Mathematical Analysis and its Contemporary Applications
Volume 6, Issue 4, 2024, 45–57
doi: 10.30495/maca.2025.2051385.1124
ISSN 2716-9898

The study of cases of fixed points of a complex function depending on the inputs and concluding examples of useful functions

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Abstract. This work aims to propose mathematical procedures that help to deduce the fixed points of a complex function. These fixed points can also be used as a basis to generate new useful complex functions that have for sure at least one fixed point. Hence, this is a good article for specialists in calculus and analysis and even for beginners in mathematics who want to improve their skills in the field of complex numbers.

1. Introduction

After my thesis on mathematical physics that discusses the classical use of discrete mathematics in Newtonian Mechanics [1], we chose to focus on subjects of pure mathematics and especially calculus and analysis [2, 3] or the number theory [4, 5] by writing my articles that respect all the rules of classical mathematics. This work aims to study two cases of fixed points of a complex function by using a complex number S that respects that $S = \frac{1}{2} + ib$, where b is a strictly positive real number. The article proposes also two examples of concluded complex functions that have for sure at least one fixed point. The choice of the number S in its form as an example

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2020 *Mathematics Subject Classification.* Primary: 32Hxx; Secondary: 32H50.

Key words and phrases. complex analysis, complex function, uniqueness, fixed-point, implications, Landau-Siegel Zero

in this proof is because we are trying to make the example of this article more useful and more attractive since the zeros of the Riemann Hypothesis are supposed to have the same form which is $S = \frac{1}{2} + ib$. The purpose is that this article proposes new useful mathematical procedures about complex functions and gives thanks to the concluded examples of functions a probable hint that may help to identify or deny for example possible Landau-Siegel Zeros [6].

These are the considerations and notations of this article:

Let S be a complex number that respects that $S = \frac{1}{2} + ib$, where b is a strictly positive real number. We can also write S as:

$$S = \frac{1}{2} + i\frac{1}{\varepsilon} = \frac{1}{2} + i\frac{\tan(\theta)}{2},$$

where $b = \frac{\tan(\theta)}{2} = \frac{1}{\varepsilon}$. In this case, we should only consider that $\frac{1}{\varepsilon} > 0$, $0 < \theta < \frac{\pi}{2}$, and we have

$$S = \frac{e^{i\theta}}{2\cos(\theta)}.$$

We have also:

$$\theta = \arctan(\frac{2}{\varepsilon}) = \frac{\pi}{2} - \arctan(\frac{\varepsilon}{2}),$$

$$\cos(\theta) = \cos(\arctan(\frac{2}{\varepsilon})) = \frac{\varepsilon}{\sqrt{\varepsilon^2 + 4}},$$

and

$$\sin(\theta) = \sin(\arctan(\frac{2}{\varepsilon})) = \frac{2}{\sqrt{\varepsilon^2 + 4}}.$$

Consequently,

$$\sin(\theta) \times \cos(\theta) = \frac{2\varepsilon}{\varepsilon^2 + 4}.$$

Let's consider also a function f of complex numbers z defined from $\mathbb{C}$ in $\mathbb{C}$ as:

$$f(z) = \frac{z^{S+1}}{S}.$$

We will use the complex logarithm $\ln(z)$ point by point during all this proof.

2. The considered complex function and its fixed-points

Theorem 2.1. *$f(z) = z$ if and only if $z^S = S$ if and only if $\frac{e^{i\theta}}{2\cos(\theta)} = z^{\frac{e^{i\theta}}{2\cos(\theta)}}$.*

Proof. We can start with a remark about two cases:

Case 1: $z = r$, where r is a positive real. In this case, we have, $r^S = S$ if and only if $r^{\frac{1}{2}+ib} = \frac{1}{2} + ib$ if and only if

$$\sqrt{r}(\cos(b\ln(r)) + i\sin(b\ln(r))) = \frac{1}{2} + ib. \tag{1}$$

This means that $\sqrt{r}\cos(b\ln(r)) = \frac{1}{2}$, and so

$$\sqrt{r}\sin(b\ln(r)) = b. \tag{2}$$

This implies that

$$\cos(b\ln(r))^2 + \sin(b\ln(r))^2 = 1 = \frac{1+4b^2}{4r}. \tag{3}$$

Hence, we get

$$r = \frac{1+4b^2}{4}. \tag{4}$$

Let us test a specific case of this fixed point with $b = 1$. We have in this case $r = \frac{5}{4}$, but we remark that

$$\begin{aligned} r^S &= \left(\frac{5}{4}\right)^{\frac{1}{2}+i} \\ &= \left(\frac{5}{4}\right)^{\frac{1}{2}} \exp\left(\ln\left(\frac{5}{4}\right) i\right) \\ &= \left(\frac{5}{4}\right)^{\frac{1}{2}} \left(\cos\left(\ln\left(\frac{5}{4}\right)\right) + i\sin\left(\ln\left(\frac{5}{4}\right)\right)\right) \\ &\neq \frac{1}{2} + i. \end{aligned} \tag{5}$$

Consequently, this means that the positive real fixed point r that has in this case the form $r = \frac{1+4b^2}{4}$ may exist for the function $f(r)$ or may not exist. The reality is that this fixed point r may also have other forms and restrictions at the same time with the form $r = \frac{1+4b^2}{4}$ but may also not exist at all because the result of these calculations of case 1 is built on an implication by the use of the trigonometric formula, $\cos(b\ln(r))^2 + \sin(b\ln(r))^2 = 1$, and not only equivalences. This means that we still should make a mathematical investigation for case 1.

