

On close-to-convex infinitesimal generators with fixed initial coefficient

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ABSTRACT. This article extends the subclasses of infinitesimal generators to $\mathcal{C}_{n,b}(\alpha)$ and $\mathcal{SS}_{n,b}^*(\alpha)$ that are associated with close-to-convex and starlike functions with fixed second coefficient. The main result of this study is a solution to an open problem in a certain specific case.

1. Introduction and preliminaries

We commonly use the symbols \mathbb{R} to represent the set of real numbers, \mathbb{N} to represent the set of positive integers, and \mathbb{C} to represent the set of complex numbers. Here, the unit disk is denoted by

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

and the closed unit disk is denoted by

$$\overline{\mathbb{U}} = \mathbb{U} \cup \{z : |z| = 1\}.$$

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Let \mathcal{H} be the class of all analytic functions in \mathbb{U} . We define two well-known classes of analytic functions:

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \dots, z \in \mathbb{U}\},$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1}, z \in \mathbb{U}\},$$

where $\mathcal{A} = \mathcal{A}_1$ denotes the set of normalized analytic functions.

The class of univalent functions is denoted by \mathcal{S} and is a subset of the class \mathcal{A} . We define the class of starlike functions of order α , where α is a real number such that $0 \leq \alpha < 1$, as

$$\mathcal{S}^*(\alpha) = \{f \in \mathcal{A} : \Re\{z f'(z)/f(z)\} > \alpha, z \in \mathbb{U}\}.$$

Let $\mathcal{S}^*(0) = \mathcal{S}$. A starlike function is a univalent function that maps the open unit disk onto a domain that is star-shaped with respect to the origin. In other words, for any point z in the unit disk, the line segment connecting 0 and z is entirely contained in the image of the unit disk under the mapping of $f(z)$.

Let define Convex functions by

$$\mathcal{K} = \{f \in \mathcal{A} : \Re\{1 + z f''(z)/f'(z)\} > 0, z \in \mathbb{U}\}.$$

A convex function is a univalent function that maps line segments in the complex plane to line segments or line segments that lie entirely within a convex set. Forthemore, we denote the class of close-to-convex functions by \mathcal{C} , consisting of all function f such that there exists a starlike function g (not necessarily normalized) satisfies

$$\Re\{1 + z f'(z)/g(z)\} > 0, \quad (z \in \mathbb{U}).$$

A close-to-convex function is one that does not stray too far from being a convex function. Even more, $\mathcal{K} \subseteq \mathcal{S}^* \subseteq \mathcal{C}$. All mentioned classes are proper subclasses of \mathcal{S} .

Differential subordination theory is concerned with the study of a specific class of complex-valued functions referred to as subordination functions. These functions map the unit disk onto itself and satisfy certain geometric conditions related to convexity and starlikeness. The origins of differential subordination theory can be traced back to 1974 with the work of Miller, Mocanu, and Reade [8]. Later, Miller and Mocanu introduced differential subordination in [9]. In definition, if $f(z)$ and $g(z) \in \mathcal{H}$, we say that $f(z)$ is subordinate to $g(z)$ and denote it as $f(z) \prec g(z)$ if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$), such that $f(z) = g(w(z))$ ($z \in \mathbb{U}$). Assuming that $g(z)$ is univalent in \mathbb{U} leads to an equivalent definition of subordination: $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

In a recent study, R.M. Ali et al. [1] enhanced the theory of subordination functions by considering functions with a fixed second coefficient. This assumption typically leads to extensions in the results related to analytical functions since

knowledge of their Taylor expansion coefficients provides more information about the function. Let us denote by

$$\mathcal{H}_\beta[a, b] = \{p \in \mathcal{H} : p(z) = a + \beta z^n + a_{n+1}z^{n+1} + \dots\}$$

and

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + \dots, z \in \mathbb{U}\}.$$

the classes of analytic functions with fixed second coefficient where β and b are fixed complex numbers and let $\mathcal{A}_b = \mathcal{A}_{1,b}$. Without loss of generality, we assume that β and b are positive real numbers.

