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Hat functions operational matrix for solving the nonlinear fractional-order integro-differential equation

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Abstract. This paper aims to construct a general formulation for the hat basis functions of the fractional integral operator. We derive the hat basis functions operational matrix of the fractional order integration, and use it to solve the fractional-order integro-differential equation. The method is described and illustrated with a numerical example. The results show that the method is accurate and easy to apply.

1. Introduction

Fractional differential and integral equations involving the Caputo fractional operator or the Riemann–Liouville fractional operator has been paid more and more attentions. There are several numerical methods for solving fractional integro-differential equations. Such as Haar wavelet method [**12**], CAS wavelets [**13**], collocation method [**14**], fractional differential transform method [**16**], block pulse operational matrix [**5**].

Integro-differential equation of fractional order has proven to be valuable tools to model the dynamics of many processes in various fields of science and engineering

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through strongly anomalous media. Indeed, we can find numerous applications in electro-chemistry, viscoelasticity, signal processing, economies, electromagnetic, etc [1, 2, 10].

Hat functions (HFs) are a powerful mathematical tool for solving various kinds of equations. Solution of stochastic Ito-Volterra integral equations based on stochastic operational matrix[3], E. Babolian et al. have applied this method for solving systems of nonlinear integral equations [4], M.H. Heydari et al. have applied hat functions for for solving nonlinear stochastic Ito integral equations [6]. F. Mirzaee and E. Hadadiyan used two-dimensional hat functions for solving space-time integral equations [7]. M. P. Tripathi et al. have applied hat functions for solving fractional differential equations [8].

The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Legendre polynomials [19], Laguerre series [20] and Block-pulse Functions [11]. The operational matrix of fractional derivatives has been determined for some types of orthogonal polynomials, such as Legendre polynomials [21], Chebyshev polynomials [22]. Triangular functions [9].

Consider the fractional-order integro-differential equation of the form

$$D^q y(x) = f(x) + \lambda_1 \int_0^x k_1(x,t,F(y(t)))dt + \lambda_2 \int_0^1 k_2(x,t,F(y(t)))dt, \quad x \in [a,b]. \tag{1}$$

With the supplementary conditions:

$$y^{(i)}(0) = \delta_i, \quad i = 0,1,\cdots,r-1, \quad \ni r-1 < \alpha \leq r, \quad r \in N \tag{2}$$

where $k \in L^2([0,1)^2)$, $f \in L^2([0,1))$ are known functions, D_*^α is the Caputo fractional differentiation operator of order α, and the unknown function $y(x)$ to be determined.

In this work, we consider that, the nonlinear function has the following form

$$F(y(t)) = (y(t))^p,$$

where p are positive integers.

In this paper, we introduce a new operational method to solve nonlinear Volterra-Fredholm integro-differential equation of fractional order. The method is based on reducing the equation to the system of algebraic equation by expanding the solution as hat functions.

2. Riemann-Liouville and Caputo fractional derivatives

There are various types of definition for the fractional derivative of order $q > 0$, the most commonly used definitions among various definitions of fractional derivatives of order $\alpha > 0$ are the Riemann-Liouville and Caputo formulas, ones which use fractional integrations and derivatives of the whole order. The difference between the two definitions is in the order of evaluation. Riemann-Liouville fractional integration of order α is defined as

$$I^{\alpha}_{x_0} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad x > 0. \tag{3}$$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively:

$$D^{\alpha}_{x_0} f(x) = \frac{d^m}{dx^m}[I^{m-\alpha}_{x_0} f(x)], \tag{4}$$

$$D^{\alpha}_{*x_0} f(x) = I^{m-\alpha}_{x_0}[\frac{d^m}{dx^m} f(x)], \tag{5}$$

where $m-1 \leq \alpha < m$ and $n \in N$. From (3) and (4), we have

$$D^{\alpha}_{x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f(t)dt, \quad x > x_0 \tag{6}$$

Lemma 1: If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, then $D^{\alpha}_{*} I^{\alpha} f(x) = f(x)$, and:

$$I^{\alpha} D^{\alpha}_{*} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \frac{x^k}{k!}, \quad x > 0.$$

