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Orthogonal projections approach to characterization of numerical ranges of elementary operators

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Abstract. Properties of elementary operators have been characterized over the years for instance numerical ranges the operators are implemented by other different types of operators. However, elementary operators implemented by orthogonal projections have not been characterized in terms of numerical ranges. In this paper, we characterize elementary operators in terms of numerical ranges, when they are implemented by orthogonal projections. The results show that the numerical range of a basic elementary operator satisfies the ellipsoidicity criterion. Moreover, the numerical range of a Jordan elementary operator satisfies the spheroidicity criterion.

1. Introduction

The term elementary operator originates from the broader framework of analysis involving operator algebras [52]. It emerged as a natural classification for operators that are constructed from a limited number of given operators using algebraic operations such as multiplication and addition. In the mid-20th century, the study of

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derivations and perturbation theory in Banach and Hilbert spaces led to the investigation of operators defined through elementary algebraic expressions [37]. Some studies introduced foundational aspects of elementary operators in the context of general algebras and von Neumann algebras, particularly in relation to derivations and automorphisms [55]. Other authors discussed the the structures of elementary operators, emphasizing their role in spectral theory. Therefore, term elementary signifies that these operators are built from simpler components rather than arising as general abstract transformations [3]. The classification of these operators was particularly useful in studying operator inequalities, numerical range and perturbation theory. There are already many great assessments on expositions of specific issues in the extensive literature on elementary operators [2]. In a series of notes published in the earlier decades, Sylvester introduced elementary operators by calculating the eigenvalues of the matrices that corresponded to the generalized derivation. The work of [20] first used the term elementary operators when they calculated the spectrum of these operators.

The length and numerical range of elementary operators are the primary focus of the intriguing field of structural theory of elementary operators research. The connection between spatial numerical range and spectrum has been studied over time. Above all, literature has demonstrated that the NR is convex. This was first proposed by [4] and the idea of numerical range of operators for matrices. Applying the classical Toeplitz theorem, the study demonstrated that the numerical range is a convex set, which is a crucial numerical range property. The work of [58] separately expanded on this idea to include the spatial numerical range.

The closure of the numerical range is equivalent to the algebraic numerical range, as seen in [3]. In [11] presented the idea of the largest numerical range whereby the research determined the norm of the inner and generalized derivation by proving that the maximal numerical range as a set is nonempty, closed, and convex. In [1] the author gave an interesting exposition in a general settings based on their research on elementary operators. The authors in [57] investigated the properties of numerical ranges of elementary operators with more interest on the various types of numerical ranges. The study focussed on the relationships between these numerical ranges and determined that the classical numerical range is equal to the algebraic numerical range. The elementary operators in consideration were the ones induced by normal operators [12]. Moreover, spectra of elementary operators were also considered in this work particularly the point spectrum and the joint spectrum. However, the work did not investigate the numerical ranges and the spectra of elementary operators implemented by orthogonal projections [6]. Since orthogonal projections are very interesting operators that leave vectors unchanged after acting on them twice, then studying elementary operators implemented by orthogonal projections remains interesting [33].

Runji [23] investigated the properties of numerical ranges of elementary operators with more interest on the various types of numerical ranges. The study focussed on the relationships between these numerical ranges and determined that the classical numerical range is equal to the algebraic numerical range. The elementary operators in consideration were the ones induced by normal operators. Moreover, spectra of elementary operators were also considered in this work particularly the point spectrum and the joint spectrum. However, the work did not investigate the numerical ranges and the spectra of elementary operators implemented by orthogonal projections. Since orthogonal projections are very interesting operators that leave vectors unchanged after acting on them twice [26], then studying elementary operators implemented by orthogonal projections remains interesting. Barraa [4] also on his study of the essential numerical range of elementary operators, carried out interesting research on a general setting of Banach algebras. The study was very particular on the essential numerical range and their properties in Banach algebras. Also, Barraa [4] considered essential numerical range and joint numerical range and their characterizations. As the name suggests, numerical ranges which are essential are very interesting in a Banach algebra setting. It was shown that The essential numerical range is a subset of the joint numerical range. This inclusion however is not strict and gives room for more studies [8]. It was left an open question to determine whether conditions can be obtained for a stricter inclusion [16]. However, with these important results, the study did not consider numerical range of elementary operators implemented by orthogonal projections. The focus was on self-adjoint operators [34] in Banach algebras.

