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Uncertainty principles and extremal functions for generalized Hartley-Gabor transform

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Abstract. The main crux of this paper is to introduce a new integral transform called the generalized Hartley-Gabor transform which generalizes the classical Gabor Fourier transform and to give some new results related to this transform as Plancherel's, Parseval's, inversion and Calderon's reproducing formulas. Next, we analyse the concentration of this transform on sets of finite measures and we give the uncertainty principle for orthonormal sequences. Last, using the best approximations and the theory of reproducing kernels, we study the extremal functions related to this transform and we give an integral representation, band est estimates of these functions on weighted Sobolev spaces.

1. Introduction

Time-frequency analysis [17] and uncertainty principles [14, 21] play a fundamental role in field of mathematics and physics, these principes appear in harmonic analysis and signal theory in a various different forms involving not only the signal f and its Fourier transform $\hat{f}$, but also every representation of a signal in the time-frequency space.

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The uncertainty principles are mathematical results that gives limitations on the simultaneous concentration of a signal and its Fourier transform and they have implications in signal analysis and quantum physics. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies of signal consists of.

Timelimited and bound limited functions are basic tools in signal analysis and imaging processing. In quantum physics they tell us that a particule's speed and position cannot both of them be measured with infinite presicion, the mathematical formulation of this principle is given by the Heisenberg-Pauli-Weyl sharp inequality [**24**]. Other uncertainty relations have been investigated among them, we refer to the papers of Benedick's [**2**], Donoho-Stark's [**12**], Jaming's [**19**]. The Hartley transform [**7, 18, 26**] is a linear operator defined for a suitable function $\psi(x)$ as follows:

$$\mathcal{H}(\psi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) \operatorname{cas}(\lambda x) dx, \tag{1}$$

where $\operatorname{cas}(x)$ is the cas function, defined as:

$$\operatorname{cas}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{n!} x^n, \tag{2}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$ being the binomial coefficient. The cas(x) function (2) can be seen as a generalization of the exponential function exp. A simple computation shows that the cas function is the unique C^∞ solution of the following differential-reflection problem see [**4**]

$$\begin{cases} R\partial_x u(x) = \lambda u(x), \\ u(0) = 0. \end{cases}$$

Here, ∂_x represents the first-order derivative, and R is the reflection operator acting on functions $f(x)$ as:

$$R(f)(x) = f(-x). \tag{3}$$

Furthermore, the function $cas(x)$ is multiplicative on $\mathbb{R}$ in the sens

$$\operatorname{cas}(x)\operatorname{cas}(y) = \frac{1}{2}(\operatorname{cas}(x+y) - \operatorname{cas}(-x-y) + \operatorname{cas}(x-y) + \operatorname{cas}(y-x)). \tag{4}$$

Inspired by the relation (4), the author in [**4**] generalized the relation (4) for the Hartley-Bessel function and introduce a generalized convolution product. This paper focuses on the generalized Hartley transform introduced in [**4, 5, 6**] called the Hartley-Bessel transform, more precisely we consider the following differential-reflection operator Δ_α defined by

$$\Delta_\alpha = R\left(\partial_x + \frac{\alpha}{x}\right) + \frac{\alpha}{x}, \quad \alpha \geq 0. \tag{5}$$

where R is the reflection operator given by the relation (3). The operator Δ_α is closely connected with the Dunkl's theory [**13**], furthermore the eigenfunctions of this operator are related to Bessel functions and they satisfies a product formula which permits to develop a new harmonic analysis associated with this operator see [4] for more information.

Uncertainty principles play a fundamental role in the field of mathematics, physics and some area of engineering such as signal processing, image processing, quantum theory and optics see [12, 14, 17, 21], in this context using the Gabor transform introduced by Gabor, using translation, modulation and convolution operators of a single Gaussian, the authors in [3, 25] gives a new uncertainty principles for the Gabor transform. Uncertainty principles associated with the Gabor was studied in the one dimensional Hankel setting [1, 15], in the multidimensional Hankel setting [9, 10, 11] and in the two-sided quaternion setting [8], motivated by these works the main purpose of this work is to introduce the Gabor transform associated with the Hartley-Bessel operator (5) called the Hartley-Gabor transform and to give some new results related to this transform as Plancherel's, Parseval's, inversion and Calderon's reproducing formulas. Next, we give some new uncertainty principles associated with this transform.

The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the Hartley-Bessel transform, in section 3 we introduce the Hartley-Gabor transform and we give some new results related to this transform , section 4 is devoted to analyse the concentration of the multidimensional Hartley-Gabor transform on sets of finite measure and to give some new uncertainty principles related to this transform. Last, we study the extremal functions related to the Hartley-Gabor transform and we give an integral representation, best estimates of these functions on weighted Sobolev spaces.

2. Harmonic Analysis Associated with the Hartley-Bessel Transform

In this section we recall some results in harmonic analysis related to the Hartley-Bessel transform, for more details we refer the reader to [4, 20].

- For $\alpha \geq 0$, μ_α is the weighted Lebesgue measure defined on $\mathbb{R}$ by

$$d\mu_\alpha(x) := \frac{|x|^{2\alpha}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}dx,$$

where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{R}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}$, satisfying

$$\|f\|_{p,\mu_\alpha} =: \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty. \end{cases}$$

In particular, for $p = 2$, $L^2_\alpha(\mathbb{R})$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_{\mu_\alpha} = \int_{\mathbb{R}} f(x)\overline{g(x)}d\mu_\alpha(x).$$

2.1. The Eigenfunctions of the Differential-reflection operator Δ_α. For $\lambda \in \mathbb{C}$ we consider the following Cauchy problem

$$(S) : \begin{cases} \Delta_\alpha(u)(x) = \lambda u(x), \\ u(0) = 1. \end{cases}$$

From [4], the Cauchy problem (S) admits a unique solution $B_\alpha(\lambda.)$ given by

$$B_\alpha(\lambda x) = j_{\alpha-\frac{1}{2}}(\lambda x) + \frac{\lambda x}{2\alpha+1} j_{\alpha+\frac{1}{2}}(\lambda x), \tag{6}$$

where j_α denotes the normalized Bessel function of order α see [4]. The function $B_\alpha(\lambda.)$ is infinitely differentiable on $\mathbb{R}$ and we have the following important result

$$\forall \lambda, x \in \mathbb{R}, \quad |B_\alpha(\lambda x)| \leq \sqrt{2}. \tag{7}$$

Furthermore from [4], the Hartley-Bessel kernel (6) is multiplicative on $\mathbb{R}$ in the sense

$$\forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^* \quad B_\alpha(\lambda x) B_\alpha(\lambda y) = \int_{\mathbb{R}} B_\alpha(\lambda z) K_\alpha(x, y, z) d\mu_\alpha(z) \tag{8}$$

where K_α is the Bessel kernel given explicitly in [4]. The product formula () generalize the relation (4) and permits to define a translation operator, convolution product and to develop a new harmonic analysis associated to the Differential-reflection operator Δ_α.

