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Decomposition of frames in Banach spaces

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Abstract. In this paper, the decomposition of the retro Banach frame (RBF) and operator Banach frame (OBF) into two infinite subsequences has been discussed. In the sequel, we further decompose $OBFs$ into Block sequences and construct $OBFs$. We demonstrate a result regarding the image of an OBF by a bounded linear operator is also an OBF. Moreover, results related to the perturbation and stability of $OBFs$ have been proved. In the end, we deal with the problem related to the Feichtinger Conjecture as an application.

1. Introduction

The study of frames for Hilbert spaces was first initiated by Duffin and Schaeffer [2] in 1952, on the occasion of a discussion on some deep problems in the nonharmonic Fourier series. The notion of Hilbert frame was extended to the Banach spaces by Gröchenig [3] in 1991, who introduced the notion of Banach frames. During the last two decades, many classes of frames in Banach spaces have been introduced and studied. In 2004, Jain et al. [6] generalized Banach frames and introduced retro Banach frames in the setting of conjugate Banach spaces. In 2015, Shekhar [12] generalized Banach frames and defined operator Banach frames. Further, certain classes of retro Banach frames and frames for bounded linear operators called Λ-Banach frame were introduced and studied in [4] and [5] respectively.

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However, there are many applications of frames that have been investigated which cannot be modelled by one single-frame system. In such cases, it is too difficult to handle data assigned numerically. This is the reason to split a large frame system into a set of smaller systems and then process the data locally within each subsystem effectively. This idea leads to the introduction of the frame of subspaces [1], in which a sequence of closed subspaces of the Hilbert space is used to handle this problem within these subsystems locally. Another method is to decompose frames in block sequences as one can see in [7]. Here we further study the decomposition of frames in Banach spaces, namely, concerning the retro Banach frame and operator Banach frame. In the sequel, we establish some stability results for operator Banach frames.

2. Preliminaries

Throughout the paper, following notations and abbreviations will be used. The Banach space and the conjugate Banach space over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$ will be denoted by E and E^* respectively. E_d and $(E^*)_d$ will denote the Banach space of scalar valued sequences associated with E and E^*, respectively. Further, $[x_n]$ denotes the closed linear subspace of E, spanned by $\{x_n\} \subset E$. $B(E, E_i)$, $i \in E_i$, be the set of all bounded linear operators from E into E_i, where $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of Banach spaces. $\mathbb{N}$ is the set of all positive integers. We shall use the short abbreviations *RBF* and *OBF* for retro Banach frames and operator Banach frames, respectively. Recall that, an associated Banach space is a linear space of sequences which is complete in the given norm so that it becomes a Banach space. In order make the paper self contained we list some required definition and results.

Definition 2.1 ([6])**.** Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar valued sequences associated with E^*, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a retro Banach frame (*RBF*) for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$, for each $f \in E^*$

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \qquad f \in E^* \tag{2.1}$$

(iii) T is bounded linear operator such that $T(\{f(x_n)\}) = f$, $f \in E^*$.

Definition 2.2. A retro Banach frame $(\{x_n\}, T)$ for E^* is said to be

(a) *exact* [6] if removal of one x_n renders the collection $\{x_n\} \subset E$ no longer a retro Banach frame for E^*.

(b) *near exact* [10] retro Banach frame for E^* if it can be transformed into an retro Banach frame by omitting a finite number of its elements.

Definition 2.3 ([12])**.** Let E be a Banach space, $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of Banach spaces and $\Lambda_i \in B(E, E_i)$, $i \in \mathbb{N}$. Let $\mathcal{A}$ be an associated Banach space and $\mathcal{S} : \mathcal{A} \to E$ be an operator. Then $(\{\Lambda_i\}, \mathcal{S})$ is called an operator Banach frame (OBF) for E with respect to $\mathcal{A}$ if

(i) $\{\Lambda_i x\} \in \mathcal{A}$, $x \in E$,

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{\Lambda_i x\}\|_{\mathcal{A}} \leq B\|x\|_E, \qquad x \in E. \tag{2.2}$$

(iii) $\mathcal{S}$ is a bounded linear operator such that $\mathcal{S}(\{\Lambda_i x\}) = x$, $x \in E$.

Definition 2.4 ([12]). Let E be a Banach space, $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of Banach spaces and $\Lambda_i \in B(E, E_i)$, for all $i \in \mathbb{N}$. Then $\{\Lambda_i\}$ is called total on E if $\{x \in E : \Lambda_i x = 0, \forall i \in \mathbb{N}\} = \{0\}$.

Definition 2.5 ([8]). A sequence $\{\alpha_n\} \subset \mathbb{R}$ is said to be positively confined if $0 < \inf_{1\leq n<\infty} \alpha_n \leq \sup_{1\leq n<\infty} \alpha_n < \infty$.

Lemma 2.1 ([14]). *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the BK-space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $||\{f_n(x)\}||_{E_d} = ||x||_E$, $x \in E$.*

Lemma 2.2 ([12]). *Let E be a Banach space, $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of Banach spaces and let $\Lambda_i \in B(E, E_i)$, $i \in \mathbb{N}$. If $\{\Lambda_i\}$ be total over E, then $\mathcal{A} = \{\{\Lambda_i x\} : x \in E\}$ is a Banach space with norm given by $\|\{\Lambda_i x\}\|_{\mathcal{A}} = \|x\|_E$.*

Theorem 2.3 ([6]). *Let $(\{x_n\}, T)$ $(\{x_n\} \subset E$, $T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$ and with frame bounds A and B. Then, the coefficient mapping $S : E^* \to (E^*)_d$ defined by $S(f) = \{f(x_n)\}$, $f \in E^*$ is a topological isomorphism onto a closed subspace of $(E^*)_d$ with $\|S\| \leq B$ and $\|S^{-1}\| \leq \frac{1}{A}$, where S^{-1} is defined on the range $R(S)$.*

3. Retro Banach frame

In this section, we investigate the decomposition of a retro Banach frame (RBF) into subsequences and using them we shall construct new $RBFs$. If we decompose an RBF into two infinite subsequences and one of the subsequence is an RBF, then we observe that the other need not be an RBF.

More precisely, let $(\{x_n\}, T)$ be a near exact RBF for E^* with respect to $(E^*)_d$. Then, it is easy to find an infinite increasing sequence $\{n_k\} \in \mathbb{N}$ (by omitting a finite number of terms in $\{x_n\}$) such that $(\{x_{n_k}\}, S)$ is an exact RBF for E^* with respect to $(E^*)_d$, where $S : (E^*)_d \to E^*$ is a reconstruction operator. However, let $\{n_k\}$ and $\{m_k\}$ be two infinite increasing sequences in $\mathbb{N}$ such that $\{n_k\} \cup \{m_k\} = \mathbb{N}$. If $(\{x_{n_k}\}, T_0)$ is an RBF for E^* then there may not exists a reconstruction operator $T_1 : (E^*)_d \to E^*$ such that $(\{x_{m_k}\}, T_1)$ is an RBF for E^* with respect to $(E^*)_d$.