In 1969, Brannan [2] defined the class of strongly starlike functions of order α , denoted by $\mathcal{SS}^*(\alpha)$, where $0 < \alpha \leq 1$, in the following way:

$$\mathcal{SS}^*(\alpha) = \left\{ f \in \mathcal{A} : \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\alpha\pi}{2}, z \in \mathbb{U} \right\}.$$

Using the fixed second coefficient method, we define a comparable class as follows:

$$\mathcal{SS}_{n,b}^*(\alpha) = \left\{ f \in \mathcal{A}_{n,b} : \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\alpha\pi}{2}, z \in \mathbb{U} \right\}.$$

Additionally, N. Tuneski et al. [11] defined a novel subclass of close-to-convex functions relative to $\mathcal{SS}^*(\alpha)$, denoted by

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \exists g \in \mathcal{SS}^*(\alpha) \text{ s.t. } \left| \arg \left\{ \frac{zf'(z)}{g(z)} \right\} \right| < \frac{\alpha\pi}{2}, z \in \mathbb{U} \right\}.$$

Similarly, by considering functions that belong to the class $\mathcal{SS}_{n,b}^*(\alpha)$, we define the class $\mathcal{C}_{n,b}(\alpha)$, which comprises close-to-convex functions with respect to strongly starlike functions with a fixed second coefficient of order α , as follows:

$$\mathcal{C}_{n,b}(\alpha) = \left\{ f \in \mathcal{A}_{n,b} : \exists g \in \mathcal{SS}_{n,b}^*(\alpha) \text{ s.t. } \left| \arg \left\{ \frac{zf'(z)}{g(z)} \right\} \right| < \frac{\alpha\pi}{2}, z \in \mathbb{U} \right\}.$$

Obviously, $\mathcal{SS}_{n,b}^*(\alpha) \subseteq \mathcal{C}_{n,b}(\alpha)$.

In what follows, we want to find a connection between the close-to-convexity and the class of so-called infinitesimal generators. Recall that by definition, $f \in \mathcal{H}$ is called an infinitesimal generator if for every $z \in \mathbb{U}$, the Cauchy problem

$$\frac{\partial u(t, z)}{\partial t} + f(u(t, z)) = 0, \quad u(0, z) = z,$$

has a unique solution $u = u(t, z)$, $z \in \mathbb{U}$ for all $t \geq 0$. The function f is referred to as the infinitesimal generator of a one-parameter continuous vector field on \mathbb{U} .

The set of all infinitesimal generators is denoted by \mathcal{G} . In prior research, several criteria have been explored by various authors [4, 5, 10] to determine whether a function is an infinitesimal generator. In one of the most significant results, Berkson and Porta [3] established the following theorem:

Theorem 1.1. *The following assertions are equivalent:*

- i) $f \in \mathcal{G}$,
- ii) $f(z) = (z - \rho)(1 - \bar{\rho}z)p(z)$ for some $\rho \in \bar{\mathbb{U}}$ and $p \in \mathcal{H}$, $\Re\{p(z)\} \geq 0$, $z \in \mathbb{U}$;

The point $\rho \in \bar{\mathbb{U}}$ in the above theorem, is called the Denjoy–Wolff [10] point of the semigroup generated by f . For $\rho = 0$, we have the following subclass

$$\mathcal{G}[0] = \{f \in \mathcal{G} : f(z) = zp(z), \Re\{p(z)\} > 0\} \quad (1)$$

Based on the aforementioned class, a function $f \in \mathcal{A}$ is deemed to be an infinitesimal generator if and only if $\Re f(z)/z > 0$. Due to a result from Bracci et al. in [5], a sufficient condition for $f \in \mathcal{A}$ to be an infinitesimal generator is $\Re f'(z) \geq 0$ for all $z \in \mathbb{U}$. In accordance with an earlier finding by Marx-Strohhacker [7], we know

$$\mathcal{K} \subseteq \mathcal{S}^*\left(\frac{1}{2}\right) \subseteq \mathcal{G}[0]. \quad (2)$$

The implication (2) can be interpreted as follows: *Every convex function $f \in \mathcal{A}$ generates a semigroup $\{u(t, 0)\}_{t \geq 0}$.* In [11], N. Tuneski et al. aimed to extend the class $\mathcal{G}[0]$ to close-to-convex functions and answered the following question:

Under what conditions on α is $\mathcal{C}(\alpha) \subseteq \mathcal{G}[0]$?