By considering a different type of numerical range that is the joint spatial numerical range, Seddik[54] in his work on the NR and norm of elementary operators came up with an interesting characterization.

In the study of special cases, [54] considered the joint spatial numerical range and the algebraic numerical range. The works showed that the joint spatial numerical range is contained in the algebraic numerical range. This work was done on general Banach algebras however, the work did not consider numerical range and the spectra of elementary operators implemented by orthogonal projections.

Okelo [47] surveyed numerical ranges of operators that are normal in Hilbert spaces and characterized elementary operators induced by them. It was confirmed that for normal operators, convexity of their numerical ranges suffices. Moreover, the elementary operators that they induce also have convex numerical ranges. This is a confirmation of one of the known results that the numerical range is always convex [18]. This holds true for general BA and also in all the spacial cases that have been considered in [7] and [15]. This characterization has also been done in other spaces like the Hardy spaces [24], Bergman spaces [13] and also in both complex and real cases [21].

In [55], the author gave an alternative proof to a useful principle on numerical ranges. The proof of Toeplitz-Hausdorff's Theorem that this study gave was more elegant, simple and easy to understand as described in [25]. This is one of the results that fall under many surveys that has been done regarding convexity for every bounded Linear Operator in all spaces [17]. In a given context an analogy [24] has been given to show that elementary operators also exhibit this property however,no work has been considered when the elementary operators are induced by orthogonal projections [5]. It is worth-noting that elementary operators have very intricate underlying algebraic structures when the inducing operators are changed and hence this makes it very interesting to consider elementary operators when their implementing operators are orthogonal projections [49].

Hwa-Long and Pei [31] also studied the shape of the boundary of numerical ranges in complex spaces and considered the elementary operators when they are induced by hyponormal operators. The shape of the boundary of a NR is a very important property that has been considered by many authors (see [15], [22], [28] and [27] for details). The authors in [31] showed that the boundary of the numerical range forms a compact spheroid. The closedness of the numerical range showed that the elements in the numerical range are limit points [50] that gave the compactness criterion for Hilbert spaces as stated in [53]. Analogously, in [29] the elementary operators induced by these operators were also considered in [45] and the same result obtained in terms of the shape of the boundary of numerical ranges but the shape was small in size compared to other operators when the real case was studied [54]. This work did not study consider elementary operators that are induced by orthogonal projections and therefore it is interesting to determine if a characterization in this perspective gives similar results or a deviation as recommended in [48].

In [25] the author also considered characterizations of in a broader sense and worked on the relationship between numerical ranges of different types. In a different study [46], the classical numerical range was compared with other types of numerical ranges in-terms of their ellipsoidity. This also forms a remarkable contribution on the study of numerical ranges as seen in [30]. However, this study did not consider elementary operators at all. Even the elementary operators that are induced by orthogonal projections were not and therefore it is interesting to determine if a characterization in this perspective gives similar results or a deviation for elementary operators that implemented by orthogonal projections.

The work of [18] showed a research done in a stricter sense for Hilbert space operators regarding the completeness of numerical ranges. The result in this study shows a characterization of convexoid operators which is also a very interesting area in the theory of operators in terms of certain fundamental properties as expressed also in [32]. This work considered the aspect of completeness. It characterized and showed that the numerical range is complete particularly for Hilbert space operators. This work considered a wide range of operators including seminormal [33], hyponormal [36], normal (see [35], [42] and [38] for details and the references therein), norm-attainable [40] among others. The study however did not focus on elementary operators. Even the elementary operators that are induced by orthogonal projections were not considered and therefore it is interesting to determine if a characterization in this perspective gives similar results or a deviation for elementary operators that implemented by orthogonal projections [41].

