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Time-frequency analysis associated with the Hartley-Wigner localization operators

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Abstract. The main crux of this paper is to introduce a new integral transform called the Hartley-Wigner transform and to give some new results related to this transform. Next, we introduce a new class of pseudo-differential operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ called localization operator which depends on a symbol σ and two admissible functions ψ_1 and ψ_2, we give criteria in terms of the symbol σ for its boundedness and compactness, we also show that these operators belong to the Schatten-Von Neumann class S^p for all $p \in [1, +\infty]$ and we give a trace formula.

1. Introduction

The Wigner transform has a long story which started in 1932 with Eugene Wigner's as a probability quasi-distribution which allows the expression of quantum mechanical expectation values in the same form as the averages of classical statistical mechanics. It is also used in signal processing as a transform in time-frequency analysis, for more information one can see [8, 15]. The Hartley transform

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is a linear operator defined for a suitable function $\psi(x)$ as follows:

$$\mathcal{H}(\psi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) \operatorname{cas}(\lambda x) dx, \tag{1.1}$$

where $\operatorname{cas}(x)$ is the cas function, defined as:

$$\operatorname{cas}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{n!} x^n, \tag{1.2}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$ being the binomial coefficient. The $\operatorname{cas}(x)$ function (1.2) can be seen as a generalization of the exponential function exp. A simple computation shows that the cas function is the unique C^∞ solution of the following differential-reflection problem, see [2]

$$\begin{cases} R\partial_x u(x) = \lambda u(x), \\ u(0) = 0. \end{cases}$$

Here, ∂_x represents the first-order derivative, and R is the reflection operator acting on functions $f(x)$ as:

$$R(f)(x) = f(-x). \tag{1.3}$$

Furthermore, the function $cas(x)$ is multiplicative on $\mathbb{R}$ in the sense

$$\operatorname{cas}(x)\operatorname{cas}(y) = \frac{1}{2}(\operatorname{cas}(x+y) - \operatorname{cas}(-x-y) + \operatorname{cas}(x-y) + \operatorname{cas}(y-x)). \tag{1.4}$$

Inspired by the relation (1.4), the author in [2] generalized the relation (1.4) for the Hartley-Bessel function and introduce a generalized convolution product. This paper focuses on the generalized Hartley transform introduced in [2, 3] called the Hartley-Bessel transform, more precisely we consider the following differential-reflection operator Δ_α defined by

$$\Delta_\alpha = R\left(\partial_x + \frac{\alpha}{x}\right) + \frac{\alpha}{x}, \quad \alpha \geq 0. \tag{1.5}$$

where R is the reflection operator given by the relation (1.3). The operator Δ_α is closely connected with the Dunkl's theory [9], furthermore the eigenfunctions of this operator are related to Bessel functions and they satisfies a product formula which permits to develop a new harmonic analysis associated with this operator see [2] for more information.

One of the aims of the Fourier transform is the study of the theory of localization operators called also Gabor multipliers, Toeplitz operators or Anti-Wick operators, Daubechies initiated this theory in [7], developed and detailed in the book [17] by Wong. Wong was the first one who defined the localization operators on the Weyl Heisenberg group in [16], Boggiatto and Wong extended these results on $L^p\left(\mathbb{R}^d\right)$ in [1]. The theory of localization operators associated with the Fourier-Wigner transform on hypergroups has been studied and known for remarkable development for example in the spherical mean hypergroups [13], in the Dunkl hypergroup [12], we have also generalized this theory in the Laguerre-Bessel hypergroups [6]. However, up to our knowledge, the localization operators for Wigner transform have not been studied for the operator Δ_α [5]. The main purpose of this paper is twofold on the one hand we introduce the Fourier-Wigner transform associated with the operator Δ_α [5] and we give some new results related to this transform on the other hand we introduce the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ associated with this transform and we give a criteria in terms of the symbol σ for its boundedness and compactness, we also show that these operators belongs to the Schatten-Von Neumann classes S^p, for all $p \in [1; +\infty]$ and we give a trace formula.

The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the operator Δ_α [5] and in Schatten-Von Neumann classes, in section 3 we will study the boundedness, compactness and the Schatten properties of the localization operator associated with the generalized Hartley-Wigner transform.

2. Harmonic Analysis Associated with the Hartley-Bessel Transform

In this section we recall some results in harmonic analysis related to the Hartley-Bessel transform, for more details we refer the reader to [2].