Case 2: $z = r$, where r is a negative real. In this case, we have again $r^S = S$ if and only if $r^{\frac{1}{2}+ib} = \frac{1}{2} + ib$. Since r is negative, we have:

$$(\pm i)e^{-b\pi}\sqrt{|r|}(\cos(b\ln(|r|)) + i\sin(b\ln(|r|))) = \frac{1}{2} + ib. \tag{6}$$

This means that $\cos(b\ln(|r|)) = \frac{\pm b e^{b\pi}}{\sqrt{|r|}}$ and that

$$\sin(b\ln(|r|)) = \frac{\pm e^{b\pi}}{2\sqrt{|r|}}. \tag{7}$$

This implies that

$$\cos(b\ln(|r|))^2 + \sin(b\ln(|r|))^2 = 1 = e^{2b\pi}\frac{1+4b^2}{4|r|}. \tag{8}$$

Hence, by (8), we get

$$r = -e^{2b\pi}\frac{1+4b^2}{4}. \tag{9}$$

However, like for Case 1, our demonstration means only that the negative real fixed point r that may exist for the function $f(r)$ has, in this case, the form $r = -e^{2b\pi\frac{1+4b^2}{4}}$. The reality is that this fixed point r may have other forms and restrictions at the same time and may also not exist at all because the result of these calculations of Case 1 is built on an implication by the use of the trigonometric formula: $\cos(b\ln(|r|))^2 + \sin(b\ln(|r|))^2 = 1$, and not only equivalences.

This means that we still should make a mathematical investigation for the Case 2. Thus, we have,

$$z^S = S \quad \text{if and only if} \quad e^{S\ln(z)} = \frac{e^{i\theta}}{2\cos(\theta)}. \tag{10}$$

By (10), we have

$$\ln(S) = \ln(z^S) \quad \text{if and only if} \quad \ln(\frac{e^{i\theta}}{2\cos(\theta)}) = \frac{e^{i\theta}}{2\cos(\theta)} \times \ln(z). \tag{11}$$

Then (11) implies that

$$\ln(z) = \frac{2\cos(\theta)(\ln(e^{i\theta}) - \ln(2\cos(\theta)))}{e^{i\theta}}. \tag{12}$$

Consequently,

$$\frac{\ln(z)}{2} = (\cos(\theta) - i\sin(\theta))\cos(\theta)(\ln(e^{i\theta}) - \ln(2\cos(\theta))). \tag{13}$$

Thus,

$$\frac{\ln(z)}{2} = \left(\cos(\theta)^2 - i\frac{\sin(2\theta)}{2}\right)(i\theta - \ln(\cos(\theta)) - \ln(2)). \tag{14}$$

We consider $\ln(e^{i\theta}) = i\theta + 2k\pi i = i\theta$ with $k = 0$. Because we have one unique complex number $\ln(z)$, for each $e^{i\theta}$ and for each $\cos(\theta)$ with $0 < \theta < \frac{\pi}{2}$, hence,

$$\begin{aligned}\frac{\ln(z)}{2} = &\left(-\cos(\theta)^2(\ln(\cos(\theta)) + \ln(2)) + \frac{\theta\sin(2\theta)}{2}\right)\\ &+ i\left(\theta\cos(\theta)^2 + \frac{\sin(2\theta)\ln(\cos(\theta))}{2} + \frac{\sin(2\theta)\ln(2)}{2}\right).\end{aligned} \tag{15}$$

By using the formulas of the introduction, we get

$$\begin{aligned}\frac{\ln(z)}{2} = &\left(\frac{-\varepsilon^2}{\varepsilon^2+4}\left(\ln\left(\frac{\varepsilon}{\sqrt{\varepsilon^2+4}}\right)\right) + \ln(2)\right) + \arctan\left(\frac{2}{\varepsilon}\right)\frac{2\varepsilon}{\varepsilon^2+4})\\ &+ i\left(\arctan\left(\frac{2}{\varepsilon}\right)\frac{\varepsilon^2}{\varepsilon^2+4} + \frac{2\varepsilon}{\varepsilon^2+4}\ln\left(\frac{\varepsilon}{\sqrt{\varepsilon^2+4}}\right) + \frac{2\varepsilon}{\varepsilon^2+4}\ln(2)\right).\end{aligned} \tag{16}$$