We illustrate this observation with the following examples.

Example 3.1. Let $\{m_k\} \subset \mathbb{N}$ be defined by $m_0 = 1$, $m_k = m_{k-1} + k$, $k \in \mathbb{N}$ and $\{n_k\} = \mathbb{N} \setminus \{m_k\}$, $k \in \mathbb{N}$. Let $\{x_n\} \subset E$ be defined by $x_{m_k} = e_k$ and $x_{n_k} = e_{k+2}$, for all $k \in \mathbb{N}$, where $\{e_k\}$ be the standard sequence of unit vectors in E. Since $[x_{m_k}] = E$, there exists a reconstruction operator $T_0 : (E^*)_{d_0} = \{\{f(x_{m_k})\} : f \in E^*\} \to E^*$ such that $(\{x_{m_k}\}, T_0)$ is an *RBF* for E^* with respect to $(E^*)_{d_0}$. However, there does not exists a reconstruction operator $T_1 : (E^*)_{d_1} = \{\{f(x_{n_k})\} : f \in E^*\} \to E^*$ such that $(\{x_{n_k}\}, T_1)$ is an *RBF* for E^* with respect to $(E^*)_{d_1}$.

Example 3.2. Let $E = c_0$. Define $x_{2k} = x_{2k-1} = e_k$, $k \in \mathbb{N}$. Let $m_k = 2k$ and $n_k = 2k - 1$, $k \in \mathbb{N}$. Since $[x_{m_k}] = E = [x_{n_k}]$, there exist reconstruction operators $T_0 : (E^*)_{d_0} = \{\{f(x_{m_k})\} : f \in E^*\} \to E^*$ and $T_1 : (E^*)_{d_1} = \{\{f(x_{n_k})\} : f \in E^*\} \to E^*$ such that $(\{x_{m_k}\}, T_0)$ and $(\{x_{n_k}\}, T_1)$ are retro Banach frame for E^* with respect to $(E^*)_{d_0}$ and $(E^*)_{d_1}$, respectively.

In view of the above observation, we prove the following result providing a necessary and sufficient condition to overcome the problem discussed.

Theorem 3.1. *Let $(\{x_n\}, T)$ be an* RBF *for E^* with respect to $(E^*)_d$. Let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences in $\mathbb{N}$ with $\{m_k\} \cup \{n_k\} = \mathbb{N}$. Let $(\{x_{m_k}\}, T_0)$ $(\{x_{m_k}\} \subset E, T_0 : (E^*)_d \to E^*)$ is an* RBF *for E^*. Then $(\{x_{n_k}\}, T_1)$ $(\{x_{n_k}\} \subset E, T_1 : (E^*)_d \to E^*)$ is an* RBF *for E^* if and only if there exists an isomorphism $V : (E^*)_d \to (E^*)_d$ such that $V(\{f(x_{n_k})\}) = \{f(x_{m_k})\}$, for each $f \in E^*$.*

Proof. Let A_0 and B_0 are frame bounds for the *RBF* $(\{x_{m_k}\}, T_0)$. Then we have

$$A_0 \|f\|_{E^*} \le \|\{f(x_{m_k})\}\|_{(E^*)_d} \le B_0 \|f\|_{E^*}. \tag{3.1}$$

Now, since $\|\{f(x_{m_k})\}\|_{(E^*)_d} \le \|V\| \|\{f(x_{n_k})\}\|_{(E^*)_d}$, we have

$$A_0 \frac{\|f\|}{\|V\|} \le \frac{1}{\|V\|} \|\{f(x_{m_k})\}\|_{(E^*)_d} \le \|\{f(x_{n_k})\}\|_{(E^*)_d} \tag{3.2}$$

Again, since $\{f(x_{n_k})\} = V^{-1}(\{f(x_{m_k})\})$, we have

$$\|\{f(x_{n_k})\}\|_{(E^*)_d} \le \|V^{-1}\| \|\{f(x_{m_k})\}\|_{(E^*)_d} \le B_0 \|V^{-1}\| \|f\|. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$A_0 \frac{\|f\|}{\|V\|} \le \|\{f(x_{n_k})\}\|_{(E^*)_d} \le B_0 \|V^{-1}\| \|f\|, \qquad f \in E^*.$$

Define $T_1 : (E^*)_d \to E^*$ by $T_1(\{f(x_{n_k})\}) = f$, $f \in E^*$. Then, $(\{x_{n_k}\}, T_1)$ is an *RBF* for E^* with respect to $(E^*)_d$.

Conversely, suppose that $(\{x_{n_k}\}, T_1)$ is an *RBF* for E^* having frame bounds A_1 and B_1. Then we have,

$$A_1 \|f\|_{E^*} \le \|\{f(x_{n_k})\}\|_{(E^*)_d} \le B_1 \|f\|_{E^*}. \tag{3.4}$$

and $T_1 : (E^*)_d \to E^*$ is given by

$$T_1(\{f(x_{n_k})\}) = f, \qquad f \in E^*.$$

Then by Theorem 2.3, the coefficient mapping $S_1 : (E^*) \to (E^*)_d$ defined by

$$S_1(f) = \{f(x_{n_k})\}, \qquad f \in E^*.$$

is a topological isomorphism onto a closed subspace of $(E^*)_d$ with $\|S_1\| \leq B_1$ and $\|S_1^{-1}\| \leq \frac{1}{A_1}$, where S_1^{-1} is defined on the range of S_1.

Notice that, $(\{x_{m_k}\}, T_0)$ is an *RBF* for E^*, with respect to $(E^*)_d$, where $T_0 : (E^*)_d \to (E^*)$ is given by

$$T_0(\{f(x_{m_k})\}) = f, \qquad f \in E^*.$$

Applying the same argument, the coefficient mapping $S_0 : E^* \to (E^*)_d$ defined by

$$S_0(f) = \{f(x_{m_k})\}, \qquad f \in E^*$$

is a topological isomorphism onto a closed subspace of $(E^*)_d$ with $\|S_0\| \leq B_0$ and $\|S_0^{-1}\| \leq \frac{1}{A_0}$, where S_0^{-1} is defined on the range of S_0. Consequently, $V = S_1 T_0 : (E^*)_d \to (E^*)_d$ is a bounded linear operator such that

$$V(\{f(x_{n_k})\}) = \{f(x_{m_k})\}, \text{ for each } f \in E^*.$$

Moreover, assume that $\{f(x_{n_k})\} \in \ker V$, then $V(\{f(x_{n_k})\}) = 0$ and so $\{f(x_{m_k})\} = 0$. Therefore, by *RBF* inequality (3.1), we get $f = 0$, and hence by *RBF* inequality (3.4) we get $\{f(x_{n_k})\} = 0$, which shows that V is an isomorphism. □

4. Decomposition of OBF

Now, we decompose an *OBF* into two infinite subsequences, as we discussed in the previous section, we observe that if one of the subsequences is an *OBF* then the other need not be an *OBF*. In the sequel, in Theorem 4.2, we provide a necessary and sufficient condition for the other subsequence to be an *OBF*. First, we need to prove the following result required for Theorem 4.2.

