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Some fixed point results on weighted rectangular b-metric spaces

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Abstract. In this paper, we prove some fixed point theorems for Reich contraction, Fisher contraction and Chatterjee contraction in the setting of rectangular b-metric spaces by relaxing the rectangular inequality to include unequal weights. Some examples are given which illustrate the newly proven results.

1. Introduction and Preliminaries

Metric fixed point theory is a relatively natural part of fixed point theory concerning methods and results that involve properties of an essentially metrical nature. The first result of this kind was proved on the setting of usual metric spaces in 1922 by the well-known Polish mathematician Banach [2] which, is often, refereed as Banach contraction principle. This principle ensures the existence and uniqueness of a fixed point for any contraction mapping defined on a complete metric space. Banach principle has been extended and generalized on several ways. One of the most famous ways to generalize Banach contraction principle is by changing the metric and defining new metric spaces. In 1989, Bakhtin [1] and Czerwik [5] introduced the notion of b-metric spaces as a generalization of metric spaces and they generalized Banach contraction theorem to such spaces. Similarly, [8] Khamsi and Hussain

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reintroduced the notion of a b-metric under the name metric-type and provided a similar version of Banach contraction principle on the setting of metric-type spaces. Branciari [3] presented the notion of a rectangular metric spaces and extended Banach result to such spaces. Thereafter, George et al. [7] introduced the notion of rectangular b-metric spaces as a generalization of the notions of metric spaces, rectangular metric spaces and b-metric spaces and proved a generalized version for Banach contraction theorem on the setting of rectangular b-metric spaces. Singh [11] provided a new generalization of a rectangular b-metric space by relaxing the rectangular inequality to include unequal weights and he extended Banach contraction principle to such spaces. In this paper, we prove some fixed point theorems for Reich contraction [10], Fisher contraction [6] and Chaterjee contraction [4] in the setting of weighted rectangular b-metric spaces. Our results extend the main results of Singh [11]. We exhibit the utility of ou results by construct some illustrative examples.

Now, let us recall some basic definitions, notions and results which will be needed in the sequel.

Definition 1.1. [11] *Let M be a nonempty set and $\Re : M \times M \to [0, \infty)$. Assume that the following assumptions are satisfied:*

- **(i):** $\Re(c, d) = 0$ *if and only if* $c = d$.
- **(ii):** $\Re(c, d) = \Re(d, c)$ *for all* $c, d \in M$.
- **(iii):** $\Re(c, d) \leq \omega_1 \Re(c, u) + \omega_2 \Re(u, v) + \omega_3 \Re(v, d)$ *for all* $c, d \in M$ *and all distinct points* $u, v \in M - \{c, d\}$, *for fixed postive real numbers* ω_1, ω_2 *and* ω_3.

Then $\Re$ is a weighted rectangular b-metric on M and $(M, \Re)$ is a weighted rectangular b-metric space. If $\omega_1 = \omega_2 = \omega_3 = 1$, we get a rectangular b-metric space and if $\omega_1 = \omega_2 = \omega_3 = s > 1$, we have a rectangular b-metric space with coefficient s.

Definition 1.2. [11] *Let $(M, \Re)$ be a weighted rectangular b-metric space and $\{c_n\}$ be a sequence in M with $c \in M$. Then*

- **(i):** *the sequence $\{c_n\}$ is called convergent in $(M, \Re)$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\Re(c_n, c) < \epsilon$ for all $n \geq N$ and is exemplified by $\lim_{n\to\infty} c_n = c$ or $c_n \to c$ as $n \to \infty$.*
- **(ii):** *the sequence $\{c_n\}$ is a Cauchy in $(M, \Re)$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(c_n, c_{n+p}) < \epsilon$ for all $n > N, p > 0$ or $\lim_{n\to\infty} \rho(c_n, c_{n+p}) = 0$, for all $p > 0$.*
- **(iii):** *$(M, \Re)$ is said to be a complete weighted rectangular b-metric space if every Cauchy sequence in M is converges to some $c \in M$.*

Singh [11] proved the following theorems.

THEOREM 1.1. [11] *Let* $(M, \Re)$ *be a complete weighted rectangular b-metric space with weights* ω_1, ω_2 *and* ω_3 *and* $T : M \to M$ *be a mapping such that*

$$\Re(Tc, Td) \leq \lambda \Re(c, d), \quad \text{for all } c, d \in M, \tag{1.1}$$

where $0 < \lambda < \min\{1, \frac{1}{\sqrt{\omega_3}}\}$. *Then* T *has a unique fixed point in* M.

THEOREM 1.2. [11] *Let* $(M, \Re)$ *be a complete weighted rectangular b-metric space with weights* ω_1, ω_2 *and* ω_3 *and* $T : M \to M$ *be a mapping such that*

$$\Re(Tc, Td) \leq \lambda[\Re(c, Tc) + \Re(d, Td)], \forall c, d \in M, \tag{1.2}$$

where $\lambda < \frac{1}{2}$ *if* $\omega_3 \leq 1$ *and* $\lambda < \frac{1}{1+\sqrt{\omega_3}}$ *if* $\omega_3 > 1$. *Then* T *has a unique fixed point in* M.

2. Main Results

In this section, we state and prove our main results as under.

THEOREM 2.1. *Let* $(M, \Re)$ *be a complete weighted rectangular b-metric space with weights* ω_1, ω_2 *and* ω_3 *and* $T : M \to M$ *be a mapping such that*

$$\Re(Tc, Td) \leq \lambda \Re(c, Tc) + \mu \Re(d, Td) + \delta \Re(c, d), \forall c, d \in M, \tag{2.1}$$

where $\lambda + \mu + \delta < 1$ *if* $\omega_3 \leq 1$ *and* $\dfrac{\lambda + \delta}{1 - \mu} < \frac{1}{\sqrt{\omega_3}}$ *if* $\omega_3 > 1$. *Then* T *has a unique fixed point in* M.