Consequently by (16), we have,

$$\begin{aligned}\frac{\ln(z)}{2} =& \frac{-\varepsilon^2}{\varepsilon^2+4}\left(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2)\right)+\arctan\left(\frac{2}{\varepsilon}\right)\frac{2\varepsilon}{\varepsilon^2+4}\\ &+i\left(\arctan\left(\frac{2}{\varepsilon}\right)\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\ln(2)\right).\end{aligned} \tag{17}$$

□

3. First investigation: If z is a real positive number

The number z can not be equal to zero because zero is not a solution to the equation $z^S = S$. Since

$$e^{S\ln(z)} = e^{S\times\ln(z)+i2k\pi} = e^{S(\ln(z)+i\frac{2k\pi}{S})} = \frac{e^{i\theta}}{2\cos(\theta)},$$

for all $k \in \mathbb{Z}$, we have k numbers that respect $z^S = S$, such that these numbers are $(\ln(z)+i\frac{2k\pi}{S})$. We can notice that $\ln(z)$ is the only real number among these k numbers, when z is a real number and this corresponds to $k = 0$. Hence, when z is a real number, we have only one unique real number $\ln(z)$, for each θ with $0 < \theta < \frac{\pi}{2}$. If z is a real strictly positive number, then $\Im(\frac{\ln(z)}{2}) = 0$. Hence, we have

$$\varepsilon\arctan(\frac{2}{\varepsilon}) = -2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2). \tag{18}$$

Thus, equation (9) becomes

$$\frac{\ln(z)}{2} = \frac{-\varepsilon^2}{\varepsilon^2+4}(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\frac{2\varepsilon}{\varepsilon^2+4}. \tag{19}$$

Hence,

$$\frac{\ln(z)}{2} = \frac{-\varepsilon^2}{\varepsilon^2+4}(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+(-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2))\frac{2\varepsilon}{\varepsilon^2+4}. \tag{20}$$

Consequently,

$$\frac{\ln(z)}{2} = \frac{-\varepsilon}{\varepsilon^2+4}(\varepsilon(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\frac{4}{\varepsilon}(\ln(\varepsilon)-\ln\frac{(\varepsilon^2+4)}{2}+\ln(2))). \tag{21}$$

Thus,

$$\frac{\ln(z)}{2} = \frac{-\varepsilon}{\varepsilon^2+4}(\varepsilon+\frac{4}{\varepsilon})(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2)). \tag{22}$$

We conclude finally that

$$\frac{\ln(z)}{2} = -\ln(\varepsilon)+\frac{\ln(\varepsilon^2+4)}{2}-\ln(2), \tag{23}$$

which is equivalent to

$$\ln(z) = \ln(\frac{\varepsilon^2+4}{4\varepsilon^2}). \tag{24}$$

Moreover, $\ln(z) = \ln(\frac{\varepsilon^2+4}{4\varepsilon^2})$ if and only if $z = \frac{\varepsilon^2+4}{4\varepsilon^2}$ if and only if $z = \frac{1}{4} + \frac{1}{\varepsilon^2}$. We proved also that

$$\varepsilon \arctan(\frac{2}{\varepsilon}) = -2\ln(\varepsilon) + \ln(\varepsilon^2+4) - 2\ln(2).$$

This means that $\varepsilon\theta = \ln(\frac{\varepsilon^2+4}{4\varepsilon^2})$. Hence, $e^{\varepsilon\theta} = \frac{1}{4} + \frac{1}{\varepsilon^2}$. Then we conclude that $z = e^{\varepsilon\theta} = e^{\varepsilon \arctan(\frac{2}{\varepsilon})} = e^{\frac{2\theta}{\tan(\theta)}} = e^{\frac{\arctan(2b)}{b}}$. We can also remark that $e^{\varepsilon\times\theta} = \frac{1}{4} + \frac{1}{\varepsilon^2}$ if and only if $\frac{\varepsilon^2}{\varepsilon^2 e^{\varepsilon\theta}-1} = 4$. This means that $\frac{\varepsilon^2}{(\varepsilon e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})})^2-1} = 4$, and we know that the maximum of $\frac{\arctan(x)}{x}$ is 1. Hence, $\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon}) < 1$, and hence, $e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})} < e$.

Finally, we conclude that we should have $\varepsilon \geq \frac{1}{e}$ which is equivalent to $\tan(\theta) \leqslant 2e$. Otherwise, we will have a contradiction in the equation $\frac{\varepsilon^2}{(\varepsilon e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})})^2-1} = 4$, because 4 can not be equal to any negative value.

