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$*$-woven frames of multipliers in Hilbert C^*-modules

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Abstract. In the current article, by using the sequence of adjointable operators from a C^*-algebra $\mathcal{A}$ into Hilbert $\mathcal{A}$-module $\mathcal{H}$, the $*$-woven frames of multipliers in Hilbert C^*-modules are introduced. Applying invertible elements in $L(\mathcal{H})$, the set of all bounded $\mathcal{A}$-module maps from $\mathcal{H}$ into itself, and co-isometry mappings, some new $*$-woven frames of multipliers in Hilbert $\mathcal{A}$-module $\mathcal{H}$ are constructed. Moreover, a woven frame of multipliers in $\mathcal{A}$ as a Hilbert C^*-module is indicated.

1. Introduction

The story of frames was commenced by Duffin and Schaeffer in [10] in which they used the frames as a generalization of bases in Hilbert space to deal with some problems in the non-harmonic Fourier series. Next, Daubechies et al. [8] reintroduced the notions of frames and characterized function spaces. In other words, they replaced the sequence of bounded linear operators instead of the sequence of elements in Hilbert space; see also [10]. Frames have many applications, such as study and characterization of function spaces, signal and image processing, wireless communications, transceiver design, data compression and so on. The theory of frames was generalized to different vectors in Hilbert spaces; for instance, see [15], [20] and [22]. After that, Frank et al. investigated the concept of frames in Hilbert

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C^*-modules as a generalization of frames in Hilbert spaces; for the details see [11]. More results for the generalizations of frames in Hilbert C^*-modules are available in [14].

The notion of woven frames in Hilbert space has been introduced and studied by Bemrose et al. [3], and more deeply investigated in [4], [5].

By allowing the inner product to take values in a C^*-algebra, Hilbert C^*-modules are natural generalizations of Hilbert spaces. Note that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. For example, the Riesz representation theorem for continuous linear functionals on Hilbert spaces does not extend to Hilbert C^*-modules [21] and there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [17]. Moreover, every bounded operator on a Hilbert space has an adjoint, while there are bounded operators on Hilbert C^*-modules which do not have any [18]. Thus, there are many essential differences between Hilbert space frames and modular frames. The problems with modular frames are more complicated than those in Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. The concept of woven frames is partially motivated by the preprocessing of Gabor frames and has potential applications in wireless sensor networks that require distributed processing under different frames [12]. More information about Hilbert C^*-modules and their applications to the study of locally compact quantum groups, complete maps between C^*-algebras, noncommutative geometry, and KK-theory. Woven frames for finitely or countably generated Hilbert C^*-modules are introduced and studied in [13].

In this paper, we introduce the $*$-woven frame of multipliers and give a concrete example. We make a $*$-woven frame employing an invertible element in $L(\mathcal{H})$, the set of all bounded $\mathcal{A}$-module maps from $\mathcal{H}$ into itself, where $\mathcal{A}$ is a C^*-algebra. We also show that the images of $*$-woven frames are again the $*$-woven frames under a co-isometry map. Furthermore, we shall construct a new woven frame of multipliers in Hilbert C^*-module $\mathcal{A}$.

2. Basic definitions and Preliminaries

Let I be a countable index set. Throughout this paper, we assume that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. For information about frames in Hilbert spaces we refer to [6]. In what follows, all definitions about C^*-algebras are taken from [7] and [9]. An element a in $\mathcal{A}$ is called *positive* and denoted by $a \geq 0$ if $a = a^*$ and $\sigma(a) \subset R^+$, where $\sigma(a)$ is the spectrum of a. Moreover, $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$. The nonzero element a is called *strictly* nonzero if zero does not belong to $\sigma(a)$, and a is said to be *strictly positive* if it is strictly nonzero and positive. The absolute value of a is defined and denoted by $|a| := (a^*a)^{\frac{1}{2}}$. The relation $\leq$ given by $a \leq b$ if and only if $b - a$ is positive, defines a partial ordering on $\mathcal{A}$. Some elementary facts about $\leq$ are given in the following statements for each $a, b, c \in \mathcal{A}$.

(i) $a \leq \|a\|$;
(ii) $0 \leq a \leq b$ implies $\|a\| \leq \|b\|$, $ab \geq 0$, $a + b \geq 0$, and $at \leq bt$ for $t \in (0, 1)$;
(iii) If $a \leq b$, then $cac^* \leq cbc^*$. Moreover, if c commutes with a and b, then $ca \leq cb$ for $c \geq 0$;
(iv) If a and b are positive invertible elements and $a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$.