The work of [56] gave another look at proved the Toeplitz-Hausdorff's Theorem and the famous Folk theorem. Different notions of numerical ranges were considered. In deed a consideration was given to the numerical radius of Hilbert space operators. This was done by doing a comparison with the norms and it was determined that the numerical radius of the operators equals the norms of such operators. The operators considered included spectraloid [39], unitary [10], convexoid [43], normal and hyponormal [2]. Since elementary operators were not considered in this work, it is therefore important to consider elementary operators and to determine if the same results hold for elementary operators particularly when they are implemented by orthogonal projections.

2. Preliminaries

In this section properties of orthogonal projections which are used in later discussion are stated and key concepts are defined.

Definition 2.1. ([1]) A mapping G is an elementary operator if its formation is $G(R) = \sum_{i=1}^{n} M_i R M_i$, for all R in an algebra $\mathcal{A}$, where M_i, B_i are fixed in $\mathcal{A}$. We have the left multiplication operator, right multiplication operator, inner derivation, generalized derivation, basic elementary operator and Jordan elementary operator as examples of elementary operators.

Definition 2.2. ([28]) The numerical range of a map F is given by:
$W(F) = \{\langle Tz, z\rangle : z \in \mathcal{H}, \text{where z is a unit vector.}\}$.

Definition 2.3. ([8]) An orthogonal projection is a linear operator P on an inner product space that maps every vector onto a subspace such that the difference between the original vector and its projection is orthogonal to that subspace.

3. Main Results

The first property considered for characterization is the numerical range of elementary operators induced by orthogonal projections. We note that all orthogonal projections are in the algebra of all orthogonal projections on a Hilbert space denoted by $B_{op}(\mathcal{H})$. We begin by auxiliary results on orthogonal projections which are crucial in the sequel. Our first characterization is given in the proposition below.

Proposition 3.1. *Let $P_1, P_2 \in B_{op}(\mathcal{H})$. Then*

$$co(W(P_1) \odot W(P_2))^- \subseteq W_0(R_q(P_1, P_2)),$$

where $q \geq 1$.

PROOF. Let f be a state in $B_{op}(\mathcal{H})$ with $f(1) = 1$. Since

$$f(R_q(P_1, P_2)) = \sum_{i=1}^{n} \langle P_{1i}x, x\rangle \cdot \langle P_{2i}y, y\rangle \in W_o((R_q(P_1, P_2)),$$

we get $W(P_1) \circ W(P_2) \subset W_0(R_q(P_1, P_2))$, and since $W_0(R_q(P_1, P_2))$ satisfies compactness criterion and Toeplitz-Hausdorff's Theorem of convexity criterion, we have that

$$co(W(P_1) \circ W(P_2))^- \subset W_o(R_q(P_1, P_2)).$$

□

Proposition 3.2. *For $P \in B_{op}(\mathcal{H})$, We have that $W_0(L_p|l_q) = W_0(R_p|l_q) = W_0(P)$, where l_q is a general Banach space.*

PROOF. The proof for the equality requires that we show both inclusions. We first note that the inclusion $W_0(P) \subseteq W_0(L_p|l_q)$, $W_0(P) \subseteq W_0(R_p|l_q)$ follows from Proposition 3.1. Therefore, $W_0(L_p|l_q) \subseteq W_0(P)$. An analogy gives $W_0(R_p|l_q) \subseteq W_0(P)$. □

Proposition 3.3. *For $P_1, P_2 \in B_{op}(H)$, we have that $W_o(\delta_{P_1,P_2}|l_q) = W_o(\delta_{P_1,P_2})$. Where $q \geq 1$*

PROOF. From Proposition 3.1 and Proposition 3.2, we get $W_0(P_1) - W_0(P_2) \subseteq W_0(\delta_{P_1,P_2}|l_q) \subseteq W_0(L_p|l_q) - W_0(R_{P_2}|l_q) = W_0(P_1) - W_0(P_2)$. □

Lemma 3.4. *For $P_i, Q_i \in B_{op}(\mathcal{H})$ we have $co[(W_e(P) \circ W(Q)) \cup W(P) \circ W_e(Q))] \subseteq V_e(R_{2,P,Q})$.*