4. Second investigation: If z is a strictly negative real number

Since,

$$e^{S\ln(z)} = e^{S\ln(z)+i2k\pi} = e^{S(\ln(z)+i\frac{2k\pi}{S})} = \frac{e^{i\theta}}{2\cos(\theta)},$$

for all $k \in \mathbb{Z}$, we have k numbers that respect $z^S = S$ and these numbers are $(\ln(z) + i\frac{2k\pi}{S})$. If z is a strictly negative number then $\ln(z)$ is a complex number. Then, there exists $k' \in \mathbb{Z}$ with $\ln(z) = \ln(-z) + i\pi + i2k'\pi$. Hence,

$$z^S = e^{S(\ln(z)+i\frac{2k\pi}{S})} = e^{S(\ln(-z)+i\pi+i2k'\pi+i\frac{2k\pi}{S})}.$$

Thus, we should have $e^{i2k'\pi S+i2k\pi} = 1$. This implies that there exist $k'', k''' \in \mathbb{Z}$ such that $k'\pi S + k\pi = k'''\pi$. Since b is a strictly positive number, then we have, $k'\pi S = k'\pi(\frac{1}{2} + ib) = 0$. Hence, we have $k' = 0$.

Finally, we conclude that we should investigate about the unique complex number $\ln(z)$ that corresponds to each θ with $0 < \theta < \frac{\pi}{2}$ and that respects $\ln(z) = \ln(-z) + i\pi$.

In this case, we use $\Im(\frac{\ln(z)}{2}) = \frac{\pi}{2}$, and we get

$$\arctan(\frac{2}{\varepsilon})\frac{\varepsilon^2}{\varepsilon^2+4} + \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) + \frac{2\varepsilon}{\varepsilon^2+4}\ln(2) = \frac{\pi}{2}.$$

Hence,

$$\arctan\left(\frac{2}{\varepsilon}\right)\frac{\varepsilon^2}{\varepsilon^2+4} = \frac{\pi}{2} - \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) - \frac{2\varepsilon}{\varepsilon^2+4}\ln(2).$$

Consequently,

$$\arctan\left(\frac{2}{\varepsilon}\right)\frac{\varepsilon^2}{\varepsilon^2+4} = \frac{\pi}{2} - \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) - \frac{2\varepsilon}{\varepsilon^2+4}\ln(2).$$

Thus,

$$\arctan\left(\frac{2}{\varepsilon}\right)\frac{2\varepsilon}{\varepsilon^2+4} = \frac{\pi}{\varepsilon} - \frac{2}{(\varepsilon^2+4)}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) - \frac{4}{\varepsilon^2+4}\ln(2).$$

This implies that

$$\theta\frac{2\varepsilon}{\varepsilon^2+4} = \frac{\pi}{\varepsilon} - \frac{2}{\varepsilon^2+4}\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})$$

which is equivalent to

$$\ln\left(\frac{4\varepsilon^2}{\varepsilon^2+4}\right) = \left(-\theta\frac{2\varepsilon}{\varepsilon^2+4} + \frac{\pi}{\varepsilon}\right)\frac{\varepsilon^2+4}{2} = -\varepsilon\theta + \frac{\varepsilon\pi}{2} + \frac{2\pi}{\varepsilon}.$$

Consequently, we get $\frac{4\varepsilon^2}{\varepsilon^2+4} = e^{-\varepsilon\theta}e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}$. Hence, $\frac{4\varepsilon^2}{e^{-\varepsilon\theta}e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}} = \varepsilon^2+4$ and thus

$$4 = \varepsilon^2\left(\frac{4}{e^{-\varepsilon\theta}e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}} - 1\right).$$

Hence, in order to avoid the contradiction, we have, $\frac{4}{e^{-\varepsilon\theta}e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}} > 1$ and this is equivalent to $\frac{(2e^{\frac{\varepsilon}{2}\theta})^2}{e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}} > 1$ if and only if $\frac{(2e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})})^2}{e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}} > 1$. This proves that $e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})} < e$. Thus, in order to avoid the contradiction, we have, $e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}} < (2e)^2$. Now, we check if this inequality is possible. We notice that $\frac{\varepsilon\pi}{2} + \frac{2\pi}{\varepsilon} - 4 = \frac{\varepsilon^2\pi-8\varepsilon+4\pi}{2\varepsilon}$ and that the discriminant of the quadratic equation $\varepsilon^2\pi - 8\varepsilon + 4\pi$ is negative. Hence, $\frac{\varepsilon^2\pi-8\varepsilon+4\pi}{2\varepsilon} > 0$, and thus $\frac{\varepsilon\pi}{2} + \frac{2\pi}{\varepsilon} > 4$.

Consequently, $e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}} > e^4$, and we have $e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}} < 4e^2$. However, $e^4 > 4e^2$, so we conclude that we can never have $e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}} < (2e)^2$ and this leads to the contradiction because we conclude that $\varepsilon^2\left(\frac{4}{e^{-\varepsilon\theta}e^{\frac{\varepsilon\pi}{2}}e^{\frac{2\pi}{\varepsilon}}} - 1\right)$ can not be equal to 4, and thus we have,

$$\arctan\left(\frac{2}{\varepsilon}\right)\frac{\varepsilon^2}{\varepsilon^2+4} + \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) + \frac{2\varepsilon}{\varepsilon^2+4}\ln(2) \neq \frac{\pi}{2}.$$

Finally, we conclude that if $z < 0$, then we have $z^S \neq S$. This result is correct despite the form $r = -e^{2b\pi\frac{1+4b^2}{4}}$ demonstrated above incase 2 for the probable negative real fixed points of the studied function f.