PROOF. Let $c_0 \in M$ be an arbitrary point in M. Define $\{c_n\}$ by $c_{n+1} = Tc_n$, for all $n \geq 0$. We show that $\{c_n\}$ is a Cauchy sequence. Now, if there exists $n \in \mathbb{N}$ such that $c_{n+1} = c_n$, then c_n is a fixed point of T. Otherwise, suppose that $c_n \neq c_{n+1}$ for all $n \geq 0$. Denote $\Re(c_n, c_{n+1}) = \Re_n$ for all $n \in \mathbb{N}$. Now, in view of (2.1), we have

$$\begin{aligned}
\Re(c_n, c_{n+1}) &= \Re(Tc_{n-1}, Tc_n) \\
&\leq \lambda \Re(c_{n-1}, Tc_{n-1}) + \mu \Re(c_n, Tc_n) + \delta \Re(c_{n-1}, c_n) \\
&= \lambda \Re(c_{n-1}, c_n) + \mu \Re(c_n, c_{n+1}) + \delta \Re(c_{n-1}, c_n) \\
(1 - \mu)\Re(c_n, c_{n+1}) &\leq (\lambda + \delta)\Re(c_{n-1}, c_n) \\
\Re(c_n, c_{n+1}) &\leq \frac{\lambda + \delta}{1 - \mu} \Re(c_{n-1}, c_n),
\end{aligned}$$

or

$$\Re_n \leq \frac{\lambda + \delta}{1 - \mu} \Re_{n-1} = \beta \Re_{n-1},$$

where $\beta = \dfrac{\lambda + \delta}{1 - \mu} < 1$. Repeating this process in (2.1), we obtain

$$\Re_n \leq \beta^n \Re_0. \tag{2.2}$$

Also, suppose that c_0 is not a periodic point of T, otherwise $c_0 = c_n$ for $n \geq 2$ which implies that

$$\begin{aligned} \Re(c_0, Tc_0) &= \Re(c_n, Tc_n) \\ \Re(c_0, c_1) &= \Re(c_n, c_{n+1}) \\ \Re_0 &= \Re_n \leq \beta^n \Re_0, \end{aligned} \tag{2.3}$$

this contradicts the fact $\Re_0 \neq 0$ or $c_1 \neq c_0$. Thus, we suppose that $c_n \neq c_m$ for all $n, m \in \mathbb{N}$. Now

$$\begin{aligned} \Re\left(c_n, c_{n+2}\right) =& \Re\left(Tc_{n-1}, Tc_{n+1}\right) \\ \leq& \lambda \Re\left(c_{n-1}, Tc_{n-1}\right) + \mu \Re\left(c_{n+1}, Tc_{n+1}\right) + \delta \Re(c_{n-1}, c_{n+1}) \\ =& \lambda \Re\left(c_{n-1}, c_n\right) + \mu \Re\left(c_{n+1}, c_{n+2}\right) + \delta \Re(c_{n-1}, c_{n+1}) \\ \leq& \lambda \Re\left(c_{n-1}, c_n\right) + \mu \Re\left(c_{n+1}, c_{n+2}\right) + \delta\left[\omega_1 \Re\left(c_{n-1}, c_n\right) + \omega_2 \Re\left(c_n, c_{n+2}\right)\right. \\ & \left. + \omega_3 \Re\left(c_{n+2}, c_{n+1}\right)\right], \end{aligned}$$

that is,

$$\Re\left(c_n, c_{n+2}\right) \leq \frac{\lambda + \delta\omega_1}{1 - \delta\omega_2} \Re_{n-1} + \frac{\mu + \delta\omega_3}{1 - \delta\omega_2} \Re_{n+1},$$

which together with (2.2) yields

$$\begin{aligned} \Re\left(c_n, c_{n+2}\right) &\leq \frac{\lambda + \delta\omega_1 + [\mu + \delta\omega_3]\beta^2}{1 - \delta\omega_2} \beta^{n-1} \Re_0 \\ &\leq \frac{\lambda + \delta\omega_1 + \mu + \delta\omega_3}{1 - \delta\omega_2} \beta^{n-1} \Re_0 \\ &= \alpha \beta^{n-1} \Re_0 \\ \Re^\star &\leq \alpha \beta^{n-1} \Re_0. \end{aligned} \tag{2.4}$$

where $\alpha = \frac{\lambda + \delta\omega_1 + \mu + \delta\omega_3}{1 - \delta\omega_2}$. We shall show that the sequence $\{c_n\}$ is a Cauchy sequence. We consider the value of $\Re\left(c_n, c_{n+k}\right)$ in two cases.

If k is odd, say $2m+1$, then using rectangular inequality and (2.2) we obtain

$$\begin{aligned}
\Re\left(c_n, c_{n+2m+1}\right) &\leq \omega_1 \Re\left(c_n, c_{n+1}\right)+\omega_2 \Re\left(c_{n+1}, c_{n+2}\right)+\omega_3 \Re\left(c_{n+2}, c_{n+2m+1}\right) \\
&\leq \omega_1 \Re_n+\omega_2 \Re_{n+1}+\omega_3\left[\omega_1 \Re\left(c_{n+2}, c_{n+3}\right)+\omega_2 \Re\left(c_{n+3}, c_{n+4}\right)\right. \\
&\left.+\omega_3 \Re\left(c_{n+4}, c_{n+2m+1}\right)\right] \\
&=\omega_1 \Re_n+\omega_2 \Re_{n+1}+\omega_3 \omega_1 \Re_{n+2}+\omega_3 \omega_2 \Re_{n+3}+\omega_3^2 \Re\left(c_{n+4}, c_{n+2m+1}\right) \\
&\leq\left(\omega_1 \Re_n+\omega_2 \Re_{n+1}\right)+\omega_3\left(\omega_1 \Re_{n+2}+\omega_2 \Re_{n+3}\right) \\
&+\omega_3^2\left(\omega_1 \Re_{n+4}+\omega_2 \Re_{n+5}\right)+\cdots+\omega_3^m \Re_{n+2m} \\
&=\sum_{q=0}^{m-1} \omega_3^q\left(\omega_1 \Re_{n+2q}+\omega_2 \Re_{n+2q+1}\right)+\omega_3^m \Re_{n+2m} \\
&\leq \sum_{q=0}^{m-1} \omega_3^q\left(\omega_1 \beta^{n+2q}+\omega_2 \beta^{n+2q+1}\right) \Re_0+\omega_3^m \beta^{n+2m} \Re_0 \\
&=\beta^n \omega_1 \Re_0 \sum_{q=0}^{m-1} \omega_3^q \beta^{2q}+\beta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-1} \omega_3^q \beta^{2q}+\beta^n \Re_0 \omega_3^m \beta^{2m} \\
&=\beta^n \omega_1 \Re_0 \sum_{q=0}^{m-1}\left(\omega_3 \beta^2\right)^q+\beta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-1}\left(\omega_3 \beta^2\right)^q+\beta^n \Re_0\left(\omega_3 \beta^2\right)^m \\
&\leq \beta^n \omega_1 \Re_0 \sum_{q=0}^{m-1}\left(\omega_3 \beta^2\right)^q+\beta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-1}\left(\omega_3 \beta^2\right)^q \\
&+\beta^n \Re_0\left(\omega_3 \beta^2<1\right) \\
&<\beta^n \Re_0\left[1+\frac{\omega_1+\omega_2 \beta}{1-\omega_3 \beta^2}\right] .
\end{aligned} \tag{2.5}$$