PROOF. Consider an element in the essential numerical range of P, that is, $\lambda \in W_e(P)$. we can have an orthonormal sequence such that $\langle R_{P,Q}(x_n \otimes y), x_n \otimes y\rangle = \sum_{i=1}^{q} \langle P_i x_n, x_n\rangle \cdot \langle Q_i y, y\rangle$. By [48] we have that $\lambda \circ \mu \in V_e(R_{P,Q})$. Hence, $co[(W_e(P) \circ W(Q)) \cup W(P) \circ W_e(Q))] \subseteq V_e(R_{2,P,Q})$. □

Theorem 3.5. *Let P_1, P_2 be a nonnegative self-adjoint operators and $P_1P_2 = P_2P_1$. Then $V_e(P_1P_2) \subseteq V_e(P_1)V_e(P_2)$.*

PROOF. We let $\lambda \in V_e(P_1P_2)$. From Lemma 3.4 we have $x_n \rightharpoonup 0$ and $\lambda = \lim\langle P_1P_2(x_n), x_n\rangle$. Next, we let $y_n = P_1^{\frac{1}{2}} x_n$. If $y_{n_k} = 0$ for some subsequence, then 0 is in both sides of $V_e(P_1P_2) \subseteq V_e(P_1)V_e(P_2)$. Let $y_n \neq 0 \; \forall n$. We take $z_n = \frac{y_n}{\|y_n\|}$. Now $z_n \rightharpoonup 0$ and $\lambda = \lim\langle P_2 z_n, z_n\rangle \cdot \langle P_1 x_n, x_n\rangle$. From [1], $\lim\langle P_2 z_n, z_n\rangle \in V_e(P_2)$ and so $\lambda \in V_e(P_1)V_e(P_2)$. Therefore, $V_e(P_1P_2) \subseteq V_e(P_1)V_e(P_2)$. □

Corollary 3.6. *Let $P_1, P_2 \in B_{op}(\mathcal{H})$. Then $V_e(M_{2,P_1,P_2} \subseteq W(P_1)^- W(P_2)^-$ if P_1, P_2 are unitary.*

Proof. By the essentiality of the numerical range, we have from [29] that $L_{P_1} R_{P_2} = R_{P_2} L_{P_1}, V_e(L_{P_1}) = W(P_1)^-$ and $V_e(R_{P_2}) = W(P_2)^-$. Let P_1, P_2 be unitary. Then from the statement of Corollary 3.6 we have that $V_e(M_{2,P_1,P_2} \subseteq W(P_1)^- W(P_2)^-$ if P_1, P_2. □

Theorem 3.7. *For $T : B_{op}(\mathcal{H}) \to B_{op}(\mathcal{H})$ defined by $T(X) = \sum_{i=1}^{n} P_i X Q_i, P_i, Q_i \in B_{op}(\mathcal{H})$, we have $W_0(T(X)) = W_0(P_i) - W_0(Q_i)$.*

Proof. From Proposition 3.1, we obtain that $W(P_i) - W(Q_i) \subset W_0(T(X))$. By closedness of NR it suffices that $(W(P_i) - W(Q_i))^- = W_0(P_i) - W_o(Q_i) \subset W_o(T(X))$. By Proposition 3.2 that $W_0(T(X)) \subset W_0(L_{P_i}) - W_0(R_{Q_i})) = W_0(P_i) - W_0(Q_i)$ which gives the equality. □

Proposition 3.8. *The numerical range of the left multiplication operator is equal to the numerical range of the right multiplication operator when they are implemented by orthogonal projections.*

Proof. Let A_1, A_2, A_3, A_4 be subsets of $B_{op}(\mathcal{H})$ which are convex. If $A_1 \oplus A_2 = A_3 \oplus A_4$, $A_1 \subseteq A_2$ and $A_3 \subseteq A_4$, then left multiplication operator that is $L : A_1 \oplus A_2$ and right multiplication operator that is $R : A_3 \oplus A_4$ have equal numerical ranges. To prove this, we first, show that if $A_1 \oplus A_2 \subseteq A_3 \oplus A_4$ then $A_1 \subseteq A_2$ and $A_3 \subseteq A_4$.