5. First conclusion

We have the following implications:

First of all, set $z^S = S$, for $z \in \mathbb{R}$ and $S = \frac{1}{2} + ib$. Thus, $z > 0$ and also $z^S = S$, for $z \in \mathbb{R}$ and $S = \frac{1}{2} + ib$ implies that

$$\begin{aligned} z =& e^{\varepsilon\theta} = e^{\varepsilon\times\arctan(\frac{2}{\varepsilon})} \\ =& e^{\frac{2\theta}{\tan(\theta)}} \\ =& e^{\frac{\arctan(2b)}{b}}, \end{aligned}$$

and we have also, $z^S = S$, for $z \in \mathbb{R}$ and $S = \frac{1}{2} + ib$. Thus, $b \leqslant e$.

The function f still may admit a fixed-point $z = r$ and we proved that z in this case can only be strictly positive, whenever z is a real number. However, we proved above in Case 1 the form of the probable positive real fixed points of the studied function f and it is $r = \frac{1}{4} + b^2$. This means that: $e^{\frac{\arctan(2b)}{b}} = \frac{1}{4} + b^2$. Thus, the strictly positive real number b can be the only solution to the equation

$$e^{\frac{\arctan(2b)}{b}} - b^2 = \frac{1}{4}.$$

Analytically of the curve of the equation implies $b \approx \pm 1,449$. But the real number b is strictly positive, then we have, $z^S = S$, where $z \in \mathbb{R}$ and $S = \frac{1}{2} + ib$. This implies that $b \approx 1,449$ and $z = r = e^{\frac{\arctan(2b)}{b}} \approx 2,323$.

6. First annex: Concluding a first example of complex functions with at least one fixed point

Let $u = v + iw \in \mathbb{C}$ and $S = \frac{1}{2} + iw$. We now consider a complex function H defined by $H(u) = H(v + iw) = f(v) + iw$. Hence, $H(u) = \frac{v^{S+1}}{S} + iw$. We can also generally express the considered function H as

$$H(v + iw) = \frac{v^{S+1}}{S} + iw = \frac{v^{\frac{3}{2}+iw}}{\frac{1}{2} + iw} + iw = \frac{v^{\frac{3}{2}+iw} - w + i\frac{w}{2}}{\frac{1}{2} + iw}.$$

We note that if we have, $u = v + iw = r + ib$, then $u = r + ib$ is a fixed point for the considered complex function H. We remind that in this case, $w = b \approx 1,449$ and $v = r = e^{\frac{\arctan(2b)}{b}} \approx 2,323$. Thus, $H(r + ib) = \frac{r^{S+1}}{S} + ib = r + ib$, because in this case, we have $S = \frac{1}{2} + ib$.

The considered complex function H, for which we concluded in this part that it has at least one fixed point can be very useful after the study of its derivability and integrability as a mapping since it gives, for example, a method to replace the complex number $S = \frac{1}{2} + iw$ that can not be a hypothetical Landau-Siegel zero with the term: $S = \frac{v^{\frac{3}{2}+iw} - w + i\frac{w}{2}}{H(v+iw)}$.

7. Third investigation: If $z = e^{iB}$ with B a real number and $B \neq k'\pi$, for all $k' \in \mathbb{Z}$

The statement $B \neq k'\pi$, for all $k' \in \mathbb{Z}$, means that $z \notin \mathbb{R}$. Then $z = e^{iB}$ implies that there exists $k \in \mathbb{Z}$ such that $\ln(z) = i \times (B + 2k\pi)$ with $z^S = S$. We proved that

$$\begin{aligned}\frac{\ln(z)}{2} =& \frac{-\varepsilon^2}{\varepsilon^2+4}\left(\ln(\varepsilon) - \frac{\ln(\varepsilon^2+4)}{2} + \ln(2)\right) + \arctan\left(\frac{2}{\varepsilon}\right)\frac{2\varepsilon}{\varepsilon^2+4} \\ & + i\left(\arctan\left(\frac{2}{\varepsilon}\right)\frac{\varepsilon^2}{\varepsilon^2+4} + \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) + \frac{2\varepsilon}{\varepsilon^2+4}\ln(2)\right).\end{aligned}$$

We know that if $z = e^{iB}$ then the real part of $\ln(z)$ is null. Hence, $\frac{\varepsilon^2}{\varepsilon^2+4}(\ln(\varepsilon) - \frac{\ln(\varepsilon^2+4)}{2} + \ln(2)) = \arctan(\frac{2}{\varepsilon})\frac{2\varepsilon}{\varepsilon^2+4}$. Consequently, $\frac{\varepsilon}{2}(\ln(\varepsilon) - \frac{\ln(\varepsilon^2+4)}{2} + \ln(2)) = \arctan(\frac{2}{\varepsilon})$. We can use this result in the imaginary part and we get,

$$\frac{\ln(z)}{2} = i(\arctan(\frac{2}{\varepsilon})\frac{\varepsilon^2}{\varepsilon^2+4} + \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) + \frac{2\varepsilon}{\varepsilon^2+4}\ln(2)).$$