Lemma 4.1. *Let $(\{\Lambda_i\}, \mathcal{S})$ $(\{\Lambda_i\} \subset B(E, E_i), \mathcal{S} : \mathcal{A} \to E)$ be an* OBF *for E with respect to $\mathcal{A}$, and with frame bounds A and B. Then the coefficient mapping $\mathcal{T} : E \to \mathcal{A}$ defined by $\mathcal{T}(x) = \{\Lambda_i(x)\}$, $x \in E$, is a topological isomorphism onto a closed subspace of $\mathcal{A}$ with $\|\mathcal{T}\| \leq B$ and $\|\mathcal{T}^{-1}\| \leq \frac{1}{A}$, where $\mathcal{T}^{-1}$ defined on the range of $\mathcal{T}$.*

Proof. By using *OBF* inequality (2.2), it can be computed that $\|\mathcal{T}\| \leq B$. Further, let $x \in \ker \mathcal{T}$. Then $\mathcal{T}(x) = 0$, thus, by definition of $\mathcal{T}$, $\Lambda_i(x) = 0$, $i \in \mathbb{N}$. Again, by *OBF* inequality (2.2), we obtain $x = 0$. Thus $\mathcal{T}$ is an injective bounded linear operator from E onto *range*$(\mathcal{T})$. Therefore, $\mathcal{T}^{-1}$ exists on *range*$(\mathcal{T})$ and $\|\mathcal{T}^{-1}\| \leq \frac{1}{A}$. Next to prove that *range*$(\mathcal{T})$ is closed, let $\{\varphi_n\} \subset$ *range*$(\mathcal{T})$ be a sequence converges to φ in $\mathcal{A}$. Let $\{y_n\}$ be a sequence in E such that $\mathcal{T}(y_n) = \varphi_n$, $n \in \mathbb{N}$. Then, $\{\mathcal{T}(y_n)\}$ is a Cauchy sequence in $\mathcal{A}$. Continuity of $\mathcal{T}^{-1}$ shows that $\{y_n\}$ is a Cauchy sequence in E. Then, there exists $y \in E$, such that $\lim_{n\to\infty} y_n = y$. Therefore, $\lim_{n\to\infty} \mathcal{T}(y_n) = \mathcal{T}(y)$. Hence, $\varphi = \mathcal{T}(y) \in \textit{range}(\mathcal{T})$.

□

Theorem 4.2. *Let* $(\{\Lambda_i\}, \mathcal{S})$ $(\{\Lambda_i\} \subset B(E, E_i), \mathcal{S} : \mathcal{A} \to E)$ *be an* OBF *for* E *with respect to* $\mathcal{A}$. *Let* $\{i_k\}$ *and* $\{j_k\}$ *be two infinite increasing sequences in* $\mathbb{N}$ *with* $\{i_k\} \cup \{j_k\} = \mathbb{N}$. *Let* $(\{\Lambda_{i_k}\}, \mathcal{S}_0)$ *(where* $\mathcal{S}_0 : \mathcal{A} \to E$*) be an* OBF *for* E. *Then,* $(\{\Lambda_{j_k}\}, \mathcal{S}_1)$ *(where* $\mathcal{S}_1 : \mathcal{A} \to E$*) is also an* OBF *for* E *if and only if there exists an isomorphism* $\mathcal{V} : \mathcal{A} \to \mathcal{A}$ *such that*

$$\mathcal{V}(\{\Lambda_{j_k}(x)\}) = \{\Lambda_{i_k}(x)\}, \qquad x \in E.$$

Proof. Let A_0 and B_0 are the frame bounds for the *OBF* $(\{\Lambda_{i_k}\}, \mathcal{S}_0)$. Then, the frame inequality for *OBF* $(\{\Lambda_{i_k}\}, \mathcal{S}_0)$ is:

$$A_0\|x\|_E \le \|\{\Lambda_{i_k}(x)\}\|_{\mathcal{A}} \le B_0\|x\|_E, \qquad x \in E. \tag{4.1}$$

Now, using (4.1) we compute

$$\begin{aligned}\|\{\Lambda_{i_k}(x)\}\|_{\mathcal{A}} &= \|\mathcal{V}(\{\Lambda j_k(x)\})\|_{\mathcal{A}}\\ &\le \|\mathcal{V}\|\|\{\Lambda_{j_k}(x)\}\|_{\mathcal{A}}\end{aligned}$$

Then,

$$\begin{aligned}\|\{\Lambda_{j_k}(x)\}\|_{\mathcal{A}} &\ge \frac{1}{\|\mathcal{V}\|}\|\{\Lambda_{i_k}(x)\}\|_{\mathcal{A}}\\ &\ge A_0\frac{\|x\|}{\|\mathcal{V}\|}. \end{aligned}\tag{4.2}$$

Again, using (4.1), we compute

$$\begin{aligned}\|\{\Lambda_{j_k}(x)\}\| &= \|\mathcal{V}^{-1}(\{\Lambda_{i_k}(x)\})\|\\ &\le \|\mathcal{V}^{-1}\|\|\{\Lambda_{i_k}(x)\}\|\\ &\le B_0\|\mathcal{V}^{-1}\|\|x\| \end{aligned}\tag{4.3}$$

Combining, (4.2) and (4.3) we obtain,

$$A_0\frac{\|x\|}{\|\mathcal{V}\|} \le \|\{\Lambda_{j_k}(x)\}\| \le B_0\|\mathcal{V}^{-1}\|\|x\|.$$

Moreover, by defining the bounded linear operator $\mathcal{S}_1 : \mathcal{A} \to E$ as $\mathcal{S}_1(\{\Lambda_{j_k}(x)\}) = x$, $x \in E$, $(\{\Lambda_{j_k}\}, \mathcal{S}_1)$ is an *OBF* for E with respect to $\mathcal{A}$.