2.2. The Hartley-Bessel transform.

Definition 2.1. ([4]) The Hartley-Bessel transform $\mathcal{H}_\alpha$ defined on $L^1_\alpha(\mathbb{R})$ by

$$\mathcal{H}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x) d\mu_\alpha(x), \quad \text{for } \lambda \in \mathbb{R}$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [4].

Proposition 2.1. (1) *For every $f \in L^1_\alpha(\mathbb{R})$ we have*

$$\|\mathcal{H}_\alpha(f)\|_{\infty,\mu_\alpha} \leq \sqrt{2} \|f\|_{1,\mu_\alpha}. \tag{9}$$

(2) *(Inversion formula) For $f \in \left(L^1_\alpha \cap L^2_\alpha\right)(\mathbb{R})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$ we have*

$$f(x) = \int_{\mathbb{R}} B_\alpha(\lambda x) \mathcal{H}_\alpha(f)(\lambda) d\mu_\alpha(\lambda), \quad a.e \quad x \in \mathbb{R}. \tag{10}$$

(3) *(Plancherel theorem) The Hartley-Bessel transform $\mathcal{H}_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{R})$ into $L^2_\alpha(\mathbb{R})$. and we have*

$$\|f\|_{2,\mu\alpha} = \|\mathcal{H}_\alpha(f)\|_{2,\mu\alpha}. \tag{11}$$

2.3. The translation operator associated with the Hartley-Bessel transform. The product formula (8) permits to define the translation operator as follows

Definition 2.2. Let $x, y \in \mathbb{R}$ and f is a measurable function on $\mathbb{R}$ the translation operator is defined by

$$\mathcal{T}_\alpha^x f(y) = \int_{\mathbb{R}} f(z) K_\alpha(x, y, z) d\mu_\alpha(z), \tag{12}$$

The following proposition summarizes some properties of the Hartley-Bessel translation operator see [4].

Proposition 2.2. *For all $x, y \in \mathbb{R}$, we have:*

(1)

$$\tau_\alpha^x f(y) = \tau_\alpha^y f(x). \tag{13}$$

(2)

$$\int_{\mathbb{K}} \mathcal{T}_\alpha^x f(y) d\mu_\alpha(y) = \int_{\mathbb{R}} f(y) d\mu_\alpha(y). \tag{14}$$

(3) *for $f \in L_\alpha^p(\mathbb{R})$ with $p \in [1; +\infty]$ $\quad \mathcal{T}_\alpha^x f \in L_\alpha^p(\mathbb{R})$ and we have*

$$\|\mathcal{T}_\alpha^x f\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}. \tag{15}$$

By using the translation, we define the generalized convolution product of f, g by

$$(f *_\alpha g)(x) = \int_{\mathbb{R}} \mathcal{T}_\alpha^x (f)(y) g(y) d\mu_\alpha(y).$$

This convolution is commutative, associative and its satisfies the following properties see [4].

Proposition 2.3. (1) *(Young's inequality) for all $p, q, r \in [1; +\infty]$ such that: $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and for all $f \in L_\alpha^p(\mathbb{R}), g \in L_\alpha^q(\mathbb{R})$ the function $f *_\alpha g$ belongs to the space $L_\alpha^r(\mathbb{R})$ and we have*

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq 4\|f\|_{p,\mu_\alpha}\|g\|_{q,\mu_\alpha} \tag{16}$$

(2) *For $f, g \in L_\alpha^2(\mathbb{R})$ the function $f *_\alpha g$ belongs to $L_\alpha^2(\mathbb{R})$ if and only if the function $\mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g)$ belongs to $L_\alpha^2(\mathbb{R})$ and in this case we have*

$$\mathcal{H}_\alpha(f *_\alpha g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g). \tag{17}$$

(3) *For all $f, g \in L_\alpha^2(\mathbb{R})$ then we have*

$$\int_{\mathbb{R}} |f *_\alpha g(x)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}} |\mathcal{H}_\alpha(f)(\lambda)|^2 |\mathcal{H}_\alpha(g)(\lambda)|^2 d\mu_\alpha(\lambda), \tag{18}$$

where both integrals are simultaneously finite or infinite.

3. Gabor Transform Associated with the Hartley-Bessel Transform

The main purpose of this section is to introduce the Hartley-Gabor transform and to give some new results related to this transform.

Notation: we denote by

- $L^p_\alpha(\mathbb{R}^2), 1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R} \times \mathbb{R}$ satisfying

$$\|f\|_{p,\mu_\alpha\otimes\mu_\alpha} := \begin{cases} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x,y)|^p \, \mathrm{d}\mu_\alpha(x) \otimes \mathrm{d}\mu_\alpha(y)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[; \\ \operatorname{ess\,sup} |f(x,y)| & \text{if } p = +\infty. \\ (x,y) \in \mathbb{R}\times\mathbb{R} & \end{cases}$$

Let u in $L^2_\alpha(\mathbb{R})$ and $y \in \mathbb{R}$, we recall that the modulation operator of u is given by

$$\mathcal{M}^y(u) := u^y := \mathcal{H}_\alpha\left(\sqrt{\tau^y_\alpha |\mathcal{H}_\alpha(u)|^2}\right).$$