Hence,

$$\begin{aligned}\frac{\ln(z)}{2} =& i\left(\frac{\varepsilon}{2}(\ln(\varepsilon) - \frac{\ln(\varepsilon^2+4)}{2} + \ln(2)\right)\frac{\varepsilon^2}{\varepsilon^2+4} \\ &+ \frac{\varepsilon}{\varepsilon^2+4}\left(2\ln(\varepsilon) - \ln(\varepsilon^2+4)) + \frac{2\varepsilon}{\varepsilon^2+4}\ln(2)\right).\end{aligned}$$

Consequently,

$$\frac{\ln(z)}{2} = i(\frac{\varepsilon}{4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4) + 2\ln(2))\frac{\varepsilon^2}{\varepsilon^2+4} + \frac{\varepsilon}{\varepsilon^2+4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4) + 2\ln(2))).$$

Thus,

$$\frac{\ln(z)}{2} = i(2\ln(\varepsilon) - \ln(\varepsilon^2+4) + 2\ln(2))\frac{\varepsilon}{\varepsilon^2+4}(1 + \frac{\varepsilon^2}{4}).$$

We conclude that

$$\frac{\ln(z)}{2} = i\frac{\varepsilon}{4}(2\ln(\varepsilon) - \ln(\varepsilon^2+4) + 2\ln(2)),$$

which means that $\frac{\ln(z)}{2} = i\frac{\varepsilon}{4}\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})$. We also proved that

$$\frac{\varepsilon}{2}(\ln(\varepsilon) - \frac{\ln(\varepsilon^2+4)}{2} + \ln(2)) = \arctan(\frac{2}{\varepsilon}) = \theta.$$

Hence, $\frac{\varepsilon}{4}\ln(\frac{4\varepsilon^2}{\varepsilon^2+4}) = \theta$. This implies that $\frac{\ln(z)}{2} = i\theta$. Hence, $z = e^{i2\theta}$, and from $\frac{\varepsilon}{4}\ln(\frac{4\varepsilon^2}{\varepsilon^2+4}) = \theta$, we conclude that $e^{4\theta} = (\frac{4\varepsilon^2}{\varepsilon^2+4})^{\varepsilon}$. We know that $\cos(\theta)^2 = \frac{\varepsilon^2}{\varepsilon^2+4}$. Hence, we have $4\cos(\theta)^2 = e^{\frac{4\theta}{\varepsilon}}$. Note that $0 < \theta < \frac{\pi}{2}$ implies that $1 < e^{4\frac{\theta}{\varepsilon}} < e^{2\frac{\pi}{\varepsilon}}$. Consequently, we have, $1 < 4\cos(\theta)^2 < e^{2\frac{\pi}{\varepsilon}}$. Since $0 < \theta < \frac{\pi}{2}$, we have $\cos(\theta)^2 > \frac{1}{4}$. This implies $\cos(\theta) > \frac{1}{2}$. We conclude that $\theta < \frac{\pi}{3}$ and since the function tan is a monotonic increasing function in $0 < \theta < \frac{\pi}{2}$, we get $b = \frac{\tan(\theta)}{2} < \frac{\tan(\frac{\pi}{3})}{2} = \frac{\sqrt{3}}{2}$. We have also, $\cos(\theta)^2 < \frac{e^{2\frac{\pi}{\varepsilon}}}{4}$, and this case should be also studied because one can have $\frac{e^{2\frac{\pi}{\varepsilon}}}{4} \leq 1$ if and only if $2\frac{\pi}{\varepsilon} \leq \ln(4)$ if and only if $b \leq \frac{\ln(2)}{\pi}$ such that $\frac{\ln(2)}{\pi} \approx 0,2206$. In this case, since $0 < \theta < \frac{\pi}{2}$, we have, $\theta > \arccos(\frac{e^{\frac{\pi}{\varepsilon}}}{2})$.

Hence, we have, $b = \frac{\tan(\theta)}{2} > \frac{\tan(\arccos(\frac{e^{\frac{\pi}{\varepsilon}}}{2}))}{2}$. Thus, $b > \frac{2\times\sqrt{1-\frac{e^{2\frac{\pi}{\varepsilon}}}{4}}}{e^{\frac{\pi}{\varepsilon}}}$. This means that $\frac{be^{b\pi}}{2} > \sqrt{1 - \frac{e^{2\pi b}}{4}}$. This implies that $\frac{b^2e^{2b\pi}}{4} + \frac{e^{2\pi b}}{4} > 1$, and this is equivalent to $\frac{e^{2b\pi}}{4}(b^2+1) > 1$. Since for any strictly positive real number x, the function $g(x) = \frac{e^{2x\pi}}{4}(x^2+1)$ is a monotonic increasing function, we can deduce analytically from the curve of the function g that $b > 0,214$.