If k is even, say $2m$, then using rectangular inequality and (2.2), we obtain

$$\begin{aligned}
\Re\left(c_n, c_{n+2m}\right) &\leq \omega_1 \Re\left(c_n, c_{n+1}\right)+\omega_2 \Re\left(c_{n+1}, c_{n+2}\right)+\omega_3 \Re\left(c_{n+2}, c_{n+2m}\right) \\
&\leq \omega_1 \Re_n+\omega_2 \Re_{n+1}+\omega_3\left[\omega_1 \Re\left(c_{n+2}, c_{n+3}\right)+\omega_2 \Re\left(c_{n+3}, c_{n+4}\right)\right] \\
&+\omega_3^2 \Re\left(c_{n+4}, c_{n+2m}\right) \\
&=\omega_1 \Re_n+\omega_2 \Re_{n+1}+\omega_3 \omega_1 \Re_{n+2}+\omega_3 \omega_2 \Re_{n+3}+\omega_3^2 \Re\left(c_{n+4}, c_{n+2m}\right) \\
&\leq\left(\omega_1 \Re_n+\omega_2 \Re_{n+1}\right)+\omega_3\left(\omega_1 \Re_{n+2}+\omega_2 \Re_{n+3}\right)+\omega_3^2\left(\omega_1 \Re_{n+4}+\omega_2 \Re_{n+5}\right) \\
&+\cdots+\omega_3^{m-2}\left(\omega_1 \Re_{n+2m-4}+\omega_2 \Re_{n+2m-3}\right)+\omega_3^{m-1} \Re\left(c_{n+2m-2}, c_{n+2m}\right) \\
&=\sum_{q=0}^{m-2} \omega_3^q\left(\omega_1 \Re_{n+2q}+\omega_2 \Re_{n+2q+1}\right)+\omega_3^{m-1} \Re_{n+2m-2}^{\star}
\end{aligned}$$

$$\leq \sum_{q=0}^{m-2} \omega_3^q \left(\omega_1 \beta^{n+2q} + \omega_2 \beta^{n+2q+1}\right) \Re_0 + \omega_3^{m-1} \alpha \beta^{n+2m-2} \Re_0$$
$$= \beta^n \omega_1 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \beta^2\right)^q + \beta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \beta^2\right)^q + \alpha \beta^{n-1} \Re_0 \left(\omega_3 \beta^2\right)^{m-1}$$
$$\leq \beta^n \omega_1 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \beta^2\right)^q + \beta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \beta^2\right)^q + \alpha \beta^{n-1} \Re_0$$
$$< \beta^{n-1} \left[\alpha \Re_0 + \frac{\left(\omega_1 \beta + \omega_2 \beta^2\right) \Re_0}{1 - \omega_3 \beta^2}\right].$$

Hence, from (2.5) and (2)that $\lim_{n\to\infty} \Re(c_n, c_{n+k}) = 0$ for all $k > 0$ and $\{c_n\}$ is Cauchy in $(M, \Re)$. By the completeness of $(M, \Re)$ there exists $z \in M$ such that $c_n \to z$. Now, we show that z is a fixed point of T. Observe that

$$\begin{aligned}
\Re(z, Tz) &\leq \omega_1 \Re(z, c_n) + \omega_2 \Re(c_n, c_{n+1}) + \omega_3 \Re(c_{n+1}, Tz) \\
&= \omega_1 \Re(z, c_n) + \omega_2 \Re_n + \omega_3 \Re(Tc_n, Tz) \\
&\leq \omega_1 \Re(z, c_n) + \omega_2 \Re_n + \omega_3 \left[\lambda \Re(c_n, Tc_n) + \mu \Re(z, Tz) + \delta \Re(c_n, z)\right] \\
&\leq \omega_1 \Re(z, c_n) + \omega_2 \Re_n + \omega_3 \lambda \Re(c_n, c_{n+1}) + \omega_3 \mu \Re(z, Tz) + \omega_3 \delta \Re(c_n, z)
\end{aligned}$$

$$(1 - \omega_3 \mu) \Re(z, Tz) \leq (\omega_1 + \omega_3 \delta) \Re(z, c_n) + (\omega_2 + \omega_3 \lambda) \Re_n. \tag{2.2}$$

Taking the limit as $n \to \infty$ in (2) we have $\rho(z, Tz) = 0$. If v is another fixed point of T. Then

$$\begin{aligned}
\Re(z, v) &= \Re(Tz, Tv) \\
&\leq \lambda \Re(z, Tz) + \mu \Re(v, Tv) + \delta \Re(z, v) \\
&= \lambda \Re(z, z) + \mu \Re(v, v) + \delta \Re(z, v) \\
&= \delta \Re(z, v) = 0.
\end{aligned}$$

Hence $\Re(z, v) = 0$ or $z = v$ which proves the uniqueness. □

Next, we state and prove our second main results as follows.