- For $\alpha \geq 0$, μ_α is the weighted Lebesgue measure defined on $\mathbb{R}$ by

$$d\mu_\alpha(x) := \frac{|x|^{2\alpha}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}dx,$$

where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{R}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}$, satisfying

$$\|f\|_{p,\mu_\alpha} =: \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty. \end{cases}$$

2.1. The Eigenfunctions of the Differential-reflection operator Δ_α.

For $\lambda \in \mathbb{C}$, we consider the following Cauchy problem

$$(S) : \begin{cases} \Delta_\alpha(u)(x) = \lambda u(x), \\ u(0) = 1. \end{cases}$$

From [2], the Cauchy problem (S) admits a unique solution $B_\alpha(\lambda.)$ given by

$$B_\alpha(\lambda x) = j_{\alpha-\frac{1}{2}}(\lambda x) + \frac{\lambda x}{2\alpha+1} j_{\alpha+\frac{1}{2}}(\lambda x), \tag{2.1}$$

where j_α denotes the normalized Bessel function of order α see [2]. The function $B_\alpha(\lambda.)$ is infinitely differentiable on $\mathbb{R}$ and we have the following important result

$$|B_\alpha(\lambda x)| \leq \sqrt{2} \qquad (\lambda, x \in \mathbb{R}). \tag{2.2}$$

Furthermore from [2], the Hartley-Bessel kernel (2.1) is multiplicative on $\mathbb{R}$ in the sense

$$B_\alpha(\lambda x)B_\alpha(\lambda y) = \int_{\mathbb{R}} B_\alpha(\lambda z)K_\alpha(x,y,z)d\mu_\alpha(z) \tag{2.3}$$

for all $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^*$, where K_α is the Bessel kernel given explicitly in [2]. The product formula (2.3) generalizes the relation (1.4) and permits to define of a translation operator, and convolution product and to develop of a new harmonic analysis associated with the Differential-reflection operator Δ_α.

2.2. The Hartley-Bessel transform.

Definition 2.1. [2] The Hartley-Bessel transform $\mathcal{H}_\alpha$ defined on $L^1_\alpha(\mathbb{R})$ by

$$\mathcal{H}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x)f(x)d\mu_\alpha(x), \qquad \text{for } \lambda \in \mathbb{R}$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [2].

Proposition 2.1.
(1) For every $f \in L^1_\alpha(\mathbb{R})$ we have

$$\|\mathcal{H}_\alpha(f)\|_{\infty,\mu_\alpha} \leq \sqrt{2}\|f\|_{1,\mu_\alpha}. \tag{2.4}$$

(2)(Inversion formula) For $f \in (L^1_\alpha \cap L^2_\alpha)(\mathbb{R})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$ we have

$$f(x) = \int_{\mathbb{R}} B_\alpha(\lambda x)\mathcal{H}_\alpha(f)(\lambda)d\mu_\alpha(\lambda), \qquad a.e \quad x \in \mathbb{R}. \tag{2.5}$$

(3) (Plancherel theorem) The Hartley-Bessel transform $\mathcal{H}_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{R})$ into $L^2_\alpha(\mathbb{R})$. and we have

$$\|f\|_{2,\mu\alpha} = \|\mathcal{H}_\alpha(f)\|_{2,\mu\alpha}. \tag{2.6}$$

2.3. The translation operator associated with the Hartley-Bessel transform.

The product formula (2.3) permits to define the translation operator as follows

Definition 2.2. Let $x, y \in \mathbb{R}$ and f is a measurable function on $\mathbb{R}$ the translation operator is defined by

$$\mathcal{T}^x_\alpha f(y) = \int_{\mathbb{R}} f(z)K_\alpha(x,y,z)d\mu_\alpha(z), \tag{2.7}$$

The following proposition summarizes some properties of the Hartley-Bessel translation operator see [2].

Proposition 2.2. *For all $x, y \in \mathbb{R}$, we have:*
(1)

$$\tau^x_\alpha f(y) = \tau^y_\alpha f(x). \tag{2.8}$$

(2)

$$\int_{\mathbb{K}} \mathcal{T}_\alpha^x f(y) d\mu_\alpha(y) = \int_{\mathbb{R}} f(y) d\mu_\alpha(y). \tag{2.9}$$

(3) for $f \in L^p_\alpha(\mathbb{R})$ *with* $p \in [1;+\infty]$, $\mathcal{T}_\alpha^x f \in L^p_\alpha(\mathbb{R})$ *and we have*

$$\|\mathcal{T}_\alpha^x f\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}. \tag{2.10}$$

By using the translation, we define the generalized convolution product of f, g *by*

$$(f *_\alpha g)(x,t) = \int_{\mathbb{R}} \mathcal{T}_\alpha^x (f)(y) g(y) d\mu_\alpha(y).$$

This convolution is commutative, associative and its satisfies the following properties see [2].