By using Plancherel's formula (11) and the relation (14), we have $u^y \in L^2_\alpha(\mathbb{R})$ and

$$\|u^y\|_{2,\alpha} = \|u\|_{2,\alpha}. \tag{19}$$

Furthermore by using inversion formula (10), we find the following important result

$$\mathcal{H}_\alpha(u^y)(\lambda) = \sqrt{\tau^y_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}. \tag{20}$$

Now, for every non-zero window function u in $L^2_\alpha(\mathbb{R})$, we consider the family $u^{x,y}$ defined by

$$u^{x,y}(z) = \tau^x_\alpha(u^y)(z), \quad \forall (x,y) \in \mathbb{R}\times\mathbb{R}. \tag{21}$$

Definition 3.1. For every f and u in $L^2_\alpha(\mathbb{R})$ we define the Hartley-Gabor transform by

$$\mathcal{W}_u(f)(x,y) := \int_{\mathbb{R}} f(z)\overline{u^{x,y}(z)}\mathrm{d}\mu_\alpha(z), \tag{22}$$

Remark 3.2. the Hartley-Gabor transform can be also expressed by

$$\mathcal{W}_u(f)(x,y) = \left(\overline{u^y} *_\alpha f\right)(x). \tag{23}$$

By using Hölder's inequality and the relations (15), (19), (21), and (22), we find that $\mathcal{W}_u(f) \in L^\infty_\alpha(\mathbb{R}^2)$ and we have

$$\|\mathcal{W}_u(f)\|_{\infty,\mu_\alpha\otimes\mu_\alpha} \leq \|f\|_{2,\alpha}\|u\|_{2,\alpha}. \tag{24}$$

Definition 3.3. Let u, v be non-zero functions in $L^2_\alpha(\mathbb{R})$, we say that the pair (u,v) is admissible if for almost all $\lambda \in \mathbb{R}$ we have

$$C_{u,v} = \int_{\mathbb{R}} \sqrt{\tau^\lambda_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(y)\tau^\lambda_\alpha\left(|\mathcal{H}_\alpha(v)|^2\right)(y)}\mathrm{d}\mu_\alpha(y) < \infty. \tag{25}$$

We have the following generalized Parseval's formula for the Hartley-Gabor transform.

Theorem 3.1. *Let (u,v) be an admissible pair then for all $f,h\in L^2_\alpha(\mathbb{R})$ we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}_u(f)(x,y)\overline{\mathcal{W}_v(h)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y)=C_{u,v}\int_{\mathbb{R}}f(x)\overline{h(x)}\mathrm{d}\mu_\alpha(x). \tag{26}$$

Proof. By using Fubini's theorem and the relations (11), (13), (17), (20) and (23), we find that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}_u(f)(x,y)\overline{\mathcal{W}_v(h)(x,y)}\mathrm{d}\mu_\alpha(x)\mathrm{d}\mu_\alpha(y)\\&=\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\left(\overline{u^y}*_\alpha f\right)(x)\overline{\left(\overline{v^y}*_\alpha h\right)(x)}\mathrm{d}\mu_\alpha(x)\right]\mathrm{d}\mu_\alpha(y)\\&=\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\sqrt{\tau^y_\alpha\left(\left|\mathcal{H}_\alpha(u)\right|^2\right)(\lambda)}\sqrt{\tau^y_\alpha\left(\left|\mathcal{H}_\alpha(v)\right|^2\right)(\lambda)}\mathcal{H}_\alpha(f)(\lambda)\overline{\mathcal{H}_\alpha(h)(\lambda)}\mathrm{d}\mu_\alpha(\lambda)\right]\mathrm{d}\mu_\alpha(y)\\&=C_{u,v}\int_{\mathbb{R}}f(x)\overline{h(x)}\mathrm{d}\mu_\alpha(x).\end{aligned}$$

and the proof is complete. □

Corollary 3.2 (Plancherel's formula for $\mathcal{W}_u$)**.** *If $u=v$ and $f=h$ we find that*

$$\|\mathcal{W}_u(f)\|_{2,\mu_\alpha\otimes\mu_\alpha}=\|f\|_{2,\alpha}\|u\|_{2,\alpha}. \tag{27}$$

Proposition 3.3. *u be non-zero window function in $L^2_\alpha(\mathbb{R})$, for all $f\in L^2_\alpha(\mathbb{R})$, the function $\mathcal{W}_u(f)$ belongs to $L^p_\alpha(\mathbb{R}^2)$ for all $p\in[2;+\infty]$ and we have*

$$\|\mathcal{W}_u(f)\|_{p,\mu_\alpha\otimes\mu_\alpha}\le\|f\|_{2,\alpha}\,\|u\|_{2,\alpha}.$$

Proof. Is a consequence of the relations (24), (27) and the Riesz-Thorin's interpolation theorem. □

In the following, we establish an inversion formula for the Hartley-Gabor transform.

Theorem 3.4. *Let (u,v) be an admissible pair in $L^2_\alpha(\mathbb{R})$,then for all $f\in L^2_\alpha(\mathbb{R})$ we have*

$$f(.)=\frac{1}{C_{u,v}}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}_u(f)(x,y)v^{x,y}(.)\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y),$$

weakly in $L^2_\alpha(\mathbb{R})$.

Proof. By using the relations (22), (26), and Fubini's theorem we find that

$$\begin{aligned}\int_{\mathbb{R}}f(z)\overline{h(z)}\mathrm{d}\mu_\alpha(z)&=\frac{1}{C_{u,v}}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}_u(f)(x,y)\overline{\mathcal{W}_v(h)(x,y)}\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y)\\&=\frac{1}{C_{u,v}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}_u(f)(x,y)v^{x,y}(z)\mathrm{d}\mu_\alpha(x)\otimes\mathrm{d}\mu_\alpha(y)\right)\overline{h(z)}\mathrm{d}\mu_\alpha(z).\end{aligned}$$

which gives the result. □

The reproducing kernels for Hilbert space play an important role in harmonic analysis [22]. In this context, we have the following result.