THEOREM 2.2. *Let $(M, \Re)$ be a complete weighted rectangular b-metric space with weights ω_1, ω_2 and ω_3 and $T : M \to M$ be a mapping such that*

$$\Re(Tc, Td) \leq \mu[\Re(c, Td) + \Re(d, Tc)], \ \forall c, d \in X \tag{2.3}$$

where μ is a non-negative real number such that $\mu < \frac{1}{2}$ if $\omega_3 \leq 1$ and $\mu < \frac{1}{1+\sqrt{\omega_3}}$ if $\omega_3 > 1$. Then T has a unique fixed point in M.

PROOF. For any arbitrary point $c_0 \in M$, define a sequence $\{c_n\}$ in M by $c_{n+1} = Tc_n$ for all $n \geq 0$. If $c_{n+1} = c_n$, then c_n is a fixed point of T. Otherwise, suppose that $c_n \neq c_{n+1}$ for all $n \geq 0$. Let $\Re(c_n, c_{n+1}) = \Re_n$, for all $n \geq 0$. Observe that

$$\begin{aligned}
\Re(c_n, c_{n+1}) &= \Re(Tc_{n-1}, Tc_n) \\
&\leq \mu[\Re(c_{n-1}, Tc_n) + \Re(c_n, Tc_{n-1})] \\
&= \mu[\Re(c_{n-1}, c_{n+1}) + \Re(c_n, c_n)] \\
&\leq \mu\Re(c_{n-1}, c_{n+1}) \\
&\leq \mu[\Re(c_{n-1}, c_n) + \Re(c_n, c_{n+1})] \\
\Re_n &\leq \frac{\mu}{1-\mu}\Re_{n-1} \qquad (2.4) \\
&= \alpha\Re_{n-1},
\end{aligned}$$

where $\alpha = \frac{\mu}{1-\mu} < 1$. Repeat this process in (2.4), we obtain

$$\Re_n \leq \alpha^n \Re_0. \qquad (2.5)$$

Also, observe that c_0 is not a periodic point of T. Otherwise $c_0 = c_n$ for $n \geq 2$ which implies that

$$\begin{aligned}
\Re(c_0, Tc_0) &= \Re(c_n, Tc_n) \\
\Re(c_0, c_1) &= \Re(c_n, c_{n+1}) \\
\Re_0 = \Re_n &\leq \alpha^n \Re_0 \qquad (2.6)
\end{aligned}$$

a contradiction as $\Re_0 \neq 0$ or $c_1 \neq c_0$. Thus, we get that $c_n \neq c_m$ for all $n, m \in \mathbb{N}$. Now,

$$\begin{aligned}
\Re(c_n, c_{n+2}) &= \Re(Tc_{n-1}, Tc_{n+1}) \\
&\leq \mu[\Re(c_{n-1}, Tc_{n+1}) + \Re(c_{n+1}, Tc_{n-1})] \\
&= \mu\Re(c_{n-1}, c_{n+2}) + \mu\Re(c_{n+1}, c_n) \\
&\leq \mu\omega_1\Re(c_{n-1}, c_n) + \mu\omega_2\Re(c_n, c_{n+2}) + \mu\omega_3\Re(c_{n+2}, c_{n+1}) + \mu\Re(c_{n+1}, c_n)
\end{aligned}$$

that is,

$$\Re(c_n, c_{n+2}) \leq \frac{\mu\omega_1}{1-\mu\omega_2}\Re_{n-1} + \frac{\mu}{1-\mu\omega_2}\Re_n + \frac{\mu\omega_3}{1-\mu\omega_2}\Re_{n+1}$$

which together with (2.5) imply that

$$\begin{aligned}
\Re(c_n, c_{n+2}) &\leq \frac{\mu\omega_1 + \mu\alpha + \mu\omega_3\alpha^2}{1-\mu\omega_2}\alpha^{n-1}\Re_0 \\
&\leq \frac{\mu\omega_1 + \mu\alpha + \mu\omega_3}{1-\mu\omega_2}\alpha^{n-1}\Re_0
\end{aligned}$$

that is,

$$\Re^\star(c_n, c_{n+2}) \leq \kappa\alpha^{n-1}\Re_0 \qquad (2.7)$$

where $\kappa = \frac{\mu\omega_1+\mu\alpha+\mu\omega_3}{1-\mu\omega_2}$. Now, we show that the sequence $\{c_n\}$ is a Cauchy sequence. To do so, consider the value of $\Re(c_n, c_{n+k})$ in two cases. If k is odd, say $2m+1$, then using rectangular inequality and (2.2) we obtain

$$\begin{aligned}
\Re(c_n, c_{n+2m+1}) &\leq \omega_1 \Re(c_n, c_{n+1}) + \omega_2 \Re(c_{n+1}, c_{n+2}) + \omega_3 \Re(c_{n+2}, c_{n+2m+1}) \\
&\leq \omega_1 \Re_n + \omega_2 \Re_{n+1} + \omega_3 \left[\omega_1 \Re(c_{n+2}, c_{n+3}) + \omega_2 \Re(c_{n+3}, c_{n+4})\right. \\
&\left.+\omega_3 \Re(c_{n+4}, c_{n+2m+1})\right] \\
&= \omega_1 \Re_n + \omega_2 \Re_{n+1} + \omega_3\omega_1 \Re_{n+2} + \omega_3\omega_2 \Re_{n+3} + \omega_3^2 \Re(c_{n+4}, c_{n+2m+1}) \\
&\leq (\omega_1 \Re_n + \omega_2 \Re_{n+1}) + \omega_3 (\omega_1 \Re_{n+2} + \omega_2 \Re_{n+3}) \\
&+ \omega_3^2 (\omega_1 \Re_{n+4} + \omega_2 \Re_{n+5}) + \cdots + \omega_3^m \Re_{n+2m} \\
&= \sum_{q=0}^{m-1} \omega_3^q (\omega_1 \Re_{n+2q} + \omega_2 \Re_{n+2q+1}) + \omega_3^m \Re_{n+2m} \\
&\leq \sum_{q=0}^{m-1} \omega_3^q \left(\omega_1 \alpha^{n+2q} + \omega_2 \alpha^{n+2q+1}\right) \Re_0 + \omega_3^m \alpha^{n+2m} \Re_0 \\
&= \alpha^n \omega_1 \Re_0 \sum_{q=0}^{m-1} \omega_3^q \alpha^{2q} + \alpha^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-1} \omega_3^q \alpha^{2q} + \alpha^n \Re_0 \omega_3^m \eta^{2m} \\
&= \alpha^n \omega_1 \Re_0 \sum_{q=0}^{m-1} \left(\omega_3 \alpha^2\right)^q + \alpha^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-1} \left(\omega_3 \alpha^2\right)^q + \alpha^n \Re_0 \left(\omega_3 \alpha^2\right)^m \\
&\leq \alpha^n \omega_1 \Re_0 \sum_{q=0}^{m-1} \left(\omega_3 \alpha^2\right)^q + \alpha^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-1} \left(\omega_3 \alpha^2\right)^q + \alpha^n \Re_0 \left(\omega_3 \alpha^2 < 1\right) \\
&< \alpha^n \Re_0 \left[1 + \frac{\omega_1 + \omega_2 \alpha}{1 - \omega_3 \alpha^2}\right].
\end{aligned} \tag{2.8}$$