Proposition 2.3.
(1)(Young's inequality) for all $p, q, r \in [1;+\infty]$ *such that:* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ *and for all* $f \in L^p_\alpha(\mathbb{R}), g \in L^q_\alpha(\mathbb{R})$ *the function* $f *_\alpha g$ *belongs to the space* $L^r_\alpha(\mathbb{R})$ *and we have*

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha} \|g\|_{q,\mu_\alpha} \tag{2.11}$$

(2) For $f, g \in L^2_\alpha(\mathbb{R})$ *the function* $f *_\alpha g$ *belongs to* $L^2_\alpha(\mathbb{R})$ *if and only if the function* $\mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g)$ *belongs to* $L^2_\alpha(\mathbb{R})$ *and in this case we have*

$$\mathcal{H}_\alpha(f *_\alpha g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g). \tag{2.12}$$

(3) For all $f, g \in L^2_\alpha(\mathbb{R})$ *then we have*

$$\int_{\mathbb{R}} |f *_\alpha g(x,t)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}} |\mathcal{H}_\alpha(f)(\lambda)|^2 |\mathcal{H}_\alpha(g)(\lambda)|^2 d\mu_\alpha(\lambda), \tag{2.13}$$

where both integrals are simultaneously finite or infinite.

2.4. The Schatten-Von Neumann classes.

Notation: we denote by

- $l^p(\mathbb{N}), 1 \leq p \leq \infty$, *the set of all infinite sequences of real (or complex) numbers* $u := (u_j)_{j\in\mathbb{N}}$, *such that*

$$\|u\|_p := \begin{cases} \left(\sum_{j=1}^\infty |u_j|^p\right)^{\frac{1}{p}} < \infty, & if \quad 1 \leq p < \infty, \\ sup_{j\in\mathbb{N}} |u_j| < \infty, & if \quad p = +\infty. \end{cases}$$

- $B(L^p_\alpha(\mathbb{R})), 1 \leq p \leq \infty$, *the space of bounded operators from* $L^p_\alpha(\mathbb{R})$ *into itself. For* $p = 2$, *we define the space* $S_\infty := B(L^2_\alpha(\mathbb{R}))$, *equipped with the norm,*

$$\|A\|_{S_\infty} := \sup_{v\in L^2_A(\mathbb{R}):\|v\|_{2,\mu_\alpha}=1} \|Av\|_{2,\mu_\alpha}. \tag{2.14}$$

Definition 2.3.
(1) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_\alpha(\mathbb{R}))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.

(2) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$\|A\|_{S_p} := \left(\sum_{n=1}^{\infty} (s_n(A))^p \right)^{\frac{1}{p}}.$$

Remark 2.4. We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Definition 2.5. Let $(\phi_n)_n$ be an orthonormal basis of $L^2_\alpha(\mathbb{R})$ the trace of an operator A in S_1 is defined by

$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle A\phi_n, \phi_n \rangle_{\mu_\alpha}, \tag{2.15}$$

Remark 2.6. If A is positive, then

$$\operatorname{tr}(A) = \|A\|_{S_1}. \tag{2.16}$$

Moreover, a compact operator A on the Hilbert space $L^2_\alpha(\mathbb{R})$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = \operatorname{tr}(A^*A) = \sum_{n=1}^{\infty} \|A\phi_n\|^2_{2,\mu_\alpha}, \tag{2.17}$$

for any orthonormal basis $(\phi_n)_n$ of $L^2_\alpha(\mathbb{R})$.

2.5. Generalized Hartley-Wigner Transform.

The main purpose of this subsection is to introduce the generalized Fourier-Wigner transform associated with the Hartley-Bessel operator (1.5) and to give some new results related to this transform.

Notation: we denote by

- $\mathcal{S}_*(\mathbb{R}^2)$ the Schwartz space defined on $\mathbb{R}^2$ equipped with it s usual topology.
- $L^p_{\theta_\alpha}(\mathbb{R}^2)$,$1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R}^2$ satisfying

$$\|f\|_{p,\theta_\alpha} := \begin{cases} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x,\lambda)|^p \mathrm{d}\theta_\alpha(x,\lambda)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[, \\ \operatorname{ess\,sup}\limits_{(x,\lambda)\in\mathbb{R}^2} |f(x,\lambda)|, & \text{if } p = +\infty. \end{cases}$$

where θ_α is the measure defined on $\mathbb{R}^2$ by

$$\mathrm{d}\theta_\alpha(x,\lambda) := d\mu_\alpha(\lambda)) \otimes d\mu_\alpha(x)$$

Definition 2.7. The Fourier-Wigner transform associated with the operator Δ_A is defined on $\mathcal{S}_*(\mathbb{R}) \times \mathcal{S}_*(\mathbb{R})$ by

$$\mathcal{W}(f,g)(x,\lambda) := \int_{\mathbb{R}} f(y)\mathcal{T}^x_\alpha(g)(y)B_\alpha(\lambda y)d\mu_\alpha(y). \tag{2.18}$$

Remark 2.8. the transform $\mathcal{W}$ is a bilinear mapping from $\mathcal{S}_*(\mathbb{R}) \times \mathcal{S}_*(\mathbb{R})$ into $\mathcal{S}(\mathbb{R}^2)$ and can be written as

$$\begin{aligned} \mathcal{W}(f,g)(x,\lambda) &= \mathcal{H}_\alpha(f\tau_\alpha^x(g))(\lambda) && (2.19)\\ &= (g *_\alpha fB_\alpha(\lambda.))(x). && (2.20)\end{aligned}$$

We have the following results.

