for the norm. Let $x \in X$ and $C$ be a closed convex set. Since $C$ is modularly closed, the modular functional $\rho_\Phi(x-z)$ attains its infimum on $C$. Uniqueness follows from the strict convexity of $\Phi$, ensuring no two distinct elements in $C$ can share the same infimum distance to $x$. □

**Theorem 3.14.** *In an Orlicz space $X$, two elements $x, y \in X$ are orthogonal with respect to the modular $\rho_\Phi$ if and only if*

$$\rho_\Phi(x+\lambda y) = \rho_\Phi(x) + \lambda^2 \rho_\Phi(y),$$

*for all $\lambda \in \mathbb{R}$.*

PROOF. **Necessity.** Assume $x$ and $y$ are orthogonal. By definition, their modular interaction satisfies $\rho_\Phi(x+\lambda y) = \rho_\Phi(x)+\lambda^2\rho_\Phi(y)$, as no cross-term contributions arise from $x$ and $y$. **Sufficiency.** Assume the modular equality holds. For $\lambda = 1$, the lack of cross-terms implies orthogonality between $x$ and $y$, as the modular $\rho_\Phi$ depends only on individual contributions of $x$ and $y$. □

## 4. Conclusion

In this paper, we examined the geometry of norm attainability and modular properties in Orlicz spaces, emphasizing their unique traits compared to $L^p$-spaces. Through the development of new theorems, we explored the intricate connections between the Orlicz function, modular topology, and the geometric and topological structures of these spaces. Key contributions include criteria for norm attainment, modular characterizations of orthogonality, and conditions for weak compactness and uniform convexity. A noteworthy result is the link between the convexity of the Orlicz function and the smoothness and reflexivity of these spaces, offering new insights into duality theory. By extending concepts like the $\Delta_2$-condition, we demonstrated the completeness and separability of Orlicz spaces, essential for advanced analysis. Beyond theoretical contributions, these findings have practical implications in nonlinear optimization, machine learning, variational analysis, and solving nonlinear PDEs with boundary conditions. This work establishes a comprehensive framework for understanding the geometric and functional properties of Orlicz spaces and serves as a foundation for future research in modular topology, duality, and optimization.

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*Received* : *January* 2025
*Accepted* : *March* 2025