If k is even, say $2m$, then using rectangular inequality and (2.2) we obtain

$$\begin{aligned}
\Re(c_n, c_{n+2m}) &\leq \omega_1 \Re(x_n, c_{n+1}) + \omega_2 \Re(c_{n+1}, c_{n+2}) + \omega_3 \Re(c_{n+2}, c_{n+2m}) \\
&\leq \omega_1 \Re_n + \omega_2 \Re_{n+1} + \omega_3 \left[\omega_1 \Re(c_{n+2}, c_{n+3}) + \omega_2 \Re(c_{n+3}, c_{n+4})\right] \\
&+ \omega_3^2 \Re(c_{n+4}, c_{n+2m}) \\
&= \omega_1 \Re_n + \omega_2 \Re_{n+1} + \omega_3\omega_1 \Re_{n+2} + \omega_3\omega_2 \Re_{n+3} + \omega_3^2 \Re(c_{n+4}, c_{n+2m}) \\
&\leq (\omega_1 \Re_n + \omega_2 \Re_{n+1}) + \omega_3 (\omega_1 \Re_{n+2} + \omega_2 \Re_{n+3}) + \omega_3^2 (\omega_1 \Re_{n+4} + \omega_2 \rho_{n+5}) \\
&+ \cdots + \omega_3^{m-2} (\omega_1 \Re_{n+2m-4} + \omega_2 \Re_{n+2m-3}) + \omega_3^{m-1} \Re(x_{n+2m-2}, x_{n+2m})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q=0}^{m-2} \omega_3^q \left(\omega_1 \Re_{n+2q} + \omega_2 \Re_{n+2q+1}\right) + \omega_3^{m-1} \Re^{\star}_{n+2m-2} \\
&\leq \sum_{q=0}^{m-2} \omega_3^q \left(\omega_1 \alpha^{n+2q} + \omega_2 \alpha^{n+2q+1}\right) \Re_0 + \omega_3^{m-1} \alpha \alpha^{n+2m-2} \Re_0 \\
&= \alpha^n \omega_1 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \alpha^2\right)^q + \alpha^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \alpha^2\right)^q + \alpha \alpha^{n-1} \Re_0 \left(\omega_3 \alpha^2\right)^{m-1} \\
&\leq \alpha^n \omega_1 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \alpha^2\right)^q + \alpha^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-2} \left(\omega_3 \alpha^2\right)^q + \alpha \alpha^{n-1} \Re_0 \\
&< \alpha^{n-1} \left[\alpha \Re_0 + \frac{\left(\omega_1 \alpha + \omega_2 \alpha^2\right) \Re_0}{1 - \omega_3 \alpha^2}\right].
\end{aligned}
$$

As, $\lim_{n\to 0} \Re\left(c_n, c_{n+k}\right) = 0$ for all $k > 0$, it follows that $\{c_n\}$ is Cauchy in $(M, \Re)$. By the completeness of $(M, \Re)$, there exists $c^\star \in M$ such that $c_n \to c^\star$ as $n \to \infty$. Next, we show that $c^\star$ is a fixed point of T. Observe that

$$
\begin{aligned}
\Re\left(c^\star, Tc^\star\right) &\leq \omega_1 \Re\left(c^\star, c_n\right) + \omega_2 \Re\left(c_n, c_{n+1}\right) + \omega_3 \Re\left(c_{n+1}, Tc^\star\right) \\
&= \omega_1 \Re\left(c^\star, c_n\right) + \omega_2 \Re_n + \omega_3 \Re\left(Tc_n, Tc^\star\right) \\
&\leq \omega_1 \Re\left(c^\star, c_n\right) + \omega_2 \Re_n + \omega_3 \mu \left[\Re(c_n, Tc^\star) + \Re\left(c^\star, Tc_n\right)\right] \\
&\leq \omega_1 \Re\left(c^\star, c_n\right) + \omega_2 \Re_n + \omega_3 \mu \Re\left(c_n, Tc^\star\right) + \omega_3 \mu \Re(c^\star, c_{n+1}) \\
&\leq \omega_1 \Re\left(c^\star, c_n\right) + \omega_2 \Re\left(c_n, c_{n+1}\right) + \omega_3 \mu \Re\left(c_n, Tc^\star\right) + \omega_3 \mu \Re(c^\star, c_{n+1}).
\end{aligned}
$$

Taking the limit as $n \to \infty$ in (2), we get $\Re\left(c^\star, Tc^\star\right) = 0$ or $Tc^\star = c^\star$. Hence, $c^\star$ is a fixed point of T. Finally, we prove the uniqueness of the fixed point of T. To do so, assume that $c^{\star\star}$ is another fixed point of T. Observe that

$$
\begin{aligned}
\Re\left(c^\star, c^{\star\star}\right) &= \Re(Tc^\star, Tc^{\star\star}) \\
&\leq \mu[\Re(c^\star, Tc^{\star\star}) + \Re(c^{\star\star}, Tc^\star)] \\
&= \mu \Re(c^\star, c^{\star\star}) + \mu \Re(c^{\star\star}, c^\star) \\
&= 2\mu \Re(c^\star, c^{\star\star}) \\
&= 0.
\end{aligned}
$$

Hence, $\Re\left(c^\star, c^{\star\star}\right) = 0$ or $c^\star = c^{\star\star}$ proving the uniqueness. As required. □

Next, we provide the following theorem.