Theorem 3.5. *The space $\mathcal{W}_u\left(L^2_\alpha(\mathbb{R})\right)$ is a reproducing kernel Hilbert space in $L^2_\alpha\left(\mathbb{R}^2\right)$ with kernel function $\mathcal{K}_u$ defined by*

$$\mathcal{K}_u\left((x',y');(x,y)\right)=\frac{1}{\|u\|^2_{2,\alpha}}\left(u^{x,y}*_\alpha\overline{u^{y'}}\right)(x').$$

Furthermore, the kernel is pointwise bounded

$$\left|\mathcal{K}_u\left((x',y');(x,y)\right)\right|\leq 1,\quad \forall (x,y);(x',y')\in\mathbb{R}^2.$$

Proof. From the relations (23) and (26), we find that

$$\begin{aligned}\mathcal{W}_u(f)(x,y)&=\frac{1}{\|u\|^2_{2,\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}_u(f)\left(x',y'\right)\overline{\mathcal{W}_u\left(u^{x,y}\right)(x',y')}\mathrm{d}\mu_\alpha\left(x'\right)\mathrm{d}\mu_\alpha\left(y'\right)\\&=\left\langle \mathcal{W}_u(f)\mid \mathcal{K}_u((.);(x,y))\right\rangle_{\mu_\alpha\otimes\mu_\alpha},\end{aligned}$$

where

$$\mathcal{K}_u\left((x',y');(x,y)\right)=\frac{1}{\|u\|^2_{2,\alpha}}\left(u^{x,y}*_\alpha\overline{u^{y'}}\right)(x').$$

On the other hand, for every $(x,y);(x',y')\in\mathbb{R}^{2d}$ and by a direct computation, we obtain

$$\|\mathcal{K}_u((.);(x,y))\|_{2,\mu_\alpha\otimes\mu_\alpha}\leq 1.$$

Finally by the Cauchy-Schwarz inequality, we get

$$\left|\mathcal{K}_u\left((x',y');(x,y)\right)\right|\leq\frac{1}{\|u\|^2_{2,\alpha}}\int_{\mathbb{R}}\left|u^{x,y}(z)\right|\left|u^{x',y'}(z)\right|\mathrm{d}\mu_\alpha(z)\leq 1.$$

This shows that the kernel $\mathcal{K}_u$ belongs to $L^2_\alpha\left(\mathbb{R}^2\right)$ and is bounded. □

The rest of this section is devoted to give Calderón's type reproducing formula for the Hartley-Gabor transform, to do this we need the help of the following result.

Proposition 3.6. *Let $0<\gamma<\delta<+\infty$ and (u,v) be an admissible pair such that $\mathcal{H}_\alpha(u)$ and $\mathcal{H}_\alpha(v)$ belongs to $L^\infty_\alpha(\mathbb{R})$. We put*

$$G_{\gamma,\delta}(x):=\frac{1}{C_{u,v}}\int_{D(\gamma,\delta)}\left(\overline{u^y}*_\alpha v^y\right)(x)\,\mathrm{d}\mu_\alpha(y)\tag{28}$$

and

$$K_{\gamma,\delta}(\lambda):=\frac{1}{C_{u,v}}\int_{D(\gamma,\delta)}\sqrt{\tau^\lambda_\alpha\left(\left|\mathcal{H}_\alpha(u)\right|^2\right)(y)\tau^\lambda_\alpha\left(\left|\mathcal{H}_\alpha(v)\right|^2\right)(y)}\mathrm{d}\mu_\alpha(y),\tag{29}$$

where

$$D(\gamma,\delta)=\{x\in\mathbb{R}:\gamma\leq x\leq\delta\}.$$

Then we have $G_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R})$ and

$$\mathcal{H}_\alpha(G_{\gamma,\delta})(\lambda)=K_{\gamma,\delta}(\lambda).\tag{30}$$

Proof. By using Hölder's inequality and the relations (11), (18) and (19), we find that

$$\begin{aligned}\|G_{\gamma,\delta}\|_{2,\alpha}^2 &\leq \frac{\mu_\alpha(D(\gamma,\delta))}{C_{u,v}^2}\int_{D(\gamma,\delta)}\left(\int_{\mathbb{R}}\left|\mathcal{H}_\alpha(\overline{u^y})(\lambda)\right|^2\left|\mathcal{H}_\alpha(v^y)(\lambda)\right|^2\,\mathrm{d}\mu_\alpha(\lambda)\right)\,\mathrm{d}\mu_\alpha(y)\\ &\leq \left(\frac{\mu_\alpha(D(\gamma,\delta))}{C_{u,v}}\right)^2\|\mathcal{H}_\alpha(f)\|_{\infty,\alpha}^2\,\|v\|_{2,\alpha}^2<\infty,\end{aligned}$$

which proves that $G_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R})$, furthermore by using the relation (11), we find that

$$\begin{aligned}\left(\overline{u^y}*_\alpha v^y\right)(x) &= \int_{\mathbb{R}}\mathcal{H}_\alpha(\overline{u^y})(\lambda)\psi_{\alpha,d}(\lambda x)\overline{\mathcal{H}}_\alpha(v^y)(\lambda)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}}\psi_{\alpha,d}(\lambda x)\sqrt{\tau_\alpha^y\left(\left|\mathcal{F}_\alpha\left(u\right)\right|^2\right)(\lambda)\tau_\alpha^y\left(\left|\mathcal{H}_\alpha\left(v\right)\right|^2\right)(\lambda)}d\mu_\alpha(\lambda)\end{aligned}$$

now, by using Fubini's theorem and the relation (13), we find that

$$\begin{aligned}G_{\gamma,\delta}(x) &= \frac{1}{C_{u,v}}\int_{\mathbb{R}}\psi_{\alpha,d}(\lambda x)\left(\int_{D(\gamma,\delta)}\sqrt{\tau_\alpha^\lambda\left(\left|\mathcal{H}_\alpha\left(u\right)\right|^2\right)(y)\tau_\alpha^\lambda\left(\left|\mathcal{H}_\alpha\left(v\right)\right|^2\right)(y)}d\mu_\alpha(y)\right)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}}\psi_{\alpha,d}(\lambda x)K_{\gamma,\delta}(\lambda)d\mu_\alpha(\lambda),\end{aligned}$$

inversion formula (10) gives the relation (29). □

In the following we establish reproducing inversion formula of Calderón's type for the Hartley-Gabor transform $\mathcal{W}_u$.