Consider the following elements: $x_1, x_2 \in A_1$, $y_1, y_2 \in A_2$, $z_1, z_2 \in A_3$ and $w_1, w_2 \in A_4$. By addition we have $x_1 + y_1 = x_2 + y_2$. So we have that $x_1 + y_1 \in A_1 \oplus A_2$. Analogously, we have that $z_1 + w_1 \in A_3 \oplus A_4$. So, $x_1 + y_1 = z_2 + w_2$. Induction gives through sequences $x_n \in A_1$, $y_n \in A_2$, $z_n \in A_3$ and $w_n \in A_4$ that $nx_1 + y_1 = (z_1 +, , , +z_n) + y_{n+1}$ for $n \geq 1$.

Now since A_3 satisfies compactness criterion and A_1 is closed and bounded, $A_1 \oplus A_2$ and $A_3 \oplus A_4$ are also closed and bounded [9]. Next we show that left multiplication operator is equal to right multiplication operator. To do this, we first show that $A_1 \oplus A_2 = A_3 \oplus A_4$. Since we have shown that $A_1 \oplus A_2 \subseteq A_3 \oplus A_4$ where $A_1 \subseteq A_2$ and $A_3 \subseteq A_4$, by [21] the reverse inclusion suffices implying that $A_3 \oplus A_4 \subseteq A_1 \oplus A_2$ where $A_2 \subseteq A_1$ and $A_4 \subseteq A_3$. Therefore, $A_1 \oplus A_2 = A_3 \oplus A_4$.

Now let the left multiplication operator be given by $L(P) = P_1$. If the number of elements in $A_1 \oplus A_2$ is equal to the number of elements in $A_3 \oplus A_4$ for the right multiplication operator, then by [46], left multiplication operator is equal to right multiplication operator. □

Remark 3.9. *The numerical range of left multiplication operator induced by P that is an OP in $B_{op}(\mathcal{H})$ is equal to the classical numerical range of $P \in B_{op}(\mathcal{H})$. Also, The numerical range of right multiplication operator induced by P that is an* *orthogonal projection in $B_{op}(\mathcal{H})$ is equal to the classical numerical range of $P \in B_{op}(\mathcal{H})$.*

Lemma 3.10. *The numerical range of generalized derivation restricted to the algebra of all orthogonal projections is equal to classical numerical range of individual orthogonal projections.*

Proof. Let P_1, P_2 be in $B_{op}(\mathcal{H})$. For the generalized derivation $\sigma_{P_1,P_2} = P_1P - PP_2$ we have that numerical range of σ_{P_1,P_2} is equal to numerical range of P. To see this, we consider Remark 3.9 and from [**44**] we obtain that the difference between numerical range of P_1 and numerical range of P_2 is a strict subset of numerical range of generalized derivation restricted to the class of all orthogonal projections which is in turn a strict subset of the difference of the numerical range of left multiplication operator and numerical range of right multiplication operator also both restricted to the class of all orthogonal projections which is equal to the numerical radius of the difference between the numerical range of P_1 and numerical range of P_2. □

Remark 3.11. *For the basic elementary operator, the result in Lemma 3.10 is not true since basic elementary operator has both P_1, P_2 fixed on the left of and the right of P. Moreover, Lemma 3.10 does not hold for Jordan elementary operator since it is the sum of the basic elementary operators.*

Remark 3.12. *For the product of numerical ranges we have that the convex hull of the product of classical numerical range of P_1 and classical numerical range of P_2 is a strict subset of classical numerical range of basic elementary operator.*

Theorem 3.13. *For P_1, P_2 fixed in $B_{op}(\mathcal{H})$, the closure of the numerical range of P_1 times the closure of the numerical range of P_2 is a subset of the algebraic numerical range of basic elementary operator.*

Proof. Let P_1, P_2 fixed in $B_{op}(\mathcal{H})$ be convexoid. From [**29**] we have that the algebraic numerical range of basic elementary operator. equals the closure of the convex hull of the product of numerical range of P_1 and numerical range of P_2. By inclusion criterion in [**37**], we have that it is a subset of the closure of the numerical range of P_1 and numerical range of P_2. This assertion holds from [**28**] if P_1 and P_2 are convexoid. □

Remark 3.14. *Theorem 3.13 holds for Jordan elementary operator when $n = 2$.*

Remark 3.15. *Theorem 3.13 also holds for an elementary operator in general when the elementary operator is induced by orthogonal projections P_i, Q_i fixed in $B_{op}(\mathcal{H})$.*

Up-to this point it is necessary to characterize in terms of the shape of the numerical ranges. We begin with the proposition on generalized derivations in a general set up.