THEOREM 2.3. *Let $(M, \Re)$ be a complete weighted rectangular b-metric space with weights ω_1, ω_2 and ω_3 and $T : M \to M$ be a mapping such that*

$$
\Re(Tc, Td) \leq \lambda \Re(c, d) + \mu[\Re(c, Tc) + \Re(d, Td)] \quad \forall c, d \in M, \tag{2.1}
$$

where $\lambda + 2\mu < 1$ *if* $\omega_3 \leq 1$ *and* $\lambda + 2\mu < \frac{1}{\sqrt{\omega_3}}$ *if* $\omega_3 > 1$. *Then* T *has a unique fixed point in* M.

PROOF. Let $c_0 \in M$ be an arbitrary point. Define a sequence $\{c_n\}$ in M by $c_{n+1} = Tc_n$, for all $n \geq 0$. If $c_{n+1} = c_n$, then c_n is a fixed point of T. Otherwise, suppose that $c_n \neq c_{n+1}$, for all $n \geq 0$. As before denote $\Re(c_n, c_{n+1}) = \Re_n$. Observe that

$$\begin{aligned}
\Re(c_n, c_{n+1}) &= \Re(Tc_{n-1}, Tc_n) \\
&\leq \lambda\Re(c_{n-1}, c_n) + \mu\left[\Re(c_{n-1}, Tc_{n-1}) + \Re(c_n, Tc_n)\right] \\
&= \lambda\Re(c_{n-1}, c_n) + \mu\left[\Re(c_{n-1}, c_n) + \Re(c_n, c_{n+1})\right] \\
\Re_n &\leq \lambda\Re_{n-1} + \mu\Re_{n-1} + \mu\Re_n \\
\Re_n &\leq \frac{\lambda + \mu}{1 - \mu}\Re_{n-1} \qquad (2.2)\\
&= \eta\Re_{n-1},
\end{aligned}$$

where $\eta = \frac{\lambda+\mu}{1-\mu} < 1$. Repeating this process in (2.2), we obtain

$$\Re_n \leq \eta^n \Re_0. \qquad (2.3)$$

Also, observe that c_0 is not a periodic point of T. Otherwise, $c_0 = c_n$ for $n \geq 2$ which implies that

$$\begin{aligned}
\Re(c_0, Tc_0) &= \Re(c_n, Tc_n) \\
\Re(c_0, c_1) &= \Re(c_n, c_{n+1}) \\
\Re_0 &= \Re_n \leq \eta^n \Re_0 \qquad (2.4)
\end{aligned}$$

a contradiction as $\Re_0 \neq 0$ or $c_1 \neq c_0$. Therefore, we have $c_n \neq c_m$, for all $n, m \in \mathbb{N}$. Now, observe that

$$\begin{aligned}
\Re(c_n, c_{n+2}) =&\Re(Tc_{n-1}, Tc_{n+1}) \\
\leq&\lambda\Re(c_{n-1}, c_{n+1}) + \mu[\Re(c_{n-1}, Tc_{n-1}) + \Re(c_{n+1}, Tc_{n+1})] \\
=&\lambda\Re(c_{n-1}, c_{n+1}) + \mu\Re(c_{n-1}, c_n) + \mu\Re(c_{n+1}, c_{n+2}) \\
\leq&\lambda\left[\omega_1\Re(c_{n-1}, c_n) + \omega_2\Re(c_n, c_{n+2}) + \omega_3\Re(c_{n+2}, c_{n+1})\right] \\
&+ \mu\Re(c_{n-1}, c_n) + \mu\Re(c_{n+1}, c_{n+2})
\end{aligned}$$

that is,

$$\Re(c_n, c_{n+2}) \leq \frac{\lambda\omega_1 + \mu}{1 - \lambda\omega_2}\Re_{n-1} + \frac{\lambda\omega_3 + \mu}{1 - \lambda\omega_2}\Re_{n+1}$$

which together with (2.3) yield that

$$\begin{aligned}\Re\left(c_n, c_{n+2}\right) &\leq \frac{\lambda\omega_1+\mu+[\lambda\omega_3+\mu]\eta^2}{1-\delta\omega_2}\eta^{n-1}\Re_0\\ &\leq \frac{\lambda\omega_1+\lambda\omega_3+2\mu}{1-\lambda\omega_2}\eta^{n-1}\Re_0\end{aligned}$$

that is,

$$\Re^{\star}\left(c_n, c_{n+2}\right) \leq \alpha\eta^{n-1}\Re_0 \tag{2.5}$$

where $\alpha = \frac{\lambda\omega_1+\lambda\omega_3+2\mu}{1-\lambda\omega_2} \geq 0$. Now, we show that the sequence $\{c_n\}$ is a Cauchy sequence. To do so, consider the value of $\Re\left(c_n, c_{n+k}\right)$ in two cases. If k is odd, say $2m+1$, then using rectangular inequality and (2.2) we obtain