8. Second conclusion

We know that $z \notin \mathbb{R}$ implies that $B \neq k'\pi$, for all $k' \in \mathbb{Z}$. Therefore, we could prove that for $S = \frac{1}{2} + ib$, there exists $B \in \mathbb{R}$ with $z = e^{iB}$, and $B \neq k'\pi$, for all $k' \in \mathbb{Z}$ such that $z^S = S$.

Thus, there exists $k \in \mathbb{Z}$ with $B = 2\theta + 2k\pi$, which is equivalent to there exists $k \in \mathbb{Z}$ with $B = 2\arctan(2b) + 2k\pi$, and we have also, $0,214 < b < \frac{\sqrt{3}}{2}$. Since the function f admits a fixed-point $z = e^{iB}$, we have also,

$$\begin{aligned} f(z) = z \quad &\text{if and only if} \quad S = z^S \\ &\text{if and only if} \quad \frac{1}{2} + ib = e^{S \times iB} \\ &\text{if and only if} \quad \frac{1}{2} + ib = e^{-Bb} \times e^{\frac{iB}{2}}, \end{aligned}$$

and we have,

$$e^{-Bb} e^{\frac{iB}{2}} = e^{-Bb}\left(\cos\left(\frac{B}{2}\right) + i\sin\left(\frac{B}{2}\right)\right).$$

Hence, we have, $e^{-Bb}\cos(\frac{B}{2}) = \frac{1}{2}$ and consequently, $e^{-Bb} \times \sin(\frac{B}{2}) = b$. Thus, $e^{-2Bb} = \frac{1}{4} + b^2$. Consequently, we have, $B = \frac{-\ln(\frac{1}{4}+b^2)}{2b}$. This means that there exists $k \in \mathbb{Z}$ with $2\arctan(2b) + 2k\pi = \frac{-\ln(\frac{1}{4}+b^2)}{2b}$, thus, the strictly positive real number b, can only be the solution of the equation:

$$\frac{\arctan(2b)}{\pi} + \frac{\ln(\frac{1}{4}+b^2)}{4b\pi} = -k.$$

Since we considered that the real number b is strictly positive, then this equation has analytically from the curve of the equation one solution for each integer k with $-k \leq 0$. But we proved that $0,214 < b < \frac{\sqrt{3}}{2}$, then we notice analytically from the curve of the equation that this equation has only one solution with $k = 0$ and $b \approx 0,371$. Consequently, $B = \frac{-\ln(\frac{1}{4}+b^2)}{2b} \approx 1,277$.

Finally, we conclude that if $S = \frac{1}{2} + ib$ and there exists $B \in \mathbb{R}$ such that $z = e^{iB}$ and $B \neq k'\pi$, for all $k' \in \mathbb{Z}$, and $z^S = S$, then we have, $b \approx 0,371$ and $B \approx 1,277$.

9. Second annex: Concluding a second example of complex functions with at least one fixed point

Let $u = v + iw$, where $u \in \mathbb{C}$ and $v, w \in \mathbb{R}$ with $-1 \leq w \leq 1$. Consider $S = \frac{1}{2} + i(v - \sqrt{1-w^2})$. We now consider a complex function G defined by

$$\begin{aligned} G(u) =& G(v + iw) \\ =& v - \sqrt{1-w^2} + f(e^{i\arcsin(w)}) \\ =& v - \sqrt{1-w^2} + \frac{e^{i\arcsin(w)(S+1)}}{S}. \end{aligned}$$

Hence,

$$G(u) = v - \sqrt{1-w^2} + \frac{e^{i\arcsin(w)(1+\frac{1}{2}+i(v-\sqrt{1-w^2}))}}{\frac{1}{2}+i(v-\sqrt{1-w^2})}.$$

This implies that

$$\begin{aligned} G(u) =& G(v+iw) \\ =& v - \sqrt{1-w^2} + \frac{e^{-\arcsin(w)(v-\sqrt{1-w^2})+\frac{i3}{2}\arcsin(w)}}{\frac{1}{2}+i(v-\sqrt{1-w^2})}. \end{aligned}$$

We can also generally express the considered function G as

$$\begin{aligned} G(u) =& G(v+iw) \\ =& \frac{\frac{v-\sqrt{1-w^2}}{2}+i(v-\sqrt{1-w^2})^2+e^{-\arcsin(w)(v-\sqrt{1-w^2})+\frac{i3}{2}\arcsin(w)}}{\frac{1}{2}+i(v-\sqrt{1-w^2})}. \end{aligned}$$

We remark that if we have, $u = v + i \times w = b + e^{iB} = b + \cos(B) + i\sin(B)$, then, $S = \frac{1}{2} + ib$ and we have,

$$G(b+\cos(B)+i\sin(B)) = b + \cos(B) - \sqrt{1-\sin(B)^2} + \frac{e^{i\arcsin(\sin(B))(S+1)}}{S}.$$

Hence,

$$\begin{aligned} b + \cos(B) - \sqrt{1-\sin(B)^2} + \frac{e^{i\arcsin(\sin(B))(S+1)}}{S} =& b + e^{iB} \\ =& b + \cos(B) + i\sin(B). \end{aligned}$$

We remind that in this case: $b \approx 0,371$ and $B \approx 1,277$. Hence, in this case, we have, $w = \sin(B) \approx 0,9571$ and $v = b + \cos(B) \approx 0,6605$. This means that $G(b+\cos(B)+i\sin(B)) = b+\cos(B)+i\sin(B)$. Thus, $u = b+\cos(B)+i\sin(B)$ is a fixed point for the considered complex function G.