Conversely, suppose that $(\{\Lambda_{j_k}\}, \mathcal{S}_1)$ is an *OBF* for E with respect to $\mathcal{A}$ having frame bounds A_1 and B_1. Then we have the *OBF* inequality

$$A_1\|x\|_E \le \|\{\Lambda_{j_k}(x)\}\|_{\mathcal{A}} \le B_1\|x\|_E, \qquad x \in E. \tag{4.4}$$

and $\mathcal{S}_1 : \mathcal{A} \to E$ is given by $\mathcal{S}_1(\{\Lambda_{j_k}(x)\}) = x$, $x \in E$. Then by Lemma 4.1, the coefficient mapping $\mathcal{T}_1 : E \to \mathcal{A}$ defined by $\mathcal{T}_1(x) = \{\Lambda_{j_k}(x)\}$, $x \in E$, is a topological isomorphism onto a closed subspace of $\mathcal{A}$ with $\|\mathcal{T}_1\| \leq B_1$ and $\|\mathcal{T}_1^{-1}\| \leq \frac{1}{A_1}$, where $\mathcal{T}_1^{-1}$ is defined on the range of $\mathcal{T}_1$.

Notice that, $(\{\Lambda_{i_k}\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$, where $\mathcal{S}_0 : \mathcal{A} \to E$ is given by $\mathcal{S}_0(\Lambda_{i_k}(x)) = x$, $x \in E$. Applying the same argument, the coefficient mapping $\mathcal{T}_0 : E \to \mathcal{A}$ defined by $\mathcal{T}_0(x) = \{\Lambda_{i_k}(x)\}$, $x \in E$ is a topological isomorphism onto a closed subspace of $\mathcal{A}$ with $\|\mathcal{T}_0\| \leq B_0$ and $\|\mathcal{T}_0^{-1}\| \leq \frac{1}{A_0}$, where $\mathcal{T}_0^{-1}$ is defined on the range of $\mathcal{T}_0$. Consequently, $\mathcal{V} = \mathcal{S}_1 o \mathcal{T}_0 : \mathcal{A} \to \mathcal{A}$ is a bounded linear operator such that

$$\mathcal{V}(\{\Lambda_{j_k}(x)\}) = \{\Lambda_{i_k}(x)\}, \qquad x \in E.$$

Moreover, assume that $\{\Lambda_{j_k}(x)\} \in \ker \mathcal{V}$, then $\mathcal{V}(\{\Lambda_{j_k}(x)\}) = 0$. This gives $\{\Lambda_{i_k}(x)\} = 0$. Therefore, by *OBF* inequality (4.1), $x = 0$, and so by *OBF* inequality (4.4), we get $\{\Lambda_{j_k}(x)\} = 0$. Hence, $\mathcal{V}$ is an isomorphism. □

5. Block sequence of OBF

Another method of decomposition is in the form of block sequences. In this section we decompose operator Banach frame in block sequences and construct operator Banach frame for these block sequences. Let us first introduce the block sequence with respect to *OBF*.

Definition 5.1. Let $(\{\Lambda_i\}, \mathcal{S})$ be an *OBF* for E with respect to $\mathcal{A}$ and let $\{\mathcal{D}_i\}$ be a sequence of finite subsets of $\mathbb{N}$ such that $\mathcal{D}_i \cap \mathcal{D}_j = \phi$, where $i \neq j$, for all $i, j \in \mathbb{N}$ and $\cup_{i=1}^{\infty} \mathcal{D}_i = \mathbb{N}$. Then the sequence $\{\Theta_n\}$ defined by

$$\Theta_n = \sum_{i \in \mathcal{D}_n} \alpha_i \Lambda_i \neq 0, \qquad n \in \mathbb{N},$$

where α_i $(i \in \mathcal{D}_n,\ n \in \mathbb{N})$ be any scalar, is called block sequence with respect to the *OBF* $(\{\Lambda_i\}, \mathcal{S})$.

We illustrate the following examples and observe that the decomposition (block sequence) of an *OBF* need not be an *OBF*.

Example 5.2. Let $E = l^p$ $(1 \leq p < \infty)$ and $E_i = \mathbb{K}$. Let $\Lambda_i \in B(E, E_i)$ be defined by

$$\Lambda_{3i-1}(x) = \Lambda_{3i}(x) = \Lambda_{3i-2}(x) = \xi_i, \qquad \forall i \in \mathbb{N},\ x = \{\xi_i\} \in E.$$

Then, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{\Lambda_i(x)\} : x \in E\}$ with norm given by $\|\{\Lambda_i(x)\}\|_{\mathcal{A}} = \|x\|_E$. Define a bounded linear operator $\mathcal{S} : \mathcal{A} \to E$ such that $\mathcal{S}(\{\Lambda_i(x)\}) = x$, for all $x \in E$. Then, $(\{\Lambda_i\}, \mathcal{S})$ is an *OBF* for

E with respect to $\mathcal{A}$. Now, let $\mathcal{D}_i = \{3i-2, 3i-1, 3i\}$, $i \in \mathbb{N}$. Then $\mathcal{D}_i \cap \mathcal{D}_j = \phi$, where $i \neq j$, for all $i, j \in \mathbb{N}$ and $\cup_{i=1}^{\infty} D_i = \mathbb{N}$. Define

$$\Theta_n = \sum_{i \in \mathcal{D}_n} \alpha_i \Lambda_i, \qquad n \in \mathbb{N},$$

where $\alpha_i = i$, $(i \in \mathcal{D}_n,\ n \in \mathbb{N})$. Then $\{\Theta_n\}$ is a block sequence with respect to the *OBF* $(\{\Lambda_i\}, \mathcal{S})$. Further, the sequence $\{\Theta_n\}$ is total over E, therefore by Lemma 2.2, there exists an associated Banach space $\mathcal{A}_0 = \{\{\Theta_n(x)\} : x \in E\}$ with norm given by $\|\{\Theta_n(x)\}\|_{\mathcal{A}_0} = \|x\|_E$. Define a bounded linear operator $\mathcal{S}_0 : \mathcal{A}_0 \to E$ by $\mathcal{S}_0(\{\Theta_n(x)\}) = x$, for all $x \in E$. Then $(\{\Theta_n\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}_0$.

Example 5.3. Let $E = l^p$ $(1 \leq p < \infty)$ and $E_i = \mathbb{K}$. Let $\Lambda_i \in B(E, E_i)$ be defined by

$$\Lambda_i(x) = \begin{cases} \dfrac{1}{i}\xi_i, & \text{if } i \text{ is odd} \\ \dfrac{-1}{i}\xi_i, & \text{if } i \text{ is even}, x = \{\xi_i\} \in E, i \in \mathbb{N}. \end{cases} \tag{5.1}$$

Then by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{\Lambda_i(x)\} : x \in E\}$ with norm given by $\|\{\Lambda_i(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$. Define a bounded linear operator $\mathcal{S} : \mathcal{A} \to E$ by $\mathcal{S}(\{\Lambda_i(x)\}) = x$, for all $x \in E$. Then $(\{\Lambda_i\}, \mathcal{S})$ is an *OBF* for E with respect to $\mathcal{A}$. Now, let $\{\mathcal{D}_i\}$ and $\{\Theta_n\}$ be defined as in the Example 5.2 and $\alpha_i = i$ for all $i \in \mathcal{D}_n$, $n \in \mathbb{N}$. Since $\{\Theta_n\}$ is not total over E, therefore, by Lemma 2.2, there does not exists a reconstruction operator $\mathcal{S}_0 : \mathcal{A}_0 = \{\{\Theta_n(x)\} : x \in E\} \to E$ such that $(\{\Theta_n\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}_0$.