$$\begin{aligned}\Re\left(c_n, c_{n+2m+1}\right) \leq&\omega_1\Re\left(c_n, c_{n+1}\right)+\omega_2\Re\left(c_{n+1}, c_{n+2}\right)+\omega_3\Re\left(c_{n+2}, c_{n+2m+1}\right)\\ \leq&\omega_1\Re_n+\omega_2\rho_{n+1}+\omega_3\left[\omega_1\Re\left(c_{n+2}, c_{n+3}\right)+\omega_2\Re\left(c_{n+3}, c_{n+4}\right)\right.\\ &\left.+\omega_3\Re\left(c_{n+4}, c_{n+2m+1}\right)\right]\\ \leq&\omega_1\Re_n+\omega_2\Re_{n+1}+\omega_3\omega_1\Re_{n+2}+\omega_3\omega_2\Re_{n+3}+\omega_3^2\Re\left(c_{n+4}, c_{n+2m+1}\right)\\ \leq&\left(\omega_1\Re_n+\omega_2\Re_{n+1}\right)+\omega_3\left(\omega_1\Re_{n+2}+\omega_2\Re_{n+3}\right)\\ &+\omega_3^2\left(\omega_1\Re_{n+4}+\omega_2\Re_{n+5}\right)+\cdots+\omega_3^m\Re_{n+2m}\\ =&\sum_{q=0}^{m-1}\omega_3^q\left(\omega_1\Re_{n+2q}+\omega_2\Re_{n+2q+1}\right)+\omega_3^m\Re_{n+2m}\\ \leq&\sum_{q=0}^{m-1}\omega_3^q\left(\omega_1\eta^{n+2q}+\omega_2\eta^{n+2q+1}\right)\Re_0+\omega_3^m\eta^{n+2m}\Re_0\\ =&\eta^n\omega_1\Re_0\sum_{q=0}^{m-1}\omega_3^q\eta^{2q}+\eta^{n+1}\omega_2\Re_0\sum_{q=0}^{m-1}\omega_3^q\eta^{2q}+\eta^n\Re_0\omega_3^m\eta^{2m}\\ =&\eta^n\omega_1\Re_0\sum_{q=0}^{m-1}\left(\omega_3\eta^2\right)^q+\eta^{n+1}\omega_2\Re_0\sum_{q=0}^{m-1}\left(\omega_3\eta^2\right)^q+\eta^n\Re_0\left(\omega_3\eta^2\right)^m\\ \leq&\eta^n\omega_1\Re_0\sum_{q=0}^{m-1}\left(\omega_3\eta^2\right)^q+\eta^{n+1}\omega_2\Re_0\sum_{q=0}^{m-1}\left(\omega_3\eta^2\right)^q+\eta^n\Re_0\left(\omega_3\eta^2<1\right)\\ <&\eta^n\Re_0\left[1+\frac{\omega_1+\omega_2\eta}{1-\omega_3\eta^2}\right].\end{aligned} \tag{2.6}$$

If k is even, say $2m$, then using rectangular inequality and (2.2) we obtain

$$\begin{aligned}
\Re\left(c_n, c_{n+2m}\right) &\leq \omega_1 \Re\left(c_n, c_{n+1}\right)+\omega_2 \Re\left(c_{n+1}, c_{n+2}\right)+\omega_3 \Re\left(c_{n+2}, c_{n+2m}\right) \\
&\leq \omega_1 \Re_n+\omega_2 \Re_{n+1}+\omega_3\left[\omega_1 \Re\left(c_{n+2}, c_{n+3}\right)+\omega_2 \Re\left(c_{n+3}, c_{n+4}\right)\right] \\
&+\omega_3^2 \Re\left(c_{n+4}, c_{n+2m}\right) \\
&=\omega_1 \rho_n+\omega_2 \Re_{n+1}+\omega_3 \omega_1 \Re_{n+2}+\omega_3 \omega_2 \Re_{n+3}+\omega_3^2 \Re\left(c_{n+4}, c_{n+2m}\right) \\
&\leq\left(\omega_1 \Re_n+\omega_2 \Re_{n+1}\right)+\omega_3\left(\omega_1 \Re_{n+2}+\omega_2 \Re_{n+3}\right)+\omega_3^2\left(\omega_1 \Re_{n+4}+\omega_2 \Re_{n+5}\right) \\
&+\cdots+\omega_3^{m-2}\left(\omega_1 \Re_{n+2m-4}+\omega_2 \Re_{n+2m-3}\right)+\omega_3^{m-1} \Re\left(x_{n+2m-2}, x_{n+2m}\right) \\
&=\sum_{q=0}^{m-2} \omega_3^q\left(\omega_1 \rho_{n+2q}+\omega_2 \Re_{n+2q+1}\right)+\omega_3^{m-1} \Re_{n+2m-2}^{\star} \\
&\leq \sum_{q=0}^{m-2} \omega_3^q\left(\omega_1 \eta^{n+2q}+\omega_2 \eta^{n+2q+1}\right) \Re_0+\omega_3^{m-1} \alpha \eta^{n+2m-2} \Re_0 \\
&=\eta^n \omega_1 \Re_0 \sum_{q=0}^{m-2}\left(\omega_3 \eta^2\right)^q+\eta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-2}\left(\omega_3 \eta^2\right)^q+\alpha \eta^{n-1} \Re_0\left(\omega_3 \eta^2\right)^{m-1} \\
&\leq \eta^n \omega_1 \Re_0 \sum_{q=0}^{m-2}\left(\omega_3 \eta^2\right)^q+\eta^{n+1} \omega_2 \Re_0 \sum_{q=0}^{m-2}\left(\omega_3 \eta^2\right)^q+\alpha \eta^{n-1} \Re_0 \\
&<\eta^{n-1}\left[\alpha \Re_0+\frac{\left(\omega_1 \eta+\omega_2 \eta^2\right) \Re_0}{1-\omega_3 \eta^2}\right] . \qquad (2.7)
\end{aligned}$$

Since $\lim_{n \rightarrow 0} \Re\left(c_n, c_{n+k}\right)=0$ for all $k>0$, we get $\{c_n\}$ is Cauchy in $(M, \Re)$. By the completeness of $(M, \Re)$ there exists $c^{\star} \in M$ such that $c_n \rightarrow c^{\star}$. Now, we show that $c^{\star}$ is a fixed point of T. Observe that

$$\begin{aligned}
\Re\left(c^{\star}, T c^{\star}\right) &\leq \omega_1 \Re\left(c^{\star}, c_n\right)+\omega_2 \Re\left(c_n, c_{n+1}\right)+\omega_3 \Re\left(c_{n+1}, T c^{\star}\right) \\
&=\omega_1 \Re\left(c^{\star}, c_n\right)+\omega_2 \Re_n+\omega_3 \Re\left(T c_n, T c^{\star}\right) \\
&\leq \omega_1 \Re\left(c^{\star}, c_n\right)+\omega_2 \Re_n+\omega_3 \lambda \Re\left(c_n, c^{\star}\right)+\omega_3 \mu\left[\Re(c_n, T c_n)+\Re\left(c^{\star}, T c^{\star}\right)\right] \\
&\leq \omega_1 \Re\left(c^{\star}, c_n\right)+\omega_2 \Re_n+\omega_3 \lambda \Re\left(c_n, c^{\star}\right)+\omega_3 \mu \Re\left(c_n, c_{n+1}\right)+\omega_3 \mu \Re(c^{\star}, T c^{\star}).
\end{aligned}$$