Proposition 2.4. *Let $f, g \in L^2_\alpha(\mathbb{R})$ then $\mathcal{W}(f,g)$ is well defined and belongs to $L^2_{\theta_\alpha}(\mathbb{R}^2) \cap L^\infty_{\theta_\alpha}(\mathbb{R}^2)$ and we have*

$$\|\mathcal{W}(f,g)\|_{2,\theta_\alpha} \le \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}, \tag{2.21}$$

and

$$\|\mathcal{W}(f,g)\|_{\infty,\theta_\alpha} \le \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}. \tag{2.22}$$

PROOF. Is a consequence of the Hölder's inequality and the relations (2.10) and (2.18). □

3. Localization operators Associated with the Generalized Hartley-Wigner transform

3.1. Introduction. *In this section, we introduce and give sufficient conditions for the boundedness, compactness and Schatten class properties of localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ associated with the generalized Hartley-Wigner transform in terms of properties of the symbol σ and the functions ψ_1 and ψ_2.*

Definition 3.1. Let ψ_1 and ψ_2 be measurable functions on $\mathbb{R}$, σ be a measurable function on the set $\mathbb{R}^2$, we define the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ associated with the generalized Dunkl-Wigner transform by

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)(y) := \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\mathcal{W}(f,\psi_1)(x,\lambda)B_\alpha(\lambda y)\overline{\mathcal{T}_\alpha^x(\psi_2)(y)}\mathrm{d}\theta_\alpha(x,\lambda). \tag{3.1}$$

Remark 3.2. In accordance with the different choices of the symbol σ and the different continuities required, we need to impose different conditions on ψ_1 and ψ_2, and then we obtain an operator on $L^p_\alpha(\mathbb{R})$ for all $1 \le p \le +\infty$. It is more convenient to interpret the definition of $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ in a weak sense, that is for all $f \in L^p_\alpha(\mathbb{R}), g \in L^q_\alpha(\mathbb{R})$ we have

$$\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_\alpha} = \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\mathcal{W}(f,\psi_1)(x,\lambda)\overline{\mathcal{W}(g,\psi_2)(x,\lambda)}\mathrm{d}\theta_\alpha(x,\lambda). \tag{3.2}$$

We have the following result.

Proposition 3.1. *Let $1 \le p \le +\infty$, the adjoint of the linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^p_\alpha(\mathbb{R}) \longrightarrow L^p_\alpha(\mathbb{R})$$

is the operator

$$\mathcal{L}^*_{\psi_1,\psi_2}(\sigma)\; L^{p'}_\alpha(\mathbb{R}) \longrightarrow L^{p'}_\alpha(\mathbb{R}),$$

where

$$\mathcal{L}^*_{\psi_1,\psi_2}(\sigma) = \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}). \tag{3.3}$$

Proof. Let $f \in L^p_\alpha(\mathbb{R}), g \in L^{p'}_\alpha(\mathbb{R})$ by using the relation (3.2), we have

$$\begin{aligned}\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_\alpha} =& \overline{\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\sigma(x,\lambda)}\mathcal{W}(g,\psi_2)(x,\lambda)\overline{\mathcal{W}(f,\psi_1)(x,\lambda)}d\theta_\alpha(x,\lambda)}\\ =&\overline{\langle \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}))(g) \mid f\rangle_{\mu_\alpha}} = \langle f \mid \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}))(g)\rangle_{\mu_\alpha},\end{aligned}$$

we get

$$\mathcal{L}^*_{\psi_1,\psi_2}(\sigma) = \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma}).$$

□

In the sequel of this section, ψ_1 and ψ_2 will be any functions in $L^2_\alpha(\mathbb{R})$ such that $\|\psi_1\|_{2,\alpha} = \|\psi_2\|_{2,\alpha} = 1$. We note that this hypothesis is not essential and the result still true up some constant depending on $\|\psi_1\|_{2,\alpha}$ and $\|\psi_2\|_{2,\alpha}$.

3.2. Boundedness for $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ in S_∞. *The main purpose of this subsection is to prove that the linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R})$$

is bounded for all symbol $\sigma \in L^p_{\theta_\alpha}(\mathbb{R}^2)$ with $1 \leq p + \infty$. We consider first the problem for $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$, next $\sigma \in L^\infty_{\theta_\alpha}(\mathbb{R}^2)$ and we conclude by using interpolation theory.