Theorem 3.7. *Let* $0<\gamma<\delta<+\infty$ *and* (u,v) *be an admissible pair such that* $\mathcal{F}_\alpha(u)$ *and* $\mathcal{H}_\alpha(v)$ *belongs to* $L^\infty_\alpha(\mathbb{R})$. *For all* $f\in L^2_\alpha(\mathbb{R})$, *the function* $f_{\gamma,\delta}$ *defined for all* $z\in\mathbb{R}$ *by:*

$$f_{\gamma,\delta}(z)=\frac{1}{C_{u,v}}\int_{D(\gamma,\delta)}\int_{\mathbb{R}}\mathcal{W}_u(f)(x,y)\overline{\tau_\alpha^x(v^y)}(z)d\mu_\alpha(x)\otimes d\mu_\alpha(y), \tag{31}$$

belongs to $L^2_\alpha(\mathbb{R})$ *and satisfies*

$$\lim_{(\gamma,\delta)\to(0,+\infty)}\|f_{\gamma,\delta}-f\|_{2,\alpha}=0. \tag{32}$$

Proof. It is easy to see that for all $f\in L^2_\alpha(\mathbb{R})$ we have $f_{\gamma,\delta}=f*_\alpha G_{\gamma,\delta}$, where $G_{\gamma,\delta}$ is the function given by the relation (28), by using the relations (25), (29), we find that

$$\|f_{\gamma,\delta}-f\|_{2,\alpha}^2=\int_{\mathbb{R}}\left|\mathcal{H}_\alpha(f)(\lambda)\right|^2\left(1-K_{\gamma,\delta}(\lambda)\right)^2\,\mathrm{d}\mu_\alpha(\lambda)$$

by using the relations (25), (29), the relation (32) follows from the dominated convergence theorem. □

4. Uncertainty Principles Associated with the Hartley-Gabor Transform

In this section, we estimate the concentration of $\mathcal{W}_u$ on subset of $\mathbb{R}\times\mathbb{R}$ of finite measure, similar results have been checked in [8, 9, 10, 11] and we establish the uncertainty principle for orthonormal sequences associated with the Hartley-Gabor transform, first we consider the following orthogonal projections

(1) *Let P_u be the orthogonal projection from $L^2_\alpha(\mathbb{R}^2)$ onto $\mathcal{W}_u(L^2_\alpha(\mathbb{R}))$ and $\operatorname{Im} P_u$ denotes the range of P_u.*

(2) *Let P_E be the orthogonal projection on $L^2_\alpha(\mathbb{R}^2)$ defined by*

$$P_E F = \chi_E F, \quad F \in L^2_\alpha\left(\mathbb{R}^2\right), \tag{33}$$

where $E \subset \mathbb{R}\times\mathbb{R}$ and $\operatorname{Im} P_E$ is the range of P_E. Also, we define

$$\|P_E P_u\| = \sup\left\{\|P_E P_u(F)\|_{2,\mu_\alpha\otimes\mu_\alpha} : F \in L^2_\alpha\left(\mathbb{R}^2\right), \|F\|_{2,\mu_\alpha\otimes\mu_\alpha} = 1\right\}.$$

We first need the following result.

Theorem 4.1. *Let $u \in L^2_\alpha(\mathbb{R})$ be a non-zero window function. Then for any $E \subset \mathbb{R}\times\mathbb{R}$ of finite measure $\mu_\alpha\otimes\mu_\alpha(E) < \infty$, the operator $P_E P_u$ is a Hilbert-Schmidt operator. Moreover, we have the following estimation*

$$\|P_E P_u\|^2 \leq \mu_\alpha \otimes \mu_\alpha(E).$$

Proof. since P_u is a projection onto a reproducing kernel Hilbert space, for any function $F \in L^2_\alpha(\mathbb{R}^{2d})$, the orthogonal projection P_u can be expressed as

$$P_u(F)(x,\xi) = \iint_{\mathbb{R}^2} F\left(x',\xi'\right) K_u\left(\left(x',\xi'\right);(x,\xi)\right) d\mu_\alpha(x') \otimes d\mu_\alpha(\xi'),$$

where $\mathcal{K}_u\left(\left(x',\xi'\right);(x,\xi)\right)$ is given in theorem 3.3, using the relation (33), we find that

$$P_E P_u(F)(x,\xi) = \iint_{\mathbb{R}^2} \chi_E(x,\xi) F\left(x',\xi'\right) \mathcal{K}_u\left(\left(x',\xi'\right);(x,\xi)\right) d\mu_\alpha(x') \otimes d\mu_\alpha(\xi').$$

This shows that the operator $P_E P_u$ is an integral operator with kernel

$$K\left(\left(x',\xi'\right);(x,\xi)\right) = \chi_E(x,\xi)\mathcal{K}_u\left(\left(x',\xi'\right);(x,\xi)\right).$$

Using the relation (24) and Fubini's theorem, we find that

$$\begin{aligned}\|P_\Sigma P_g\|^2_{HS} &= \iint_{\mathbb{R}^2}\iint_{\mathbb{R}^2} |\chi_\Sigma(x,\xi)|^2 \left|\mathcal{K}_u\left(\left(x',\xi'\right);(x,\xi)\right)\right|^2 d\mu_\alpha(x') \otimes d\mu_\alpha(\xi') d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) \\ &\leq \frac{\|u\|^4_{2,\mu_\alpha}}{\|u\|^4_{2,\mu_\alpha}} \iint_E d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) = \mu_\alpha \otimes \mu_\alpha(E) < \infty. \end{aligned} \tag{34}$$

Thus, the operator $P_E P_u$ is a Hilbert-Schmidt operator. Now, the proof follows from the fact that $\|P_E P_u\| \leq \|P_E P_u\|_{HS}$. □

In the following, we obtain the uncertainty principle for orthonormal sequences associated with the Hartley-Gabor transform.

Theorem 4.2. *Let $u \in L^2_\alpha(\mathbb{R})$ be a non-zero window function and $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R})$. Then for any subset $E \subset \mathbb{R}\times\mathbb{R}$ of finite measure $\mu_\alpha \otimes \mu_\alpha(E) < \infty$, we have*

$$\sum_{n=1}^{N}\Big(1 - \|\chi_{E^c}\mathcal{W}_u(\phi_n)\|_{2,\mu_\alpha\otimes\mu_\alpha}\Big) \le \mu_\alpha \otimes \mu_\alpha(E),$$

for every $N \in \mathbb{N}$.