In the study of *OBF*, it would be interesting to find a necessary and sufficient condition under which the image of an *OBF* (by a bounded linear operator) is also an *OBF*. In this direction, we have the following result.

Theorem 5.1. *Let $(\{\Lambda_i\}, \mathcal{S})$ $(\{\Lambda_i\} \subset B_1 = B(E, E_i), \mathcal{S} : \mathcal{A} \to E)$ be an* OBF *for E with respect to $\mathcal{A}$ an let $\Theta \in B(B_1, B_1)$. Assume that $\Phi \in B(\mathcal{A}, \mathcal{A})$ is such that $\Phi(\{\Lambda_i(x)\}) = \{(\Theta\Lambda_i)(x)\}$, for all $x \in E$. Then there exists a bounded linear operator $\mathcal{S}_0$ such that $(\{(\Theta\Lambda_i)\}, \mathcal{S}_0)$ is an* OBF *for E with respect to $\mathcal{A}$ if and only if*

$$\|\Phi(\{\Lambda_i(x)\})\|_{\mathcal{A}} \geq \gamma \|\Psi(\{(\Theta\Lambda_i)(x)\})\|_{\mathcal{A}}, \text{ for all } x \in E, \tag{5.2}$$

where γ is a positive constant and $\Psi \in B(\mathcal{A}, \mathcal{A})$ is an operator such that, for all $x \in E$,

$$\Psi(\{(\Theta\Lambda_i)(x)\}) = \{\Lambda_i(x)\}.$$

Proof. Assume that $(\{\Theta\Lambda_i\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$ having frame bounds A_0 and B_0. Let $\vartheta : E \to \mathcal{A}$ be the analysis operator associated with the *OBF* $(\{\Lambda_i\}, \mathcal{S})$ is given by $\vartheta(x) = \{\Lambda_i(x)\}$, $x \in E$. Put $\Psi : \vartheta\mathcal{S}_0$ and $\gamma = \frac{A_0}{\|\vartheta\|} > 0$. Then, the forward part follows by computing, for each $x \in E$,

$$\begin{aligned}\|\Phi(\{\Lambda_i(x)\})\|_{\mathcal{A}} &= \|\{(\Theta\Lambda_i)(x)\}\|_{\mathcal{A}}\\ &\geq A_0\|x\|_E\\ &= \gamma\|\vartheta\|\|x\|_E\\ &\geq \gamma\|\{\Lambda_i(x)\}\|_{\mathcal{A}}\\ &= \gamma\|\Psi(\{(\Theta\Lambda_i)(x)\})\|_{\mathcal{A}}.\end{aligned}$$

Now, assume that the inequality (5.2) holds. Then we compute for each $x \in E$,

$$\|\{(\Theta\Lambda_i)(x)\}\|_{\mathcal{A}} = \|\Phi(\{\Lambda_i(x)\})\|_{\mathcal{A}} \leq \|\Phi\|\|\vartheta\|\|x\|.$$

Let A be the lower bound of $(\{\Lambda_i\}, \mathcal{S})$, then by using (5.2) we obtain,

$$\begin{aligned}\gamma A\|x\| &\leq \gamma\|\{\Lambda_i(x)\}\|_{\mathcal{A}}\\ &= \gamma\|\Psi(\{(\Theta\Lambda_i)(x)\})\|_{\mathcal{A}}\\ &\leq \|\Phi(\{\Lambda_i(x)\})\|_{\mathcal{A}}\\ &= \|\{(\Theta\Lambda_i)(x)\}\|_{\mathcal{A}}.\end{aligned}$$

Put $A_0 = \gamma A$ and $B_0 = \|\Phi\|\|\vartheta\|$. Then the required frame inequality is

$$A_0\|x\| \leq \|\{(\Theta\Lambda_i)(x)\}\|_{\mathcal{A}} \leq B_0\|x\|, \text{ for each } x \in E.$$

Moreover, Let $\mathcal{S}_0 = \mathcal{S}\Psi$. Then $\mathcal{S}_0 \in B(\mathcal{A}, E)$ is such that $\mathcal{S}_0(\{(\Theta\Lambda_i)(x)\}) = x$, for each $x \in E$. Hence, $(\{(\Theta\Lambda_i)\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$ having frame bounds A_0 and B_0. □

6. Perturbation and stability of *OBF*

The potential of perturbation theory shows that it is an important tool for various disciplines of mathematics. The study of perturbations in the basis theory began with the classical perturbation result by Paley and Wiener [11]. In this section, we prove perturbation and stability results for *OBFs* by a sequence of bounded linear operators. Our results extend some results of [8] and [13] for *OBFs*.

The result below gives a sufficient condition for the perturbation of an operator Banach frame by a sequence of bounded linear operators.

Theorem 6.1. *Let* $(\{\Lambda_i\}, \mathcal{S})$ $(\{\Lambda_i\} \subset B(E, E_i), \mathcal{S} : \mathcal{A} \to E)$ *be an* OBF *for* E *with respect to* $\mathcal{A}$ *and having a couple of frame bounds* A *and* B*. Let* $\{\Theta_i\} \subset B(E, E_i)$ *be such that* $\{\Theta_i(x)\} \in \mathcal{A}$*, for each* $x \in E$*,* $i \in \mathbb{N}$ *and for some constant* $M > 0$*,*

$$\|\{\Theta_i(x)\}\|_{\mathcal{A}} \leq M\|x\|_E, \qquad x \in E.$$

Let $\mathcal{Q} : \mathcal{A} \to \mathcal{A}$ *be a bounded linear operator defined for any nonzero constant* γ *by,* $\mathcal{Q}(\{(\Lambda_i + \gamma\Theta_i)(x)\}) = \{\Lambda_i(x)\}$*,* $x \in E$*. Then there exists a reconstruction* *operator* $\mathcal{S}_0 : \mathcal{A} \to E$ *such that* $(\{\Lambda_i + \gamma\Theta_i\}, \mathcal{S}_0)$ *is an* OBF *for* E *with respect to* $\mathcal{A}$ *and having a couple of frame bounds* $(A - M|\gamma|)$ *and* $(B + M|\gamma|)$.