3. Review of hat functions (HFs)

A set of hat functions (HFs) are usually defined on [0,1] as:

$$\phi_0(t) = \begin{cases} \dfrac{h-t}{h}, & 0 \leq t < h, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(t) = \begin{cases} \frac{t-(i-1)h}{h}), & (i-1)h \leq t < ih, \\ \frac{(i+1)h-t}{h}), & ih \leq t < (i+1)h, \quad i = 1, 2, \cdots, n-1. \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_n(t) = \begin{cases} \frac{t-(1-h}{h}), & T-h \leq t < T, \\ 0, & \text{otherwise}, \end{cases}$$

where $h = \frac{1}{n}$ and n is an arbitrary positive integer. Indeed, the unit interval [0,1] is divided into n equidistant subintervals. According to the definition of HFs:

$$\phi_i(jh) = \delta_{ij}, \tag{7}$$

where δ denotes the kronecker delta function and

$$\phi_i(t)\phi_j(t) = 0, \quad | \, i-j \, | \geq 2, \tag{8}$$

and

$$\sum_{i=0}^{n} \phi_i(t) = 1. \tag{9}$$

An arbitrary function $f \in C^2([0,1])$ can be expanded in vector form as:

$$f(t) \simeq F^T \Phi(t) = \Phi^T(t) F, \tag{10}$$

where $F = [f_0, f_1, \cdots, f_n]^T$, $\Phi(t) = [\phi_0(t), \phi_1(t), \cdots, \phi_n(t)]^T$ and

$$f_i = f(ih), \quad i = 0, 1, \cdots, n. \tag{11}$$

The positive integer powers of $f(x)$ may be approximated by HFs as follows:

$$[f(x)]^p \simeq F_p^T \cdot \Phi(t). \tag{12}$$

Now let $k(x, y)$ be an arbitrary function of two variables defined on $L^2([0,1] \times [0,1])$. It can be expanded by HFs as:

$$k(x, y) \simeq \Phi^T(x) K \Phi(y), \tag{13}$$

where $\Phi(x)$ and $\Phi(y)$ are (n_1+1) and (n_2+1) dimensional HFs vectors, respectively, and K is the $((n_1+1) \times (n_2+1))$ HFs coefficients matrix with entries $a_{ij}, i = 0, 1, \cdots, n_1, j = 0, 1, \cdots, n_2$ as follows:

$$a_{ij} = k(ih, jh). \tag{14}$$

In this paper, for convenience, we put $n_1 = n_2 = n$. Moreover, relation (8) follows:

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_0^2(t) & \phi_0(t)\phi_1(t) & & & \\ \phi_0(t)\phi_1(t) & \phi_1^2(t) & \phi_0(t)\phi_1(t) & & \\ & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \phi_{n-1}(t)\phi_n(t) \\ & & & \phi_{n-1}(t)\phi_n(t) & \phi_n^2(t) \end{pmatrix},$$

and

$$\int_0^1 \Phi(t)\Phi^T(t) dt = E_1, \tag{15}$$

where E_1 is the following $((n+1) \times (n+1))$ matrix

$$E_1 = \frac{h}{6} \begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}.$$

By considering (7) and expanding entries of $\Phi(t)\Phi^T(t)$ by HFs we have

$$\Phi(t)\Phi^T(t) \simeq diag(\Phi), \tag{16}$$

Now suppose that Λ is an (n+1)-vector by using (16) we obtain:

$$\Phi(t)\Phi^T(t)\Lambda \simeq \tilde{\Lambda}\Phi(t), \tag{17}$$

where $\tilde{\Lambda} = diag(\Lambda)$ is an $(n+1)\times(n+1)$ diagonal matrix. Furthermore, if A is an $(n+1)\times(n+1)-$matrix, we have

$$\Phi^T(t)A\Phi(t) \simeq \Phi^T(t)\hat{A}, \tag{18}$$

where $\hat{A}$ is an $(n+1)$-vector with elements equal to diagonal entries of matrix A. Now we have

$$\int_0^x \Phi(t)\Phi^T(t)dt = E_2, \tag{19}$$

where E_2 is the following $((n+1)\times(n+1))$ matrix

$$E_2 = \frac{h}{6}\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & 2 & \cdot & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