Proof. Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2_\alpha(\mathbb{R}^2)$. Since $P_E P_u$ is a Hilbert-Schmidt operator, and satisfied the relation (34), we have

$$\sum_{n\in\mathbb{N}} \langle P_u P_E P_u e_n, e_n\rangle_{\mu_\alpha\otimes\mu_\alpha} = \|P_E P_u\|^2_{HS} \le \mu_\alpha \otimes \mu_\alpha(E) < \infty.$$

According to the paper [16], the positive operator $P_u P_E P_u$ is a trace class operator and we have

$$\operatorname{tr}(P_u P_E P_u) = \|P_E P_u\|^2_{HS} \le \mu_\alpha \otimes \mu_\alpha(E) < \infty$$

where $\operatorname{tr}(P_u P_E P_u)$ denotes the trace of the operator $P_u P_E P_u$. Since $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R})$, from the orthogonality relation (27), we obtain that $\{\mathcal{W}_u(\phi_n)\}_{n\in\mathbb{N}}$ is also an orthonormal sequence in $L^2_\alpha(\mathbb{R}^2)$ thus, we find that

$$\sum_{n=1}^{N} \langle P_E\mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha} \le \mu_\alpha \otimes \mu_\alpha(E) < \infty$$

Moreover, for any n with $1 \le n \le N$, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\langle P_E\mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha} =& 1 - \langle P_{E^c}\mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha}\\ \ge& 1 - \|\chi_{E^c}\mathcal{W}_u(\phi_n)\|_{2,\mu_\alpha\otimes\mu_\alpha}.\end{aligned}$$

Thus, we obtain

$$\sum_{n=1}^{N}\Big(1 - \|\chi_{E^c}\mathcal{W}_u(\phi_n)\|_{2,\mu_\alpha\otimes\mu_\alpha}\Big) \le \sum_{n=1}^{N} \langle P_E\mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n)\rangle_{\mu_\alpha\otimes\mu_\alpha} \le \mu_\alpha \otimes \mu_\alpha(E).$$

This completes the proof of the theorem. □

5. Extremal Functions Associated with the Hartley-Gabor Transform

By using the theory of reproducing kernels [22, 23], the main purpose of this section is to study the extremal functions associated with the Hartley-Gabor transform and to give an integral representation and best estimate of these functions on weighted Sobolev spaces.

5.1. Sobolev type spaces Associated with the Hartley-Bessel Transform.

Definition 5.1. Let $s \in \mathbb{R}$, we define the Hartley-Sobolev space of order s that will be denoted by

$$H_\alpha^s(\mathbb{R}) := \left\{ f \in L_\alpha^2(\mathbb{R}) / \left(1+|\lambda|^2\right)^{s/2} \mathcal{H}_\alpha(f) \in L_\alpha^2(\mathbb{R}) \right\}$$

We provide $H_\alpha^s(\mathbb{R})$ with the inner product given by

$$\langle f, g \rangle_{H_\alpha^s} := \int_{\mathbb{R}} \left(1+|\lambda|^2\right)^s \mathcal{H}_\alpha(f)(\lambda) \overline{\mathcal{H}_\alpha(g)(\lambda)} d\mu_\alpha(\lambda), \tag{35}$$

and the norm

$$\|f\|_{H_\alpha^s}^2 := \langle f, f \rangle_{H_\alpha^s} = \int_{\mathbb{R}} \left(1+|\lambda|^2\right)^s |\mathcal{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda), \tag{36}$$

Definition 5.2. Let u be a window function in $L_\alpha^2(\mathbb{R})$, we introduce the inner product in the Hilbert space $H_\alpha^s(\mathbb{R})$ for any fixed $\beta > 0$ by

$$\langle f, g \rangle_{H_{\beta,u}^s} := \beta \langle f, g \rangle_{H_\alpha^s} + \langle \mathcal{W}_u(f), \mathcal{W}_u(g) \rangle_{\theta_\alpha}. \tag{37}$$

The norm associated to this inner product is defined by

$$\|f\|_{H_{\beta,u}^s}^2 := \beta \|f\|_{H_\alpha^s}^2 + \|\mathcal{W}_u(f)\|_{2,\theta_\alpha}^2. \tag{38}$$

We have the following result:

Proposition 5.1. *For $s > \alpha + 1$ and u be a window function in $L_\alpha^2(\mathbb{R})$ and $\beta > 0$ then we have*

$$f \in H_{\beta,u}^s(\mathbb{R}) \quad \Rightarrow \quad \mathcal{H}_\alpha(f) \in L_\alpha^1(\mathbb{R}). \tag{39}$$

PROOF. By using the relations (11), (26), (35), and (37), we find that

$$\|f\|_{H_{\beta,u}^s}^2 = \int_{\mathbb{R}} \left[\beta \left(1+|\lambda|^2\right)^s + \|u\|_{2,\alpha}^2\right] |\mathcal{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \tag{40}$$

by using Hölder's inequality and the fact that $s > \alpha + 1$ we find that

$$\|\mathcal{H}_\alpha(f)\|_{1,\mu_\alpha} \leq \|f\|_{H_{\beta,u}^s} \left(\int_{\mathbb{R}} \frac{d\mu_\alpha(\lambda)}{\left[\beta\left(1+|\lambda|^2\right)^s + \|u\|_{2,\alpha}^2\right]} \right)^{\frac{1}{2}} < \infty$$

which gives the result. □

Theorem 5.2. *Let $s > \alpha + 1, u$ be a window function in $L_\alpha^2(\mathbb{R})$ and $\beta > 0$ then the space $\left(H_{\beta,u}^s(\mathbb{R}), \langle , \rangle_{H_{\beta,u}^s}\right)$ is a reproducing kernel Hilbert space with kernel given by*

$$\mathcal{K}_{\beta,u}(x, y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x) B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|_{2,\alpha}^2} d\mu_\alpha(\lambda)$$

that is for every $y \in \mathbb{R}$

(i) *the function* $x \to \mathcal{K}_{\beta,u}(x,y) \in H^s_{\beta,u}(\mathbb{R})$.