Proof. Clearly $\{(\Lambda_i + \gamma\Theta_i)(x)\} \in \mathcal{A}$, for each $x \in E$. The *OBF* inequality for $(\{\Lambda_i\}, \mathcal{S})$ is given by

$$A\|x\|_E \leq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} \leq \|x\|_E, \qquad x \in E. \tag{6.1}$$

Now, using (6.1) we compute, for each $x \in E$,

$$\begin{aligned} \|\{(\Lambda_i + \gamma\Theta_i)(x)\}\|_{\mathcal{A}} &\leq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} + |\gamma|\|\{\Theta_i(x)\}\|_{\mathcal{A}} \\ &\leq B\|x\|_E + M|\gamma|\|x\|_E \\ &= (B + M|\gamma|)\|x\|_E \end{aligned}$$

and

$$\begin{aligned} \|\{(\Lambda_i + \gamma\Theta_i)(x)\}\|_{\mathcal{A}} &\geq \|\{\Lambda_i(x)\}\| - |\gamma|\|\{\Theta_i(x)\}\|_{\mathcal{A}} \\ &\geq A\|x\|_E - M|\gamma|\|x\|_E \\ &= (A - M|\gamma|)\|x\|_E. \end{aligned}$$

Thus, the required frame inequality is

$$(A - M|\gamma|)\|x\|_E \leq \|\{(\Lambda_i + \gamma\Theta_i)(x)\}\|_{\mathcal{A}} \leq (B + M|\gamma|)\|x\|_E.$$

Put $\mathcal{S}_0 = \mathcal{S}\mathcal{Q}$. Then $\mathcal{S}_0 : \mathcal{A} \to E$ is a bounded linear operator such that

$$\mathcal{S}_0(\{(\Lambda_i + \gamma\Theta_i)(x)\}) = x, \qquad x \in E.$$

Hence, $(\{\Lambda_i + \gamma\Theta_i\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$ and having required frame bounds. □

The following result gives a sufficient condition for the perturbation of an operator Banach frame by a sequence of type $\{\alpha_i\Lambda_i\} \subset B(E, E_i)$ (where $\{\Lambda_i\} \subset B(E, E_i)$ and $\{\alpha_i\}$ is a positively confined sequence) to be an operator Banach frame.

Theorem 6.2. *Let* $(\{\Lambda_i\}, \mathcal{S})$ *be an* OBF *for* E *with respect to* $\mathcal{A}$ *and having a couple of frame bounds* A *and* B. *Let* $\{\Theta_i\} \subset B(E, E_i)$ *and* $\{\alpha_i\} \subset \mathbb{R}$ *be any positively confined sequence such that* $\{(\alpha_i\Theta_i)(x)\} \in \mathcal{A}$, $x \in E$. *Let* $\mathcal{Q} : \mathcal{A} \to \mathcal{A}$ *be a bounded linear operator defined by,* $\mathcal{Q}(\{(\Lambda_i + \alpha_i\Theta_i)(x)\}) = \{\Lambda_i(x)\}$, $x \in E$. *If* $\mathcal{W} : E \to \mathcal{A}$ *is a coefficient mapping defined by* $\mathcal{W}(x) = \{\Theta_i(x)\}, x \in E$, *and* $\|\mathcal{W}\| < \frac{A}{(\sup_{1\leq i<\infty} \alpha_i)}$ *then there exists a reconstruction operator* $\mathcal{S}_0 : \mathcal{A} \to E$ *such that* $(\{\Lambda_i + \alpha_i\Theta_i\}, \mathcal{S}_0)$ *is an* OBF *for* E *with respect to* $\mathcal{A}$ *with frame bounds* $[A - \|\mathcal{W}\|(\sup_{1\leq i<\infty} \alpha_i)]$ *and* $[B + \|\mathcal{W}\|(\sup_{1\leq i<\infty} \alpha_i)]$.

Proof. Notice that for each $x \in E$, $\{(\Lambda_i + \alpha_i\Theta_i)(x)\} \in \mathcal{A}$. We have inequality (6.1) for the *OBF* $(\{\Lambda_i\}, \mathcal{S})$. Now, using (6.1) we compute, for each $x \in E$,

$$\begin{aligned}\|\{(\Lambda_i + \alpha_i\Theta_i)(x)\}\|_{\mathcal{A}} &\leq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} + \|\{(\alpha_i\Theta_i)(x)\}\|_{\mathcal{A}} \\ &\leq \left[B + \|\mathcal{W}\|\left(\sup_{1\leq i<\infty} \alpha_i\right)\right]\|x\|_E\end{aligned}$$

and

$$\begin{aligned}\|\{(\Lambda_i + \alpha_i\Theta_i)(x)\}\|_{\mathcal{A}} &\geq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} - \|\{(\alpha_i\Theta_i)(x)\}\|_{\mathcal{A}} \\ &\geq \left[A - \|\mathcal{W}\|\left(\sup_{1\leq i<\infty} \alpha_i\right)\right]\|x\|_E.\end{aligned}$$

Thus, the required frame inequality is

$$A_0 \leq \|\{(\Lambda_i + \alpha_i\Theta_i)(x)\}\|_{\mathcal{A}} \leq B_0,$$

where

$$A_0 = \left[A - \|\mathcal{W}\|\left(\sup_{1\leq i<\infty} \alpha_i\right)\right],$$

$$B_0 = \left[B + \|\mathcal{W}\|\left(\sup_{1\leq i<\infty} \alpha_i\right)\right].$$

Put $\mathcal{S}_0 = (\mathcal{S}\mathcal{Q})$. Then, $\mathcal{S}_0 : \mathcal{A} \to E$ is a bounded linear operator such that $\mathcal{S}_0(\{(\Lambda_i + \alpha_i\Theta_i)(x)\}) = x$, $x \in E$. Hence, $(\{(\Lambda_i + \alpha_i\Theta_i)(x)\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$ and with required frame bounds. □

Stability of Banach frames has been discussed by Jain, Kaushik and Vashisht in [9]. Here, we present stability results for *OBFs*. In this direction, we have the following results.

Theorem 6.3. *Let* $(\{\Lambda_i\}, \mathcal{S})$ *be an* OBF *for* E *with respect to* $\mathcal{A}$ *having a couple of frame bounds* A *and* B. *Let* $\{\Theta_i\} \subset B(E, E_i)$ *be such that* $\{\Theta_i(x)\} \in \mathcal{A}$. *Let* $\mathcal{Q} : \mathcal{A} \to \mathcal{A}$ *defined by* $\mathcal{Q}(\{\Theta_i(x)\}) = \{\Lambda_i(x)\}$, $x \in E$. *If* t_1 *and* t_2 *be nonzero constants satisfying*

(i) $t_2 < (1 - t_1)A$
(ii) $\|\{(\Lambda_i - \Theta_i)(x)\}\|_{\mathcal{A}} \leq t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} + t_2\|x\|_E$, $x \in E$,

then there exists an operator $\mathcal{S}_0$ *such that* $(\{\Theta_i\}, \mathcal{S}_0)$ *is an* OBF *having a couple of frame bounds as* $[(1 - t_1)A - t_2]$ *and* $[(1 + t_1)B + t_2]$.