Proposition 3.2. *Let $\sigma \in L^2_\theta(\mathbb{R}^2)$ then the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is in S_∞ and we have*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{1,\theta_\alpha}. \tag{3.4}$$

Proof. Let $f, g \in L^2_\alpha(\mathbb{R})$ by using the relation (3.2) we have

$$| \langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_\alpha} | \leq \|\mathcal{W}(f,\psi_1)\|_{\infty,\theta}\|\mathcal{W}(g,\psi_2)\|_{\infty,\theta}\|\sigma\|_{1,\theta_\alpha},$$

by using the relation (2.22), we get

$$\left|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_\alpha}\right| \leq \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}\|\sigma\|_{1,\theta_\alpha},$$

by (2.14), we find that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{1,\theta_\alpha}.$$

□

Proposition 3.3. *Let $\sigma \in L^\infty_{\theta_\alpha}(\mathbb{R}^2)$, the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is in S_∞ and we have*

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta_\alpha}. \tag{3.5}$$

Proof. Let $f, g \in L^2_\alpha(\mathbb{R})$ by using the relation (3.2) and Hölder's inequality we find that

$$|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_\alpha}| \leq \|\sigma\|_{\infty,\theta_\alpha}\|\mathcal{W}(f,\psi_1)\|_{2,\theta}\|\mathcal{W}(g,\psi_2)\|_{2,\theta_\alpha},$$

by using the relation (2.21), we get

$$|\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) \mid g\rangle_{\mu_\alpha}| \leq \|\sigma\|_{\infty,\theta_\alpha}\|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha},$$

thus

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta_\alpha}.$$

□

We can now associate a localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ *to every symbol* σ *in* $L^p_{\theta_\alpha}(\mathbb{R}^2)$, *for all* $1 \leq p \leq +\infty$, *and prove that* $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ *belongs to* S_∞.

Theorem 3.4. *Let* $\sigma \in L^p_\theta(\mathbb{R}^2)$,$1 \leq p \leq +\infty$ *then there exists a unique bounded linear operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R})$$

such that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{p,\theta_\alpha}. \tag{3.6}$$

Proof. Let $\sigma \in L^p_\theta(\mathbb{R}^2), 1 \leq p \leq +\infty$ and $f \in L^2_\alpha(\mathbb{R})$ we consider the following operator

$$T : L^1_{\theta_\alpha}(\mathbb{R}^2) \cap L^\infty_{\theta_\alpha}(\mathbb{R}^2) \longrightarrow L^2_\alpha(\mathbb{R}),$$

given by

$$T(\sigma) = \mathcal{L}_{\psi_1,\psi_2}(\sigma)(f),$$

then by using the relations (3.4) and (3.5), we have

$$\|T(\sigma)\|_{2,\alpha} \leq \|f\|_{2,\mu_\alpha}\|\sigma\|_{1,\theta_\alpha} \tag{3.7}$$

and

$$\|T(\sigma)\|_{2,\alpha} \leq \|f\|_{2,\mu_\alpha}\|\sigma\|_{\infty,\theta_\alpha}, \tag{3.8}$$

by using the relations (3.7), (3.8) and the Riesz-Thorin interpolation Theorem see [**14, 17**], the operator T may be uniquely extended to a linear operator on $L^p_{\theta_\alpha}(\mathbb{R}^2)$, for all $1 \leq p \leq +\infty$ and we have

$$\|T(\sigma)\|_{2,\alpha} = \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)\|_{2,\mu_\alpha} \leq \|f\|_{2,\mu_\alpha}\|\sigma\|_{p,\theta_\alpha}, \tag{3.9}$$

since (3.6) is true for all $f \in L^2_\alpha(\mathbb{R})$, which gives the desired result. □

3.3. L^p_α-Boundedness of localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.

Using Schur's technique [10] our main purpose of this subsection is to prove that the linear operator

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^p_\alpha(\mathbb{R}) \longrightarrow L^p_\alpha(\mathbb{R}),$$

is bounded for all $1 \leq p \leq +\infty$, we have the following result.

Theorem 3.5. *Let $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$ and ψ_1, $\psi_2 \in L^1_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ then the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ extend to a unique bounded linear operator from $L^p_\alpha(\mathbb{R})$ into itself for all $1 \leq p \leq +\infty$, furthermore we have*

$$\|\mathcal{L}_{u,v}(\sigma)\|_{B\left(L^p_\alpha(\mathbb{R})\right)} \leq \max(\|\psi_1\|_{\infty,\alpha}\|\psi_2\|_{1,\alpha}\|\sigma\|_{1,\theta_\alpha}, \|\psi_1\|_{1,\alpha}\|\psi_2\|_{\infty,\alpha}\|\sigma\|_{1,\theta_\alpha}).$$