(ii) *for every* $f \in H^s_{\beta,u}(\mathbb{R})$ *and* $y \in \mathbb{R}$ *we have*

$$f(y) = \langle f, \mathcal{K}_{\beta,u}(\cdot,y)\rangle_{H^s_{\beta,u}}.$$

Proof. Let $y \in \mathbb{R}$, by using the fact that $s > \alpha+1$ and the relation (7), the function

$$\lambda \to \frac{B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^2_\alpha(\mathbb{R})$, by using Plancherel's theorem for the Hartley-Bessel transform, there exist a unique function in $L^2_\alpha(\mathbb{R})$, which we denote by $\mathcal{K}_{\beta,u}(\cdot,y)$ such that

$$\mathcal{H}_\alpha\left(\mathcal{K}_{\beta,u}(\cdot,y)\right)(\lambda) = \frac{B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}}, \tag{41}$$

by using inversion formula (10), we find that

$$\mathcal{K}_{\beta,u}(x,y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}} d\mu_\alpha(\lambda)$$

furthermore, by using the relations (7), (39), and (41), we find that

$$\|\mathcal{K}_{\beta,u}(\cdot,y)\|^2_{H^s_{\beta,u}} \leq \int_{\mathbb{R}} \frac{d\mu_\alpha(\lambda)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}} < \infty$$

which proves that $\mathcal{K}_{\beta,u}(\cdot,y) \in H^s_{\beta,u}(\mathbb{R})$, now let $f \in H^s_{\beta,u}(\mathbb{R})$ by using the relation (40), we find that

$$\langle f, \mathcal{K}_{\beta,u}(\cdot,y))\rangle_{H^s_{\beta,u}} = \int_{\mathbb{R}} \mathcal{H}_\alpha(f)(\lambda)B_\alpha(\lambda y)\mathrm{d}\mu_\alpha(\lambda),$$

inversion formula (10) gives the desired result. □

In the following we give the main result of this section.

Theorem 5.3. *Let* $s > \alpha+1, u$ *be a window function in* $L^2_\alpha(\mathbb{R})$ *and* $g \in L^2_\alpha\left(\mathbb{R}^2\right)$, $\beta > 0$ *then the infimum*

$$\inf_{f \in H^s_\alpha(\mathbb{R})} \left\{\beta\|f\|^2_{H^s_\alpha} + \|g - \mathcal{W}_u(f)\|^2_{2,\theta_\alpha}\right\} \tag{42}$$

is attained by a unique function $f^*_{g,u,\beta}$ *given exactly by*

$$f^*_{g,u,\beta}(y) = \iint_{\mathbb{R}^2} g(x,z)\phi_{u,\beta}(x,y,z)d\mu_\alpha(x) \otimes d\mu_\alpha(z)), \tag{43}$$

where $\phi_{u,\beta}$ *is given by*

$$\phi_{u,\beta}(x,y,z) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)\sqrt{\tau^z_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}} d\mu_\alpha(\lambda). \tag{44}$$

PROOF. The existence and unicity of the extremal function $f^*_{g,u,\beta}$, satisfying the relation (42) is given in [23] and this function is given by the following relation

$$f^*_{g,u,\beta}(y) = \langle g, \mathcal{W}_u(\mathcal{K}_{\beta,u}(\cdot, y))\rangle_{\theta_\alpha}, \tag{45}$$

where $\mathcal{K}_{\beta,u}$ is the kernel function given the relation (41), on the other hand, by using the relations (11),(18),(20) and (22) we find that

$$\mathcal{W}_u(\mathcal{K}_{\beta,u}(\cdot, y)(x,z) = \int_{\mathbb{R}} \mathcal{H}_\alpha(\mathcal{K}_{\beta,u}(\cdot, y))(\lambda)\overline{\mathcal{H}_\alpha(u^{x,z})(\lambda)}\mathrm{d}\mu_\alpha(\lambda)$$

$$\phi_{u,\beta}(x,y,z) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda x)\sqrt{\tau^z_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}}\, d\mu_\alpha(\lambda) \tag{46}$$

by using the relations (45) and (46), we find the result. □

We have the following results:

Theorem 5.4. *Let $s > \alpha+1$, u be a window function in $L^2_\alpha(\mathbb{R})$ and $g \in L^2_\alpha(\mathbb{R}^2)$, $\beta > 0$ then we have*

(i)

$$f^*_{g,u,\beta}(y) = \iint_{\mathbb{R}^2} \frac{B_\alpha(\lambda y)\mathcal{H}_\alpha\left(g(\cdot, z)\right)(\lambda)\sqrt{\tau^z_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}}\, d\mu_\alpha(\lambda)\otimes d\mu_\alpha(z) \tag{47}$$

(ii)

$$\mathcal{H}_\alpha(f^*_{g,u,\beta})(\lambda) = \int_{\mathbb{R}} \frac{\mathcal{H}_\alpha\left(g(\cdot, z)\right)(\lambda)\sqrt{\tau^z_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}}\, d\mu_\alpha(z) \tag{48}$$

(iii)

$$\|f^*_{g,u,\beta}\|_{H^s_\alpha} \le \frac{\|g\|_{2,\theta_\alpha}\|u\|_{2,\mu_\alpha}}{\beta}. \tag{49}$$