Proof. Since $(\{\Lambda_i\}, \mathcal{S})$ is an *OBF* for E with respect to $\mathcal{A}$ with frame bounds A and B, we have

$$A\|x\|_E \leq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} \leq B\|x\|_E, \qquad x \in E.$$

Now, using condition (ii) we compute

$$\begin{aligned}\|\{\Theta_i(x)\}\|_{\mathcal{A}} &\leq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} + \|\{(\Lambda_i - \Theta_i)(x)\}\|_{\mathcal{A}} \\ &\leq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} + t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} + t_2\|x\|_E \\ &\leq [(1+t_1)B + t_2]\|x\|_E.\end{aligned}$$

For lower bounds, by using (ii) we compute,

$$\begin{aligned}\|\{\Theta_i(x)\}\|_{\mathcal{A}} &\geq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} - \|\{(\Lambda_i - \Theta_i)(x)\}\|_{\mathcal{A}} \\ &\geq \|\{\Lambda_i(x)\}\|_{\mathcal{A}} - t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} - t_2\|x\|_E \\ &\leq [(1-t_1)A - t_2]\|x\|_E.\end{aligned}$$

Hence the required frame inequality is

$$[(1-t_1)A - t_2]\|x\|_E \leq \|\{\Theta_i(x)\}\|_{\mathcal{A}} \leq [(1+t_1)B + t_2]\|x\|_E.$$

Finally, put $\mathcal{S}_0 = \mathcal{S}\mathcal{Q}$. Then $\mathcal{S}_0 : \mathcal{A} \to E$ is a bounded linear operator such that $\mathcal{S}_0(\{\Theta_i(x)\}) = x$, $x \in E$. Hence, $(\{\Theta_i\}, \mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$ and having the required frame bounds. □

We finish with a stability result for *OBF*. In fact, we shall show that *OBF* are stable under perturbation of frame elements by positively confined sequence of scalars.

Theorem 6.4. *Let* $(\{\Lambda_i\}, \mathcal{S})$ *be an* OBF *for* E *with respect to* $\mathcal{A} \subset l^\infty$ *and having a couple of frame bounds* A *and* B. *Let* $\{\Theta_i\} \subset B(E, E_i)$ *be such that* $\{\Theta_i(x)\} \in \mathcal{A}$, $x \in E$ *and let* $\mathcal{Q} : \mathcal{A} \to \mathcal{A}$ *be a bounded linear operator such that* $\mathcal{Q}(\{\Theta_i(x)\}) = \{\Lambda_i(x)\}$, $x \in E$. *Let* $\{\alpha_i\}$, $\{\beta_i\} \subset \mathbb{R}$ *be two positively confined sequences. If there exists nonnegative scalars* p, q, r, t_1 *and* t_2 *such that*

(i) $((1-p)(\inf_{1\leq i<\infty} \alpha_i) - t_1)A > r$

(ii) $(1-q)(\inf_{1\leq i<\infty} \beta_i) > t_2$

(iii) $\|\{(\alpha_i\Lambda_i - \beta_i\Theta_i)(x)\}\|_{\mathcal{A}} \leq p\|\{(\alpha_i\Lambda_i)(x)\}\|_{\mathcal{A}} + q\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} + t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} + t_2\|\{\Theta_i(x)\}\|_{\mathcal{A}} + r\|x\|_E, \quad x \in E,$

then there exists a reconstruction operator $\mathcal{S}_0$ *such that* $(\{\Theta_i\}, \mathcal{S}_0)$ *is an* OBF *for* E *with respect to* $\mathcal{A}$ *and with a couple of frame bounds*

$$\left(\frac{((1-p)(\inf_{1\leq i<\infty} \alpha_i) - t_1)A - r}{(1+q)(\sup_{1\leq i<\infty} \beta_i) + t_2}\right) \text{ and } \left(\frac{((1+p)(\sup_{1\leq i<\infty} \alpha_i) + t_1)B + r}{(1-q)(\inf_{1\leq i<\infty} \beta_i) - t_2}\right).$$

Proof. We have the inequality (6.1) for the *OBF* $(\{\Lambda_i\}, \mathcal{S})$. Now, we compute, for each $x \in E$,

$$\begin{aligned}\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} &\leq \|\{(\alpha_i\Lambda_i)(x)\}\|_{\mathcal{A}} + \|\{(\alpha_i\Lambda_i - \beta_i\Theta_i)(x)\}\|_{\mathcal{A}} \\ &\leq \|\{(\alpha_i\Lambda_i)(x)\}\|_{\mathcal{A}} + p\|\{(\alpha_i\Lambda_i)(x)\}\|_{\mathcal{A}} + q\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} \\ &\quad + t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} + t_2\|\{\Theta_i(x)\}\|_{\mathcal{A}} + r\|x\|_E.\end{aligned}$$

This gives, using (6.1)

$$\begin{aligned}
&(1-q)\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} - t_2\|\{\Theta_i(x)\}\|_{\mathcal{A}}\\
&\le (1+p)\|\{(\alpha_i\Lambda_i)(x)\}\|_{\mathcal{A}} + t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} + r\|x\|_E\\
&\le \left((1+p)\left(\sup_{1\le i<\infty}\alpha_i\right)+t_1\right)\|\{\Lambda_i(x)\}\|_{\mathcal{A}} + r\|x\|_E\\
&\le \left[\left((1+p)\left(\sup_{1\le i<\infty}\alpha_i\right)+t_1\right)B+r\right]\|x\|_E.
\end{aligned}$$

Since $\mathcal{A}\subset l^\infty$, we get

$$\begin{aligned}
\left((1-q)\left(\inf_{1\le i<\infty}\beta_i\right)-t_2\right)\|\{\Theta_i(x)\}\|_{\mathcal{A}} &\le (1-q)\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} - t_2\|\{\Theta_i(x)\}\|_{\mathcal{A}}\\
&\le \left[\left((1+p)\left(\sup_{1\le i<\infty}\alpha_i\right)+t_1\right)B+r\right]\|x\|_E.
\end{aligned}$$

Similarly, using condition (iii) we get

$$\begin{aligned}
&(1+q)\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} + t_2\|\{\Theta_i(x)\}\|_{\mathcal{A}}\\
&\ge (1-p)\|\{(\alpha_i\Lambda_i)(x)\}\|_{\mathcal{A}} - t_1\|\{\Lambda_i(x)\}\|_{\mathcal{A}} - r\|x\|_E\\
&\ge \left((1-p)\left(\inf_{1\le i<\infty}\alpha_i\right)-t_1\right)\|\{\Lambda_i(x)\}\|_{\mathcal{A}} - r\|x\|_E\\
&\ge \left[\left((1-p)\left(\inf_{1\le i<\infty}\alpha_i\right)-t_1\right)A-r\right]\|x\|_E.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left((1+q)\left(\sup_{1\le i<\infty}\beta_i\right)+t_2\right)\|\{\Theta_i(x)\}\|_{\mathcal{A}} &\ge (1+q)\|\{(\beta_i\Theta_i)(x)\}\|_{\mathcal{A}} + t_2\|\{\Theta_i(x)\}\|_{\mathcal{A}}\\
&\ge \left[\left((1-p)\left(\inf_{1\le i<\infty}\alpha_i\right)-t_1\right)A-r\right]\|x\|_E.
\end{aligned}$$