Proof. Let F be the function defined on $\mathbb{R}^2$ by

$$F(y,s) = \int_\mathbb{R}\int_\mathbb{R} \sigma(x,\lambda)\varphi_\lambda(y)\overline{\mathcal{T}^x_\alpha(v)(y)}B_\alpha(\lambda s)\mathcal{T}^x_\alpha(u)(s)\mathrm{d}\theta_\alpha(x,\lambda),$$

by using Fubini's theorem we find that

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f)(y) = \int_\mathbb{R} F(y,s)f(s)d\mu_\alpha(s),$$

furthermore by using the relation (2.8) and Fubini's theorem we find that

$$\int_\mathbb{R} |F(y,s)|d\mu_\alpha(y) \leq \|\psi_1\|_{\infty,\mu_\alpha}\|\psi_2\|_{1,\mu_\alpha}\|\sigma\|_{1,\theta_\alpha} \tag{3.10}$$

and

$$\int_\mathbb{R} |F(y,s)|d\mu_\alpha(s) \leq \|\psi_1\|_{1,\mu_\alpha}\|\psi_2\|_{\infty,\mu_\alpha}\|\sigma\|_{1,\theta_\alpha} \tag{3.11}$$

by using the relations (3.10), (3.11) and Schur's lemma [10] we can conclude that the linear operator

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^p_\alpha(\mathbb{R}) \longrightarrow L^p_\alpha(\mathbb{R}),$$

is bounded for all $1 \leq p \leq +\infty$ and we have

$$\|\mathcal{L}_{u,v}(\sigma)\|_{B\left(L^p_\alpha(\mathbb{R})\right)} \leq \max(\|\psi_1\|_{\infty,\alpha}\|\psi_2\|_{1,\mu_\alpha}\|\sigma\|_{1,\theta_\alpha}, \|\psi_1\|_{1,\mu_\alpha}\|\psi_2\|_{\infty,\mu_\alpha}\|\sigma\|_{1,\theta_\alpha}).$$

□

3.4. Trace of the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.

The main result of this subsection is to prove that the localization operator

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R}),$$

is in the Schatten-Von Neumann class S^p for all $1 \leq p \leq +\infty$, firstly we have the following result

Theorem 3.6. *Let $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R})$$

is an Hilbert-Schmidt operator in particular it is compact and we have

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{HS} \leq 1 + \|\sigma\|^2_{1,\theta_\alpha}.$$

Proof. Let $(\phi_k)_k$ be an orthonormal basis of $L^2_\alpha(\mathbb{R})$, by using Fubini's theorem and the relations (2.19) and (3.2), we get

$$\begin{aligned} &\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(\phi_k)\|^2_{2,\alpha} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\mathcal{H}_\alpha\left(\phi_k \mathcal{T}^x_\alpha(\psi_1)\right)(\lambda)\overline{\mathcal{H}_\alpha\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(\phi_k)\,\mathcal{T}^x_\alpha(\psi_2)\right)(\lambda)}\mathrm{d}\theta_\alpha(x,\lambda), \end{aligned}$$

by using Fubini's theorem we find that

$$\begin{aligned} \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|^2_{HS} \leq & \frac{1}{2}\int_{\mathbb{R}^2} |\sigma(x,\lambda)| \left[\sum_{k=1}^{+\infty} \left|\langle B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1) \mid \phi_k\rangle_{\mu_\alpha}\right|^2\right. \\ & \left. + \left|\left\langle \mathcal{L}_{\psi_2,\psi_1}(\bar{\sigma})\overline{\mathcal{T}^x_\alpha(\psi_2)B_\alpha(\lambda.)}|\phi_k\right\rangle_{\mu_\alpha}\right|^2\right] \mathrm{d}\theta_\alpha(x,\lambda). \end{aligned}$$

By using Parseval's identity, the relations (2.2), (2.10), (3.4) and the fact that $\|\psi_1\|_{2,\alpha} = \|\psi_2\|_{2,\alpha} = 1$ we find that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|^2_{HS} \leq \frac{1}{2}\|\sigma\|_{1,\theta}\left(1 + \|\sigma\|^2_{1,\theta_\alpha}\right) \leq \left(1 + \|\sigma\|^2_{1,\theta_\alpha}\right)^2 < \infty$$

which proves that $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is an Hilbert-Schmidt operator so compact and we have

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{HS} \leq 1 + \|\sigma\|^2_{1,\theta_\alpha}.$$

□

We have the following result:

Proposition 3.7. *Let $\sigma \in L^p_{\theta_\alpha}(\mathbb{R}^2), 1 \leq p < +\infty$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R})$$

is compact.

Proof. Let $\sigma \in L^p_{\theta_\alpha}(\mathbb{R}^2)$ with $1 \leq p < +\infty$ and let $(\sigma_n)_n$ be a sequence of functions in $L^1_{\theta_A}(\mathbb{R}^2) \cap L^\infty_{\theta_\alpha}(\mathbb{R}^2)$ such that $\sigma_n \longrightarrow \sigma$ in $L^p_{\theta_\alpha}(\mathbb{R}^2)$ as $n \longrightarrow \infty$ then by using the relation (33) we find that

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma_n) - \mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma_n - \sigma\|_{p,\theta_\alpha},$$

hence $\mathcal{L}_{\psi_1,\psi_2}(\sigma_n) \longrightarrow \mathcal{L}_{\psi_1,\psi_2}(\sigma)$ in S_∞ as $n \longrightarrow \infty$ on the other hand by Theorem 3.6, we have $\mathcal{L}_{\psi_1,\psi_2}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is compact. □

In the next theorem we obtain a L^1_α-compactness result for the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.