3.1. Operational matrix of the fractional order integration. Our purpose is to derive the Hat operational matrix of the fractional order integration. For this purpose, we used the Riemann-Liouville fractional order integration, as following

$$(I^\alpha y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * y(t), \tag{20}$$

where $\alpha \in R$ is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * y(t)$ denotes the convolution product of $t^{\alpha-1}$ and $y(t)$. Now if $y(t)$ is expanded in Hat functions, as shown in Eq.(10), the Riemann-Liouville fractional integration becomes

$$(I^\alpha y)(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * y(t) \approx C^T\frac{1}{\Gamma(\alpha)}(t^{\alpha-1} * \Phi(t)). \tag{21}$$

Thus if $t^{\alpha-1} * y(t)$ can be integrated, then expanded in Hat functions, the Riemann-Liouville fractional order integration is solved the Hat functions. Also, we define an m-set of Block Pulse Function (BPF) as

$$b_i(x) = \begin{cases} 1, & \frac{i}{m} \le x < \frac{(i+1)}{m}, \\ 0, & otherwise, \end{cases}$$

where $i = 0, 1, 2, \cdots, (m-1)$. The function $b_i(t)$ are disjoint and orthogonal. That is

$$b_i(x)b_j(x) = \begin{cases} b_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{22}$$

The Hat functions can be expanded in to m-set of block-pulse Functions as

$$\Phi(x) = \Psi_{m\times m}B_m(x), \tag{23}$$

where $B_m(x) = (b_0(x), b_1(x), \ldots, b_i(x), \ldots, b_{m-1}(x))^T$, [12, 13], and Ψ is an $MN \times MN$ product operational matrix. Kilicman and Al Zhour (see[17]) with

$$(I^\alpha B_m)(x) \simeq F^\alpha B_m(x), \tag{24}$$

$$F^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \ldots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \ldots & \xi_{m-1} \\ 0 & 0 & 1 & \xi_1 & \ldots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, \tag{25}$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. Next, we derive the Hat operational matrix of the fractional order integration. Let

$$(I^\alpha \Phi)(x) \approx P^\alpha_{m\times m}\Phi(x), \tag{26}$$

where matrix $P^\alpha_{m\times m}$ is called the Hat operational matrix of the fractional order integration. Using Eqs.(23), and (24), we have

$$(I^\alpha \Phi)(x) \approx (I^\alpha \Psi_{m\times m} B_m)(x) = \Psi_{m\times m}(I^\alpha B_m)(x) \approx \Psi_{m\times m} F^\alpha B_m(x). \tag{27}$$

By Eqs.(26) and (27) we get

$$P^\alpha_{m\times m}\Phi(x) = P^\alpha_{m\times m}\Psi_{m\times m} B_m(x) = \Psi_{m\times m} F^\alpha B_m(x). \tag{28}$$

Then, the Hat operational matrix of the fractional order integration $P^\alpha_{m\times m}$ is given by

$$P^\alpha_{m\times m} = \Psi_{m\times m} F^\alpha \Psi^{-1}_{m\times m}, \tag{29}$$

4. Solving fractional-order integro–differential equation

Consider the nonlinear Volterra Fredholm fractional integro-differential equation of the second kind given by Eq. (1). If we approximate the function $f(x)$ with the second kind Hat functions, it can be writhen as

$$f(x) \simeq f^T \Phi(x). \tag{30}$$

Now, let:

$$D^\alpha_* y(x) \simeq C^T \Phi(x). \tag{31}$$

By using Eqs.(31) and (26) and Lemma 1.1, we have:

$$f(x) \cong C^T P^\alpha_{m\times m}\Phi(x) + \sum_{k=0}^{m-1} f^k(0^+)\frac{x^k}{k!}. \tag{32}$$

Hence, by substituting the supplementary initial conditions (2), in the above summation in the above equations and approximating it by Hat functions, we have:

$$f(x) \cong (c^T P^\alpha_{m\times m} + c_1^T)\Phi(x), \tag{33}$$

where c_1 is a column m-vector. Define:

$$e = [e_0, e_1, \cdots, e_{m-1}] = (c^T P^{\alpha}_{m\times m} + c_1^T),$$

so, $f(x) \cong eB(x)$. We have:

$$\begin{aligned}(f(x))^2 \cong (eB(t))^2 &= (e_0 b_0(x), e_1 b_1(x), \cdots, e_{m-1} b_{m-1}(x))^2 \\ &= e_0^2 b_0(x), e_1^2 b_1(x), \cdots, e_{m-1}^2 b_{m-1}(x) = \tilde{e}_2 B(x)\end{aligned}$$

where $\tilde{e}_2 = (e_0^2, e_1^2, \cdots, e_{m-1}^2)$, we get

$$[f(x)]^q \cong [e_0^q, e_1^q, \cdots, e_{m-1}^q] B(t) = \tilde{e}_q B(t). \tag{34}$$