That is,

$$\left(1-\omega_3 \mu\right) \Re\left(c^{\star}, T c^{\star}\right) \leq\left(\omega_1+\omega_3 \lambda\right) \Re\left(c^{\star}, c_n\right)+\left(\omega_2+\omega_3 \mu\right) \Re_n . \qquad (2.8)$$

Taking the limit as $n \rightarrow \infty$ in (2), we obtain $\Re\left(c^{\star}, T c^{\star}\right)=0$. Therefore, $c^{\star}$ is a fixed point of T in M. Finally, we prove the uniqueness of the fixed point of T. To do so, assume that $c^{\star\star} \in M$ is another fixed point of T. Observe that

$$\begin{aligned}\Re(c^\star, c^{\star\star}) &= \Re(Tc^\star, Tc^{\star\star}) \\ &\leq \lambda\Re(c^\star, c^{\star\star}) + \mu[\Re(c^\star, Tc^\star) + \Re(c^{\star\star}, Tc^{\star\star})] \\ &= \lambda\Re(c^\star, c^{\star\star}) + \mu\Re(c^\star, c^\star) + \mu\Re(c^{\star\star}, c^{\star\star}) \\ &= \lambda\Re(c^\star, c^{\star\star}) \\ &= 0.\end{aligned}$$

Hence, $c^\star = c^{\star\star}$ proving the uniqueness. This ends the proof. □

3. Illustrative Examples

This section deals with the following example to highlight the realized improvements accomplished via our proved results

EXAMPLE 3.1. *Let $M = [0, 1]$ and define $\Re(c, d)$ by*

$$\Re(c, d) = \begin{cases} e^{|c-d|} & c \neq d, \\ 0, & c = d. \end{cases}$$

$$\begin{aligned}\Re(c, d) &= e^{|c-u+u-v+v-d|} \\ &\leq e^{|c-u|+|u-v|+|v-d|} \\ &= e^{\frac{1}{2}|c-u|+\frac{2}{5}|u-v|+\frac{1}{10}|v-d|}e^{\frac{1}{2}|c-u|+\frac{3}{5}|u-v|+\frac{9}{10}|v-d|} \\ &\leq \left(\frac{1}{2}e^{|c-u|} + \frac{2}{5}e^{|u-v|} + \frac{1}{10}e^{|v-d|}\right) \sup_M e^{\frac{1}{2}|c-u|+\frac{3}{5}|u-v|+\frac{9}{10}|v-d|} \qquad (3.1) \\ &= \left(\frac{1}{2}e^{|c-u|} + \frac{2}{5}e^{|u-v|} + \frac{1}{10}e^{|v-d|}\right) e^2 \\ &= \frac{e^2}{2}e^{|c-u|} + \frac{2e^2}{5}e^{|u-v|} + \frac{e^2}{10}e^{|v-d|} \\ &= \omega_1\Re(c, u) + \omega_2\Re(u, v) + \omega_3\Re(v, d). \qquad (3.2)\end{aligned}$$

Then $\Re(c, d)$ is a weighted rectangular b-metric and in (3.1) we have used Jenkin's inequality [9]. Let $T : M \to M$ be defined by

$$T(c) = \begin{cases} \frac{7c}{20} & \text{for } 0 \leq c \leq \frac{1}{2} \\ \frac{3c}{10} & \text{for } \frac{1}{2} < c \leq 1 \end{cases}$$

then T has a unique fixed point.

- *for* $c = \frac{1}{2}, d = 0, 1,$ *we have*

$$\begin{aligned}\Re\left(T\frac{1}{2}, T0\right) &= e^{\frac{7}{40}} \\ &\leq \frac{1}{2}\Re\left(\frac{1}{2}, 0\right) + \frac{1}{3}\Re\left(\frac{1}{2}, T\frac{1}{2}\right) + \frac{1}{4}\Re(0, T0) \\ &= \frac{1}{2}e^{\frac{1}{2}} + \frac{1}{3}e^{\frac{13}{40}} + 0.\end{aligned}$$

Hence,we can see that the condition of Theorem 2.1 is satisfied with $\lambda = \frac{1}{2}, \mu = \frac{1}{3}, \delta = 0$. *Then* T *has a unique fixed point* $c = 0$. *But does not satisfy condition (1.1) and condition (1.2) .Indeed, for* $c = \frac{1}{2}, d = 0,$ *we have the condition (1.2)*

$$\begin{aligned}\Re\left(T\frac{1}{2}, T0\right) &= e^{\frac{7}{40}} \\ &\nleq \frac{1}{3}\left[\Re\left(\frac{1}{2}, T\frac{1}{2}\right) + \Re\left(0, T0\right)\right] \\ &= \frac{1}{3}e^{\frac{13}{40}}.\end{aligned}$$

- *for* $c = \frac{1}{2}, d = 1,$ *we have*

$$\begin{aligned}\Re\left(T\frac{1}{2}, T1\right) &= e^{\frac{1}{8}} \\ &\leq \frac{1}{3}\left[\Re\left(\frac{1}{2}, T1\right) + \Re\left(1, T\frac{1}{2}\right)\right] \\ &= \frac{1}{3}\left[e^{\frac{1}{5}} + e^{\frac{33}{40}}\right].\end{aligned}$$

Hence, we can see that the condition of Theorem 2.2 is satisfied for $\lambda = \frac{1}{3}$. *But does not satisfy condition (1.1) and condition (1.2).*

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