Theorem 3.8. *Let $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$, ψ_1 and ψ_2 in $L^1_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^1_\alpha(\mathbb{R}) \longrightarrow L^1_\alpha(\mathbb{R})$$

is compact.

Proof. By using Theorem 3.5 the linear operator

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^1_\alpha(\mathbb{R}) \longrightarrow L^1_\alpha(\mathbb{R})$$

is well defined, let $(f_n) \subset L^1_\alpha(\mathbb{R})$ such that $f_n \longrightarrow 0$ weakly in $L^1_\alpha(\mathbb{R})$ as $n \longrightarrow \infty$, it is enough to prove that $\lim_{n\to+\infty} \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(f_n)\|_{1,\mu_\alpha} = 0$. By using the relation (3.1), we have

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(f_n)\|_{1,\alpha} \leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |\sigma(x,\lambda)\, \|\mathcal{W}(f_n,\psi_1)(x,\lambda)\|\, \mathcal{T}^x_\alpha(\psi_2)(y)|\, \mathrm{d}\theta_\alpha(x,\lambda) \right] d\mu_\alpha(y). \tag{3.12}$$

Using the fact that $f_n \longrightarrow 0$ weakly in $L^1_\alpha(\mathbb{R})$ as $n \longrightarrow \infty$, we deduce that

$$\lim_{n\to+\infty} |\mathcal{W}(f_n,\psi_1)(x,\lambda)\|\mathcal{T}^x_\alpha(\psi_2)(y)| = 0, \tag{3.13}$$

for all $x, y, \lambda \in \mathbb{R}$, on the other hand as $f_n \longrightarrow 0$ weakly in $L^1_\alpha(\mathbb{R})$ as $n \longrightarrow \infty$, there exists a positive conctant c such that $\|f_n\|_{1,\alpha} \leq c$, so we find that

$$|\, \mathcal{W}(f_n,\psi_1)\,((x,\lambda)\, \|\mathcal{T}^x_\alpha(\psi_2)(y)| \leq c|\sigma(x,\lambda)\|\, \psi_1\|_{\infty,\mu_\alpha} |\psi_2(y)|, \tag{3.14}$$

by using Fubuni's theorem we get

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} |\sigma(x,\lambda)\, \|\mathcal{W}(f_n,\psi_1)(x,\lambda)\|\, \mathcal{T}^x_\alpha(\psi_2)(y)|\, \mathrm{d}\theta_\alpha(x,\lambda) \right] d\mu_\alpha(y) < \infty. \tag{3.15}$$

Thus, from the relations (3.12), (3.13), (3.14), (3.15) and the Lebesgue dominated convergence theorem we deduce that $\lim_{n\to+\infty} \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(f_n)\|_{1,\alpha} = 0$ and the proof is complete. ☐

In the following we show that the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is in the trace class S^1.

Theorem 3.9. *Let $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$ then the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R})$$

is in the trace class operators S_1 and we have

$$\|\tilde{\sigma}\|_{1,\theta_\alpha} \leq \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1} \leq \|\sigma\|_{1,\theta_\alpha} \tag{3.16}$$

where $\tilde{\sigma}$ is the function given by

$$\tilde{\sigma}(x,\lambda) = \left\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)\,(B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1)) \mid B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2) \right\rangle_{\mu_\alpha}.$$

Proof. Let $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$ by using Theorem 3.6, we have $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ is a compact operator, using [**17**], there exists an orthonormal basis ϕ_j for $j = 1, 2, \dots$ for the orthogonal complement of the kernel of the operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ consisting of eigenvectors of $|\mathcal{L}_{\psi_1,\psi_2}(\sigma)|$ and (h_j), $j = 1, 2, \dots$, an orthonormal set in $L^2_\alpha(\mathbb{R})$ such that the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ can be diagonalized as

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma)(f) = \sum_{j=1}^{+\infty} s_j \left\langle f \mid \phi_j \right\rangle_{\mu_\alpha} h_j, \tag{3.17}$$

where s_j's for $j = 1, 2, \dots$ are the positive singular values of $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ corresponding to ϕ_j, then we get