5. Applying the method

Now considering the following fractional-Volterra Fredholm integro-differential

$$D^q y(x) = f(x) + \lambda_1 \int_0^x k_1(x, t, y(t))dt + \lambda_2 \int_0^1 k_2(x, t, y(t))dt. \tag{35}$$

Using Eqs.(17), (18), (26), (28), (30) and (31), we have

$$\begin{aligned}C^T\Phi(x) &= f^T\Phi(x) + \int_0^x \Phi^T(x) K_1 \Phi(t)\Phi^T(t) e_q^T dt + \int_0^1 \Phi^T(x) K_2 \Phi(t)\Phi^T(t) e_q^T dt \\ &= f^T\Phi(x) + \Phi^T(x) K_1 \int_0^x \Phi(t)\Phi^T(t) e_q^T dt + \Phi^T(x) K \int_0^1 \Phi(t)\Phi^T(t) e_q^T dt \\ &= f^T\Phi(x) + \Phi^T(x) K_1 \int_0^x \tilde{e}_q^T \Phi(t) dt + \Phi^T(x) K_2 E_1 e_q^T \\ &= f^T\Phi(x) + \Phi^T(x) K_1 \tilde{e}_q P^{\alpha}_{m\times m} \Phi(x) + \Phi^T(x) K_2 E_1 e_q^T \\ &= f^T\Phi(x) + \hat{B_1}^T \Phi(x) + \hat{B_2}^T \Phi(x),\end{aligned}$$

where $\hat{B}_1 = \hat{B}_2$ is an m-vector with elements equal to the diagonal entries of the following matrix:

$$\hat{B}_1 = \hat{B}_2 = K diag(\tilde{e}_q) P^1_{m\times m} \tag{36}$$

we obtain:

$$C^T\Phi(x) = f^T\Phi(x) + \hat{B_1}^T \Phi(x) + \hat{B_2}^T \Phi(x), \tag{37}$$

hence, we have

$$C^T = f^T + \hat{B_1}^T + \hat{B_2}^T, \tag{38}$$

which is a system of algebraic equations. By solving this system we can obtain the approximate solution of Eq. (1) according to Eq. (32).

6. Numerical examples

To illustrate the effectiveness of the proposed method in the present paper, two test examples are carried out in this section.

Example 6.1. Consider the fourth-order, nonlinear fractional integro-differential equation [18]

$$D^{\alpha}y(x) = 1 + \int_0^x e^{-t}y^2(t)dt, \quad 0 < x < 1, \quad 3 < \alpha \leq 4 \tag{39}$$

subject to boundary conditions

$$y(0) = 1, \quad y^{''}(0) = 1, \quad y(1) = e, \quad y^{''}(1) = e. \tag{40}$$

When $\alpha = 4$ the exact solution is known and given by $y(x) = e^x$. To solve this example, we implement the TFs method for $\alpha = 3.25$ and $\alpha = 3.75$ Numerical results is presented in Table (1).

Example 6.2. Consider the linear fourth-order fractional integro-differential equation [16]

$$D^{\alpha}y(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)dt, \quad 0 < x < 1, \quad 3 < \alpha \leq 4, \tag{41}$$

with the following boundary conditions:

$$y(0) = 1, \quad y^{''}(0) = 2, \quad y(1) = 1 + e, \quad y^{''}(1) = 3e. \tag{42}$$

In the case $\alpha = 4$, the exact solution is known and it is given by $y(x) = 1 + xe^x$. To solve this example, for $\alpha = 3.25$ with $m = 20$ and for $\alpha = 3.75$ with $m = 18$. Numerical results for this solution is presented in Table (2)

Example 6.3. As the second example, we consider the following linear fourth-order fractional integro-differential equation [15]

$$D^{0.5}u(t) = g(t)u(t) + h(t) + \sqrt{t}\int_0^t u^2(x)dx, \tag{43}$$

where

$$g(t) = 2\sqrt{t} + 2t^{\frac{3}{2}} - (\sqrt{t} + t^{\frac{3}{2}})\ln(1 + t), \quad h(t) = \frac{2\,\mathrm{arcsin\,h}(\sqrt{t})}{\sqrt{\pi}\sqrt{1 + t}} - 2t^{\frac{3}{2}} \tag{44}$$

with the initial condition

$$u(0) = 0. \tag{45}$$

The exact solution, is $u(t) = \ln(1 + t)$. Figs. 3 illustrate the absolute errors for $m = 12, m = 32$ and $m = 64$ respectively.