Hence, the required *OBF* inequality is

$$A_0\|x\|_E \le \|\{\Theta_i(x)\}\|_{\mathcal{A}} \le B_0\|x\|_E, \qquad x\in E$$

where

$$\begin{aligned}
A_0 &= \left(\frac{((1-p)(\inf_{1\le i<\infty}\alpha_i)-t_1)A-r}{(1+q)(\sup_{1\le i<\infty}\beta_i)+t_2}\right),\\
B_0 &= \left(\frac{((1+p)(\sup_{1\le i<\infty}\alpha_i)+t_1)B+r}{(1-q)(\inf_{1\le i<\infty}\beta_i)-t_2}\right).
\end{aligned}$$

Put $\mathcal{S}_0 = \mathcal{S}\mathcal{Q}$. Then $\mathcal{S}_0:\mathcal{A}\to E$ be a bounded linear operator such that $\mathcal{S}_0(\{\Theta_i(x)\}) = x$, $x\in E$. Hence, $(\{\Theta_i\},\mathcal{S}_0)$ is an *OBF* for E with respect to $\mathcal{A}$ and having required frame bounds. □

Remark 6.1. In case of Banach frames, when $t_1 = t_2 = 0$, Theorem 6.4 reduces to Theorem 3.3 of [9].

7. Application

Virender et al.[15] posed a problem related to the Feichtinger Conjecture as:
Problem: Let $\{M_n\}$ be a sequence of finite subsets of $\mathbb{N}$ such that $\mathbb{N} = \cup_{i=1}^{\infty} M_i$, $M_i \cap M_j = \phi$, for all $i \neq j$, $\{x_n\}$ be a bounded below frame for a Hilbert space H and $V_n = [x_i]_{i \in M_n}$. Is $H = \oplus_{n \in \mathbb{N}} V_n$?

In this section we deal this problem in case of retro Banach frame with the following illustrative examples.

Example 7.1. Let $E = l^1$ and $\{x_n\} \subset E$ be the sequence defined by

$$f(x_{3n-1}) = f(x_{3n}) = f(x_{3n-2}) = \xi_n, \qquad n \in \mathbb{N}, f = \{\xi_n\} \in E^*.$$

Then by Lemma 2.1, there exists an associated Banach space $(E^*)_d = \{\{f(x_n)\} : f \in E^*\}$ with norm $\|\{f(x_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$. Define a bounded linear operator $T : (E^*)_d \to E^*$ by $T(\{f(x_n)\}) = f$, for all $f \in E^*$. Then $(\{x_n\}, T)$ is an *RBF* for E^* with respect to $(E^*)_d$.

Now, let $D_k = \{3k-2, 3k-1, 3k\}$, $k \in \mathbb{N}$. So that $D_i \cap D_j = \phi$, for all $i \neq j$, and $\cup_{k=1}^{\infty} D_k = \mathbb{N}$. Let

$$y_n = \sum_{i \in D_n} \alpha_i x_i, \qquad n \in \mathbb{N},$$

where $\alpha_i = i$, $i \in D_n$, $n \in \mathbb{N}$. Then $\{y_n\}$ is a block sequence with respect to the *RBF* $(\{x_n\}, T)$. Since $\{y_n\}$ is total over E, by Lemma 2.1, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ with norm given by

$$\|f\|_{E^*} = \|\{f(y_n)\}\|_{(E^*)_{d_0}},$$

for all $f \in E^*$. Define $S : (E^*)_{d_0} \to E^*$ by $S(\{f(y_n)\}) = f$, $f \in E^*$. Then $(\{y_n\}, S)$ is an *RBF* for E^* with respect to $(E^*)_{d_0}$. If we take $F_k = [y_i] = [x_{3i-2}, x_{3i-1}, x_{3i}]$, $i \in D_k$, $k \in \mathbb{N}$. Then $E = \bigoplus_{n \in \mathbb{N}} F_n$.

Example 7.2. Let $E = l^p$, $(1 < p < \infty)$ and $\{x_n\} \subset E$ be the sequence defined by $x_{2n} = e_n$, $x_{2n-1} = e_n + e_1$ for all $n \in \mathbb{N}$, where $\{e_n\}$ be a sequence of unit vectors in E. Then, by Lemma 2.1, there exists an associated Banach space $(E^*)_d = \{\{f(x_n)\} : f \in E^*\}$ with norm $\|\{f(x_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$. Define a bounded linear operator $T : (E^*)_d \to E^*$ by $T(\{f(x_n)\}) = f$, $f \in E^*$. Then $(\{x_n\}, T)$ is an *RBF* for E^* with respect to $(E^*)_d$.

Now, let $D_k = \{2k-1, 2k\}$, $k \in \mathbb{N}$. Then $D_i \cap D_j = \phi$, for all $i \neq j$ and $\cup_{i=1}^{\infty} D_i = \mathbb{N}$, where $\alpha_i = 1$, for all i. Then $\{y_n\}$ is a block sequence with respect to the *RBF* $(\{x_n\}, T)$. Since $\{y_n\}$ is total over E, therefore, by Lemma 2.1, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ with norm given by $\|f\|_{E^*} = \|\{f(y_n)\}\|_{(E^*)_{d_0}}$, $f \in E^*$. Define $S : (E^*)_{d_0} \to E^*$ by $S(\{f(y_n)\}) = f$, $f \in E^*$. Then $(\{y_n\}, S)$ is an RBF for E^* with respect to $(E^*)_{d_0}$.

Let D_1 and D_2 be two subsets of $\mathbb{N}$ satisfying $D_1 \cup D_2 = \mathbb{N}$, $D_1 \cap D_2 = \phi$ and $E = F_1 \bigoplus F_2$, where $F_1 = [y_i]_{i \in D_1}$ and $F_2 = [y_i]_{i \in D_2}$. Suppose that $y_1 \in F_1$. Then for each even $n \in \mathbb{N}$, we have $< y_1, y_n >= 3 \neq 0$. Thus $y_n \in F_1$, for all even n. Also, for each odd n, we have $< y_1, y_n > \neq 0$. Hence $y_n \in F_1$, for all odd n, that is $F_2 = 0$. Hence E cannot be decomposed into two or more disjoint subspaces.

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