The considered complex function G for which we concluded in this part that it has at least one fixed point can be very useful after the study of its derivability and integrability as a mapping since it gives for example a method to replace the complex number $S = \frac{1}{2} + i(v - \sqrt{1-w^2})$ that can not be a hypothetical Landau-Siegel zero with the term

$$S = \frac{\frac{v-\sqrt{1-w^2}}{2}+i(v-\sqrt{1-w^2})^2+e^{-\arcsin(w)(v-\sqrt{1-w^2})+\frac{3i}{2}\arcsin(w)}}{G(v+iw)},$$

where $-1 \leq w \leq 1$.

10. Summary and final conclusion

We considered in this work a complex number S that respects that $S = \frac{1}{2} + ib$, where b is a strictly positive real number. Also, we considered a function f defined from $\mathbb{C}$ into $\mathbb{C}$ by $f(z) = \frac{z^{S+1}}{S}$. We studied in this article in two cases the fixed points of this function which are the solution of the equation $z^S = S$.

Theorem 10.1. *The first case is when z is a real number and we demonstrated that the unique fixed point of the function f in this case is with $b \approx 1,449$ and $z = r = e^{\frac{\arctan(2b)}{b}} \approx 2,323$.*

Corollary 10.2. *We could consequently build an example of complex function H that has at least one fixed point that is $u = r + ib$ and the expression of this complex function is*

$$\begin{aligned} H(u) =& H(v+iw) \\ =& \frac{v^{S+1}}{S} + iw = \frac{v^{\frac{3}{2}+iw}}{\frac{1}{2}+iw} + iw \\ =& \frac{v^{\frac{3}{2}+iw} - w + i\frac{w}{2}}{\frac{1}{2}+iw}, \end{aligned}$$

where u is a complex number but v and w are real numbers.

Theorem 10.3. *The second case is when z is a complex number respecting there exists $B \in \mathbb{R}$ with $z = e^{iB}$ and $B \neq k'\pi$, for all $k' \in \mathbb{Z}$. We demonstrated that the unique fixed point of the function f in this case is with $b \approx 0,371$ and $B = \frac{-\ln(\frac{1}{4}+b^2)}{2b} \approx 1,277$. Hence, $z = e^{iB} \approx e^{1,277i}$.*

Corollary 10.4. *We could consequently build an example of complex function G that has at least one fixed point that is $u = v + iw = b + \cos(B) + i\sin(B)$ and the expression of this complex function is*

$$\begin{aligned} G(u) =& G(v+iw) \\ =& \frac{\frac{v-\sqrt{1-w^2}}{2} + i(v-\sqrt{1-w^2})^2 + e^{-\arcsin(w)(v-\sqrt{1-w^2})+\frac{i3}{2}\arcsin(w)}}{\frac{1}{2} + i(v-\sqrt{1-w^2})} \end{aligned}$$

where u is a complex number but v and w are real numbers with $-1 \leq w \leq 1$.

Thus, we conclude that the complex function $f(z) = \frac{z^{S+1}}{S}$ has indeed two special different fixed points depending on the nature of the complex inputs when $S = \frac{1}{2}+ib$. However, to know all the fixed points of this function when $S = \frac{1}{2} + ib$, we should study another case which needs a future longer proof. It is the case of: There exists $B \in \mathbb{R}$ with $z = xe^{iB}$ and for all $k' \in \mathbb{Z}, B \neq k'\pi$, where x is a real number with $x \neq 0$ and $x \neq 1$.

Furthermore, the concluded functions H and G are only examples of complex functions that have fixed points, which can be concluded from the complex statements of this article. However, any concluded complex function can be studied further regarding their derivability, integrability, and respect for Banach principles or mapping contractions. This study makes any concluded function from this article's complex statements more useful and valuable.

Finally, we hope that this work will be useful to give new hints for the study of more general mathematics problems such as the case of Landau-Siegel Zeros and Riemann Hypothesis thanks to the restriction of the form $S = \frac{1}{2} + ib$ that is considered in this article and that can be changed depending on the purpose of the other proofs.

Acknowledgment

The author wish to thank the Editorial office of Mathematical Analysis and its Contemporary Applications for their kind review and acceptance of this article.

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Received : *November* 2024
Accepted : *December* 2024