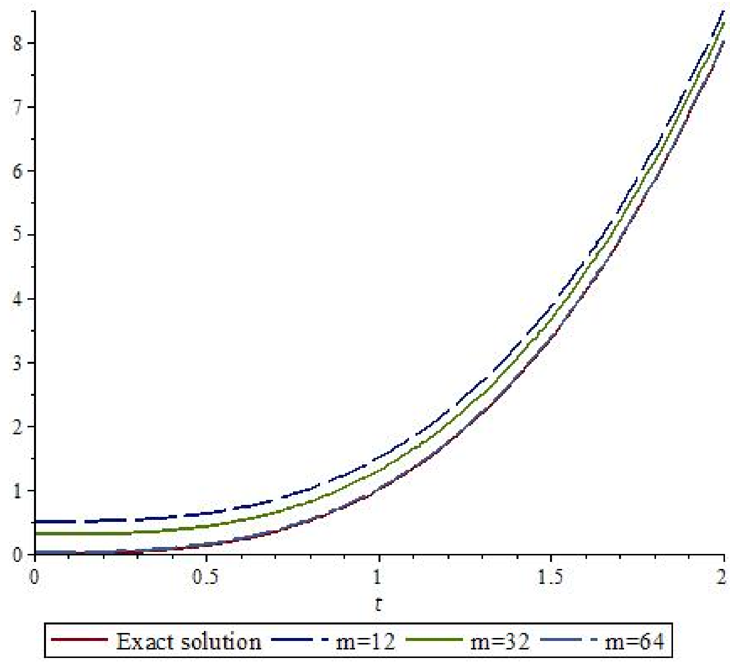

FIGURE 1. Comparison of the exact solution and the present method for Example 3.

TABLE 1. Numerical results for Example 1

	$\alpha = 3.25$	$m = 20$	$\alpha = 3.75$	$m = 16$
x	u_{ADM}	u_{HF}	u_{ADM}	u_{HF}
0.0	1.000000	1.000000	1.000000	1.0000000
0.1	1.106551	1.106245	1.106151	1.106242
0.2	1.223931	1.223687	1.223127	1.223125
0.3	1.353200	1.353225	1.352308	1.353020
0.4	1.495 600	1.495872	1.494636	1.495620
0.5	1.652553	1.652325	1.651615	1.651258
0.6	1.825654	1.825325	1.824823	1.824821
0.7	2.016687	2.016254	2.016023	2.015054
0.8	2.227634	2.227254	2.227176	2.227214
0.9	2.460691	2.460212	2.460458	2.460421
1	2.718281	2.718215	2.718281	2.718325

7. Conclusion

In this paper a Hat operational matrix of fractional order integration is obtained and is used to solve fractional integro-differential equation. A general procedure of forming this matrix $P^{\alpha}_{m\times m}$ is summarized. For more investigation, some examples

TABLE 2. Numerical results for Example 2

	$\alpha = 3.25$	$m = 20$	$\alpha = 3.75$	$m = 18$
x	u_{FDT}	u_{HF}	u_{FDT}	u_{HF}
0.0	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.1202485	1.1262265	1.1155715	1.1152654
0.2	1.2624209	1.2732584	1.2537181	1.2542154
0.3	1.4291559	1.4442258	1.4177572	1.4182142
0.4	1.6242571	1.6425132	1.6115127	1.6121051
0.5	1.8531719	1.8452311	1.8397280	1.8403245
0.6	2.1202510	2.1202154	2.1077151	2.1082514
0.7	2.4321620	2.4324521	2.4211566	2.4221524
0.8	2.7972914	2.8065481	2.7892142	2.7892152
0.9	3.2221499	3.2274125	3.2189011	3.2112151
1	3.7182818	3.7182818	3.7181818	3.7181812

have presented. As the numerical results showed, the proposed method is accurate and effective method to solve the fractional integro-differential equations.

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