They concluded that if $f \in \mathcal{C}(\alpha)$, where $0 < \alpha \leq (2/\pi) \arctan(1/2) = 0.295 \dots$, then

$$\Re\{f(z)/z\} > 0.$$

Finally, they also raised this open question that the $\alpha = 0.295 \dots$ is best possible value can be obtained with the approach used in their paper. Here, we utilize the fixed initial coefficient approach. This will allow us to expand the range of values α , in a such a way that α can be sufficiently close to 1.

2. preliminaries

First we need a result related to differential subordination for functions with fixed second coefficient as follows.

Lemma 2.1. *Let $q(z)$ be univalent in \mathbb{U} , and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D \supseteq q(\mathbb{U})$, when $\phi(w) \neq 0$ for $w \in q(\mathbb{U})$. Also, let $0 < b \leq |q'(0)|$. Set*

$$Q(z) = zq'(z)\phi(q(z))$$

and

$$h(z) = \theta(q(z)) + \left(n + \frac{|q'(0)| - b}{|q'(0)| + b}\right) zq'(z)\phi(q(z)).$$

Suppose that

- i) h is convex, or $Q(z)$ is starlike.
- ii) $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$, ($z \in \mathbb{U}$).

If $p \in \mathcal{H}_b[a, n]$, $p(\mathbb{U}) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + \left(n + \frac{|q'(0)| - b}{|q'(0)| + b} \right) zq'(z)\phi(q(z)), \quad (3)$$

then $p \prec q$.

PROOF. As we have applied a comparable technique to the one described in Theorem 3.4.h of [9], we will not delve into the specifics of the proof here. However, it should be noted that in the case of (ii), substituting the equation

$$L(z, t) = \theta(q(z)) + \left(n + \frac{|q'(0)| - b}{|q'(0)| + b} \right) zq'(z)\phi(q(z)),$$

instead of (3.4-21) in [9] is adequate, and, so proceed the proof. \square

Lemma 2.2 ([5], Corollary 4.3). *Let $f : \mathbb{U} \rightarrow \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f'(0) = 1$. If $\Re\{f'(z)\} \geq 0$ for all $z \in \mathbb{U}$, then $f \in \mathcal{G}[0]$.*

3. Main results

In what follows, we will introduce two novel subclasses, $\mathcal{SS}_{n,b}^*(\alpha)$ and $\mathcal{C}_{n,b}(\alpha)$, as infinitesimal generators using the fixed second coefficient method. By employing this technique, we can obtain a solution in a special case, to the open problem posed in [11] and achieve α close to 1.

Theorem 3.1. *Let $g \in \mathcal{SS}_{n,b}^*(\alpha)$ with $0 < \alpha \leq 1$, $n \in \mathbb{N}$ and $0 \leq b \leq 2\beta$. Then*

$$\left| \arg \left\{ \frac{g(z)}{z} \right\} \right| < \frac{\pi}{2} \beta \quad (4)$$

where β is the positive solution of

$$(2\beta + b) \tan \frac{\pi}{2} \alpha - \beta((2n + 2)\beta + b(n - 1)) = 0. \quad (5)$$

PROOF. Let verify that conditions (i) and (ii) from Lemma 2.2 hold; By choosing

$$p(z) = g(z)/z, \quad q(z) = ((1 + z)/(1 - z))^\beta$$

note that the principal value of β is considered here, we see that

$$p(z) \in \mathcal{H}_b[1, n] \quad \text{and} \quad q'(0) = 2\beta.$$

Furthermore, suppose $\phi(w) = 1/w$ and $\theta(w) = 1$. Under these conditions, the function q is both convex and univalent in the open unit disk \mathbb{U} , while the function p is analytic in \mathbb{U} and satisfies $p(0) = q(0) = 1$.