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} = \sum_{j=1}^{+\infty} s_j = \sum_{j=1}^{+\infty} \left\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\phi_j\right) \mid h_j \right\rangle_{\mu_\alpha},$$

by using the relations (3.1) and (3.2), we find that

$$\begin{aligned}\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} \le & \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(x,\lambda)| \left[\sum_{j=1}^{+\infty} \left|\left\langle B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1) \mid \phi_j\right\rangle_{\mu_\alpha}\right|^2 \right.\\ & \left. + \sum_{j=1}^{+\infty} \left|\left\langle B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2) \mid h_j\right\rangle_{\mu_\alpha}\right|^2 \right] \mathrm{d}\theta_\alpha(x,\lambda)\end{aligned}$$

by using Parseval's identity we get

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} \le \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(x,\lambda)| \|B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1)\ \|^2_{2,\mu_\alpha} + \|\ B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2)\|^2_{2,\mu_\alpha} \mathrm{d}\theta_\alpha(x,\lambda).$$

By using the relations (2.2), (2.10) and the fact that $\|\psi_1\|_{2,\alpha} = \|\psi_2\|_{2,\alpha} = 1$, we get

$$\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1} \le \|\sigma\|_{1,\theta_\alpha}.$$

Now we prove that $|\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ satisfies the first member of (3.16), it is easy to see that $\tilde{\sigma} \in L^1_{\theta_\alpha}(\mathbb{R}^2)$, using the relation (3.17) and Fubini's theorem, we find that

$$\begin{aligned} & \|\tilde{\sigma}\|_{1,\theta_\alpha} \\ \le & \frac{1}{2}\sum_{j=1}^{+\infty} s_j \left[\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\left|\left\langle B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1) \mid \phi_j\right\rangle_{\mu_\alpha}\right|^2 + \left|\left\langle h_j \mid B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2)\right\rangle_{\mu_\alpha}\right|^2\right) \mathrm{d}\theta_\alpha(x,\lambda)\right] \\ = & \frac{1}{2}\sum_{j=1}^{+\infty} s_j \left[\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{W}\left(\phi_j,\psi_1\right)(x,\lambda)\right|^2 + \left|\mathcal{W}\left(h_j,\psi_2\right)(x,\lambda)\right|^2\right] \mathrm{d}\theta_\alpha(x,\lambda), \end{aligned}$$

by using the relation (2.21) and the fact that $\|\psi_1\|_{2,\mu_\alpha} = \|\psi_2\|_{2,\mu_\alpha} = 1$ we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\tilde{\sigma}(x,\lambda)| d\theta_\alpha(x,\lambda) \le \frac{1}{2}\sum_{j=1}^{+\infty} s_j \left(\|\psi_1\|^2_{2,\alpha} + \|\psi_2\|^2_{2,\alpha}\right) = \sum_{j=1}^{+\infty} s_j = \|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1},$$

the proof is complete. $\square$

In the following we give a trace formula for the localization operators $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$.

Theorem 3.10. *Let* $\sigma \in L^1_{\theta_\alpha}(\mathbb{R}^2)$ *we have the following trace formula*

$$\operatorname{Tr}\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\left\langle B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1) \mid B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2)\right\rangle_{\mu_\alpha} \mathrm{d}\theta_\alpha(x,\lambda). \tag{3.18}$$

Proof. Let $\{\phi_j, j = 1, 2, \ldots\}$ be an orthonormal basis for $L^2_\alpha(\mathbb{R})$. From Theorem 3.5, the localization operator $\mathcal{L}_{\psi_1,\psi_2}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (2.15), Fubini's theorem and Parseval's identity, we get

$$\begin{aligned}\operatorname{Tr}\left(\mathcal{L}_{\psi_1,\psi_2}(\sigma)\right) &= \sum_{j=1}^{\infty} \left\langle \mathcal{L}_{\psi_1,\psi_2}(\sigma)\left(\phi_j\right), \phi_j\right\rangle_{\mu_\alpha} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda) \sum_{j=1}^{\infty} \langle \phi_j, \overline{B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1)}\rangle_{\mu_\alpha} \langle \overline{B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2)}, \phi_j\rangle_{\mu_\alpha} \mathrm{d}\theta_\alpha(x,\lambda) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\lambda)\left\langle B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_1) \mid B_\alpha(\lambda.)\mathcal{T}^x_\alpha(\psi_2)\right\rangle_{\mu_\alpha} \mathrm{d}\theta_\alpha(x,\lambda),\end{aligned}$$

and the proof is complete. $\square$

In the following we give the main result of this section.

Corollary 3.11. *Let* σ *in* $L^p_{\theta_A}(\mathbb{R}^2), 1 \leq p \leq +\infty$ *then, the localization operator*

$$\mathcal{L}_{\psi_1,\psi_2}(\sigma) : L^2_\alpha(\mathbb{R}) \longrightarrow L^2_\alpha(\mathbb{R})$$

is in S^p *and we have*

$$\left\|\mathcal{L}_{\psi_1,\psi_2}(\sigma)\right\|_{S_p} \leq \|\sigma\|_{p,\theta_\alpha}.$$

Proof. The result follows from (3.5) and (3.16) and by interpolation theory see [16]. $\square$

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