Theorem 3.16. *The numerical range of a generalized derivation satisfies the spheroidicity criterion.*

Proof. For the generalized derivation given by $\mathcal{D}_{A,B}(X) = AX - XB$, for all $X \in \mathcal{B}(\mathcal{H})$, the numerical range of $\mathcal{D}_{A,B}$,

$$W(\mathcal{D}_{A,B}) = \{\langle (AX - XB)x, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\},$$

is a spheroid. In deed, from the definition of the numerical range of $\mathcal{D}_{A,B}$ we have $W(\mathcal{D}_{A,B}) = \{\langle (AX - XB)x, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}$. Carrying out the expansion of the inner product, we obtain $\langle AXx, x\rangle - \langle XBx, x\rangle$. Applying the adjoint property, gives $\langle XBx, x\rangle = \langle Bx, X^*x\rangle$, which can be rewritten as $\langle AXx, x\rangle - \langle Bx, X^*x\rangle$.

If A and B are normal operators, their eigenvalues contribute significantly to the shape of $W(\mathcal{D}_{A,B})$. Now from [**14**] and the Toeplitz-Hausdorff's Theorem, the numerical range is satisfies convexity criterion. But $\mathcal{D}_{A,B}$ is linear [**19**], so $W(\mathcal{D}_{A,B})$ lies within an elliptical or spheroidal region. For finiteness case where A and B are $n \times n$ matrices, $A = U\Lambda_A U^*, \quad B = V\Lambda_B V^*$, where Λ_A and Λ_B are diagonal matrices. Then, $\mathcal{D}_{A,B}(X) = AX - XB$. The expression of the eigenvalues take the representation $(\lambda_i - \mu_j)x_{ij}$. This show that the numerical range satisfies a quadratic relation, leading to an spheroid. □

Theorem 3.17. *The numerical range of a basic elementary operator satisfies the ellipsoidicity criterion.*

Proof. For the basic elementary operator denoted by $\mathcal{E}_{A,B}$ and defined by $\mathcal{E}_{A,B}(X) = AXB$, the numerical range of $\mathcal{E}_{A,B}$, given by: $W(\mathcal{E}_{A,B}) = \{\langle AXBx, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}$, is spheroidal. To see this, numerical range of $\mathcal{E}_{A,B}$ is

$$W(\mathcal{E}_{A,B}) = \{\langle AXBx, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}.$$

From Toeplitz-Hausdorff's Theorem the numerical range of basic elementary operator satisfies convexity criterion. Therefore, $W(\mathcal{E}_{A,B})$ is necessarily convex in $\mathbb{C}$. Now, let $A = U\Lambda_A U^*$, $B = V\Lambda_B V^*$ be their spectral decompositions, where U, V are unitary matrices and Λ_A, Λ_B are diagonal matrices containing eigenvalues. We have $\mathcal{E}_{A,B}(X) = AXB = U\Lambda_A U^* X V\Lambda_B V^*$. This map preserves the quadratic nature and structure of X and gives ellipticity on the numerical range and so it is generally a spheroid. □

Remark 3.18. *The numerical range of a Jordan elementary operator satisfies the spheroidicity criterion.*

4. Conclusion

Properties of elementary operators have been characterized over the years for instance numerical ranges the operators are implemented by other different types of operators. However, elementary operators implemented by orthogonal projections have not been characterized in terms of numerical ranges. In this paper, we characterize elementary operators in terms of numerical ranges, when they are implemented by orthogonal projections. We have shown that the numerical range of a basic elementary operator satisfies the ellipsoidicity criterion. Moreover, the numerical range of a Jordan elementary operator satisfies the spheroidicity criterion.

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