Additionally, the functions $\theta(w)$ and $\phi(w)$ are analytic in the domain $D = \mathbb{C} \setminus \{0\}$, where

$$q(\mathbb{U}) = w : |\arg\{w\}| < \pi\beta/2 \quad \text{and} \quad \phi(w) = 0$$

when w is in $q(\mathbb{U})$. Moreover, the image of \mathbb{U} under p is a subset of D . Considering these assumptions, we can write

$$Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)} = \frac{2\beta z}{1-z^2} \in \mathcal{S}^*,$$

$$\begin{aligned} h(z) &= \theta(q(z)) + \left(n + \frac{|q'(0)| - b}{|q'(0)| + b} \right) zq'(z)\phi(q(z)) \\ &= 1 + \left(n + \frac{2\beta - b}{2\beta + b} \right) \frac{2\beta z}{1-z^2}, \end{aligned}$$

where

$$h'(z) = \left(n + \frac{2\beta - b}{2\beta + b} \right) (2\beta) \frac{1+z^2}{(1-z^2)^2}.$$

Thus,

$$\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \left(n + \frac{2\beta - b}{2\beta + b} \right) \frac{1+z^2}{1-z^2} \right\} > 0.$$

So, h is close-to-convex function and conditions (i) and (ii) hold. Also, the relation (4) is true because $g \in \mathcal{SS}_{n,b}^*(\alpha)$ and from the definition of subordination we have

$$\begin{aligned} \theta(p(z)) + zp'(z)\phi(p(z)) &= 1 + \frac{zp'(z)}{p(z)} = \frac{zg'(z)}{g(z)} \\ &\prec q_1(z) = \left(\frac{1+z}{1-z} \right)^\alpha \\ &\prec h(z) \\ &= 1 + \left(n + \frac{2\beta - b}{2\beta + b} \right) \frac{2\beta z}{1-z^2}. \end{aligned}$$

The implication $q_1(z) \prec h(z)$ is also true, because $q(0) = h(0) = 1$ and q and h are univalent. Also, for $\theta \in \mathbb{R}$, we have

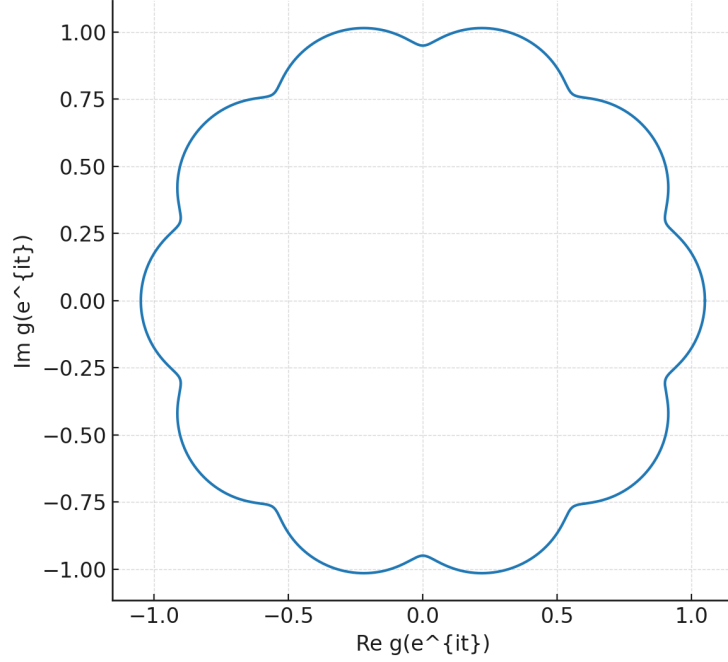
$$\begin{aligned} h(e^{i\theta}) &= 1 + \left(n + \frac{2\beta - b}{2\beta + b} \right) (2\beta) \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\ &= 1 + \beta \left(n + \frac{2\beta - b}{2\beta + b} \right) \frac{i}{\sin \theta}. \end{aligned}$$

This implies that

$$h(\mathbb{U}) = \mathbb{C} \setminus \left\{ 1 + iy : y \in \mathbb{R} \wedge |y| \geq \frac{((2n+2)\beta + b(n-1))\beta}{2\beta + b} \right\}.$$

From (5), we receive that $q_1(\mathbb{U}) = \{w : |\arg\{w\}| < \pi\alpha/2\} \subseteq h(\mathbb{U})$ and by the Lemma 2.2 we have $p(z) \prec q(z)$, hence the proof is done. \square

Remark 3.1. By taking $n = 1$ and $b = 2\beta$ in equation (4), we arrive at a result due to N. Tuneski et al. [11].