PROOF. (i) Is a consequence of (43), (44) and Fubini's theorem.
(ii) Is a consequence of Fubini's theorem and the relations (47) and (11).
(iii) By using the relation (36), we find that

$$\|f^*_{g,u,\beta}\|^2_{H^s_\alpha} = \int_{\mathbb{R}} \left(1+|\lambda|^2\right)^s \left|\mathcal{H}_\alpha(f^*_{g,u,\beta})(\lambda)\right|^2 d\mu_\alpha(\lambda)$$

by using Holder's inequality and the relations (11), (13) and (48), we find that

$$\left|\mathcal{H}_\alpha(f^*_{g,u,\beta})(\lambda)\right|^2 \le \frac{\|u\|^2_{2,\alpha}}{(\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha})^2} \int_{\mathbb{R}} |g(\lambda, z)|^2\, d\mu_\alpha(z)$$

so we find that

$$\|f^*_{g,u,\beta}\|^2_{H^s_\alpha} \le \frac{\left(1+|\lambda|^2\right)^s \|u\|^2_{2,\alpha}\|g\|^2_{2,\theta_\alpha}}{(\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha})^2} \le \frac{\|g\|^2_{2,\theta_\alpha}\|u\|^2_{2,\mu_\alpha}}{\beta^2},$$

which gives the desired result. □

Corollary 5.5. *Let $s>\alpha+1, u$ be a window function in $L^2_\alpha(\mathbb{R})$ and $\beta>0$, for all $f\in H^s_\alpha(\mathbb{R})$ and $g=\mathcal{W}_u(f)$, the extremal function $f^*_{\mathcal{W}_u(f),u,\beta}$ satisfies the following properties*

(i)

$$\mathcal{H}_\alpha(f^*_{\mathcal{W}_u(f),u,\beta})(\lambda)=\frac{\mathcal{H}_\alpha(f)(\lambda)\|u\|^2_{2,\alpha}}{\beta\left(1+|\lambda|^2\right)^s+\|u\|^2_{2,\alpha}}. \tag{50}$$

(ii)

$$\|f^*_{\mathcal{W}_u(f),u,\beta}\|_{H^s_\alpha}\leq\frac{\|f\|_{2,\mu_\alpha}\|u\|_{2,\mu_\alpha}}{\beta^2}. \tag{51}$$

Proof. (i) By using the relations (23) and (48), we find that

$$\begin{aligned}\mathcal{H}_\alpha(f^*_{\mathcal{W}_u(f),u,\beta})(\lambda)&=\int_{\mathbb{R}}\frac{\mathcal{H}_\alpha\left(\mathcal{W}_u(f)(\cdot,z)\right)(\lambda)\sqrt{\tau^z_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}}{\beta\left(1+|\lambda|^2\right)^s+\|u\|^2_{2,\alpha}}d\mu_\alpha(z)\\&=\int_{\mathbb{R}}\frac{\mathcal{F}_\alpha\left(\overline{u^z}\right)(\lambda)\mathcal{H}_\alpha\left(f\right)(\lambda)\sqrt{\tau^z_\alpha\left(|\mathcal{H}_\alpha(u)|^2\right)(\lambda)}}{\beta\left(1+|\lambda|^2\right)^s+\|u\|^2_{2,\alpha}}d\mu_\alpha(z)\end{aligned}$$

the relation (14) and Plancherel's formula (11) gives the relation (50).

(ii) By using the relations (27) and (49) we find the relation (51). □

Theorem 5.6 (Second Calderon's reproducing formula)**.** *Let $s>\alpha+1, u$ be a window function in $L^2_\alpha(\mathbb{R})$ and $\beta>0$, for all $f\in H^s_\alpha(\mathbb{R})$ the extremal function $f^*_{\mathcal{W}_u(f),u,\beta}$ satisfies*

$$\lim_{\beta\to0^+}\left\|f^*_{\mathcal{W}_u(f),u,\beta}-f\right\|_{H^s_\alpha}=0.$$

Moreover we have $f^*_{\mathcal{W}_u(f),u,\beta}\longrightarrow f$ uniformly when $\beta\longrightarrow0^+$.

Proof. By using the relation (50), we find that

$$\mathcal{H}_\alpha(f^*_{\mathcal{W}_u(f),u,\beta}-f)(\lambda)=\frac{-\beta\left(1+|\lambda|^2\right)^s\mathcal{H}_\alpha(f)(\lambda)}{\beta\left(1+|\lambda|^2\right)^s+\|u\|^2_{2,\alpha}} \tag{52}$$

consequently we find that

$$\left\|f^*_{\mathcal{W}_u(f),u,\beta}-f\right\|_{H^s_\alpha}=\int_{\mathbb{R}}\frac{\beta^2\left(1+\|\lambda\|\right)^{3s}|\mathcal{H}_\alpha(f)(\lambda)|^2}{\beta\left(1+|\lambda|^2\right)^s+\|u\|^2_{2,\alpha}}d\mu_\alpha(\lambda)$$

by using the dominated convergence theorem and the fact that

$$\frac{\beta^2\left(1+\lambda|^2\right)^{3s}|\mathcal{H}_\alpha(f)(\lambda)|^2}{\beta\left(1+|\lambda|^2\right)^s+\|u\|^2_{2,\alpha}}\leq\left(1+|\lambda|^2\right)^s|\mathcal{H}_\alpha(f)(\lambda)|^2,$$

we deduce that

$$\lim_{\beta\to0^+}\left\|f^*_{\mathcal{W}_u(f),u,\beta}-f\right\|_{H^s_\alpha}=0$$

on the other hand by using inversion formula (10) and the relation (52), we find that

$$\begin{aligned} f^*_{\mathcal{W}_u(f),u,\beta}(y) - f(y) &= \int_{\mathbb{R}} \mathcal{H}_\alpha(f^*_{\mathcal{W}_u(f),u,\beta} - f)(\lambda) B_\alpha(\lambda y) \mathrm{d}\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} \frac{-\beta\left(1+|\lambda|^2\right)^s \mathcal{H}_\alpha(f)(\lambda) B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}} d\mu_\alpha(\lambda) \end{aligned}$$

again by dominated convergence theorem and the fact that

$$\left|\frac{-\beta\left(1+\|\lambda\|\right)^s \mathcal{H}_\alpha(f)(\lambda) B_\alpha(\lambda y)}{\beta\left(1+|\lambda|^2\right)^s + \|u\|^2_{2,\alpha}}\right| \le |\mathcal{H}_\alpha(f)(\lambda)|$$

we deduce that

$$\lim_{\beta\to 0^+} \left\| f^*_{\mathcal{W}_u(f),u,\beta} - f \right\|_{\infty,\alpha} = 0$$

which proves that $f^*_{\mathcal{W}_u(f),u,\beta} \longrightarrow f$ uniformly when $\beta \longrightarrow 0^+$. □

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