FIGURE 1. For $\alpha = 0.9422841234$

Example 3.2. Since $0 \leq b \leq 2\beta \leq 2$, by taking $n = 10$ in the equation (5), we observe that $\alpha \leq \frac{2}{\pi} \tan^{-1}(11) = 0.9422841234$. This means that the function

$$g(z) = z + bz^{11} + \sum_{n=12}^{\infty} a_n z^n$$

belongs to the class $\mathcal{SS}_{10,b}^*(\alpha)$ with

$$\left| \arg \left\{ \frac{g(z)}{z} \right\} \right| < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}),$$

where $\alpha = 0.9422841234$.

Theorem 3.2. Let $f \in \mathcal{C}_{n,b}(\alpha)$, $0 < \alpha \leq \alpha_0(n,b) < 1$ where $\alpha_0(n,b)$ is the positive solution of

$$\begin{aligned} & \sqrt{4\left(\tan \frac{\pi}{2} \alpha\right)^2 + 4b(n+3) \tan \frac{\pi}{2} \alpha + b^2(n-1)^2} \\ & + 2 \tan \frac{\pi}{2} \alpha + (4\alpha - b - 4)n + 4\alpha + b - 4 = 0 \end{aligned} \quad (6)$$

with $n \in \mathbb{N}$ and $0 \leq b \leq 2$. Then

$$\Re \left\{ \frac{f(z)}{z} \right\} > 0, \quad (z \in \mathbb{U}).$$

PROOF. Let $f \in \mathcal{C}_{n,b}(\alpha)$, $0 < \alpha \leq \alpha_0(n, b) < 1$. Then there exists $g \in \mathcal{SS}_{n,b}^*(\alpha)$ such that

$$\left| \arg \left\{ \frac{zf'(z)}{g(z)} \right\} \right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{U}). \quad (7)$$

From Theorem 3.1 we have

$$\left| \arg \left\{ \frac{g(z)}{z} \right\} \right| < \frac{\pi}{2}\beta, \quad (8)$$

where $0 < \beta \leq 1$ is positive solution of the relation given by (5). By applying (7) and (8) we can write

$$\begin{aligned} |\arg\{f'(z)\}| &= \left| \arg \left\{ \frac{zf'(z)}{g(z)} \frac{g(z)}{z} \right\} \right| \\ &= \left| \arg \left\{ \frac{zf'(z)}{g(z)} \right\} + \arg \left\{ \frac{g(z)}{z} \right\} \right| \\ &< \frac{\pi}{2}(\alpha + \beta). \end{aligned}$$

According to Lemma 2.2, we must have $\alpha + \beta \leq 1$. From the combining of this and the relation (5), we get $\alpha_0(n, b)$ as a solution of equation (6), hence proof is done. \square

Remark 3.3. By taking $b = 2\beta$ and $n = 1$ (i.e. $f \in \mathcal{A}_{1,b} = \mathcal{A}_b$) in the (5) together with (6), we get $\alpha = 0.3616777377$, which is better value of α in answer to the open problem raised in [11].

As we see in Figure 1, when n is large enough, α can take values close to 1. It is necessary to mention that choosing high value of n in (6) means that in the Taylor series expansion of function $f \in \mathcal{C}_{n,b}(\alpha)$, all coefficients between z and z^n are zero, which leads to a certain function.

Example 3.4. Since $0 \leq b \leq 2$, if we take $n = 10$ in the relation (6), we get $\alpha \leq 0.7585237770$, which shows that choosing a higher value of n leads to a better result in α with this method. Therefore the function

$$g(z) = z + bz^{11} + \sum_{n=12}^{\infty} a_n z^n$$

belongs to the class $\mathcal{C}_{10,b}(\alpha)$ with

$$\Re \left\{ \frac{f(z)}{z} \right\} > 0, \quad (z \in \mathbb{U}),$$

where $\alpha = 0.7585237770$.

Corollary 3.3. $\mathcal{C}_{n,b}(\alpha) \subseteq \mathcal{G}[0]$ for $0 < \alpha \leq \alpha_0(n, b)$, $0 \leq b \leq 2$ and $n \in \mathbb{N}$.

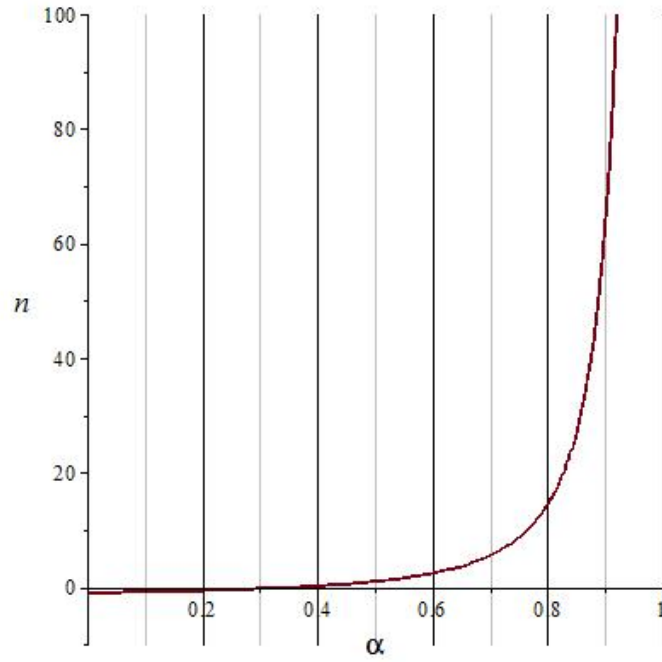


FIGURE 2. Values of $\alpha_0(n, b)$ with $b = 0$

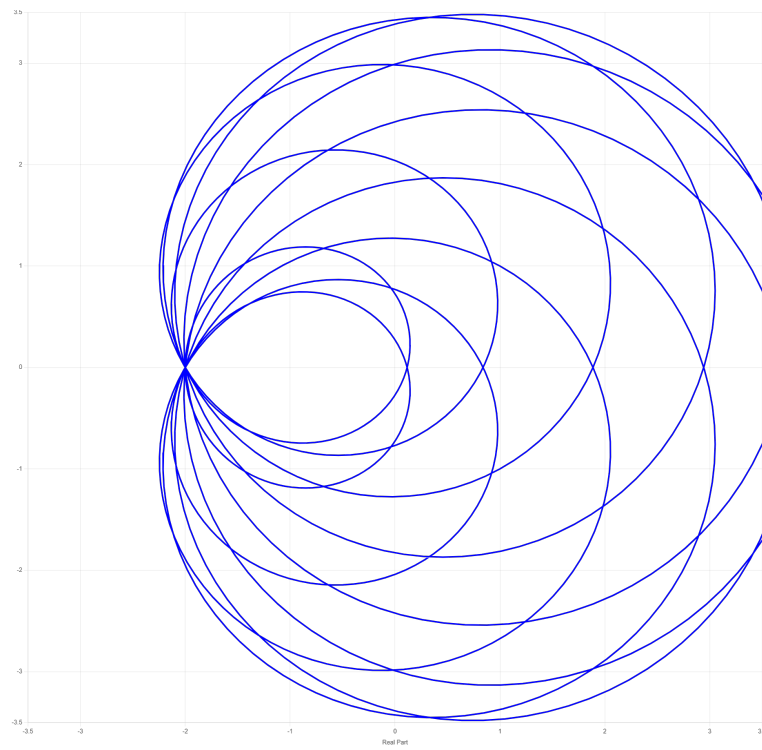


FIGURE 3. $\alpha = 0.7585237770$ and $g(z) = z + 2z^{11} + z^{12}$.

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