In this paper, the notation $a < b$ denotes $a \leq b$ with $a \neq b$.

Definition 2.1. [16] Let $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. Then, $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if it is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words

(i) $\langle x, y \rangle^* = \langle y, x \rangle$;
(ii) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;
(iii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, then it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle^{\frac{1}{2}}$. A C^*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b \rangle = ab^*$. The standard Hilbert $\mathcal{A}$-module $l_2(\mathcal{A})$ is defined by

$$l_2(A) := \{\{a_j\}_{j \in N} \subseteq A : \sum_{j \in N} a_j a_j^* \text{ converges in } \mathcal{A}\},$$

with $\mathcal{A}$-inner product $\langle \{a_j\}_{j \in N}, \{b_j\}_{j \in N} \rangle = \sum_{j \in N} a_j b_j^*$.

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$, $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. An $\mathcal{A}$-module map $T : \mathcal{H} \longrightarrow \mathcal{K}$ is said to be *adjointable* if there exists an operator $T^* : \mathcal{K} \longrightarrow \mathcal{H}$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

holds for all $x \in \mathcal{H}$, $y \in \mathcal{K}$. We denote by $\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$, the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and moreover $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}) = \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$.

The class of all adjointable maps from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $L(\mathcal{H}, \mathcal{K})$, the set of all bounded $\mathcal{A}$-module maps from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $B(\mathcal{H}, \mathcal{K})$. It is known that $L(\mathcal{H}, \mathcal{K}) \subseteq B(\mathcal{H}, \mathcal{K})$. We denote $L(\mathcal{H}, \mathcal{H})$ and $B(\mathcal{H}, \mathcal{H})$ with $L(\mathcal{H})$ and $B(\mathcal{H})$, respectively.

For C^*-algebra $\mathcal{A}$ and Hilbert $\mathcal{A}$-module $\mathcal{H}$, the set $L(\mathcal{A}, \mathcal{H})$ is a Hilbert $L(\mathcal{A})$-module with the action of $L(\mathcal{A})$ on $L(\mathcal{A}, \mathcal{H})$ defined by $t.s = t \circ s$, for $\langle t, s \rangle = t^* \circ s$. Since $L(\mathcal{A})$ can be identified with the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ ([19]),

$L(\mathcal{A},\mathcal{H})$ becomes a Hilbert $M(\mathcal{A})$-module, called the multiplier module of $\mathcal{H}$, and it is denoted by $M(\mathcal{H})$. For each $h \in M(\mathcal{H})$ and $x \in \mathcal{H}$, we denote $\langle h, x\rangle_{M(\mathcal{H})} = h^*(x)$.

To achieve our purpose in this paper we need the following lemma that illustrates lower and upper bounds of operators corresponding to a given operator T concerning $\mathcal{A}$-valued inner products.

Lemma 2.1. [1] *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in L(\mathcal{H},\mathcal{K})$.*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \le T^*T \le \|T\|^2$;*

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \le TT^* \le \|T\|^2.$$

Definition 2.2. Let $\mathcal{H}$ be a Hilbert C^*-module. A sequence $\{h_i\}_{i\in I}$ in $M(\mathcal{H})$ is called a standard frame of multipliers in $\mathcal{H}$ if for each $x \in \mathcal{H}$, the series

$$\sum_{i\in I}\langle x, h_i\rangle_{M(\mathcal{H})}\langle h_i, x\rangle_{M(\mathcal{H})}$$

is convergent in $\mathcal{A}$ and there exist two positive nonzero numbers C and D in such that

$$C\langle x, x\rangle_{\mathcal{H}} \le \sum_{i\in I}\langle x, h_i\rangle_{M(\mathcal{H})}\langle h_i, x\rangle_{M(\mathcal{H})} \le D\langle x, x\rangle_{\mathcal{H}},$$

for all $x \in \mathcal{H}$. The elements C and D are called *frame bounds* for $\{h_i\}_{i\in I}$. The frame of multipliers is called *tight* if $C = D$ and called a *Parseval* or *normalized* if $C = D = 1$. If in the above we only need to have the upper bound, then $\{h_i\}_{i\in I}$ is called a *Bessel* sequence.

It should be noted that $*$-frames are C^*-algebra version of frames. In other words, it was used the strictly positive elements of C^*-algebra $\mathcal{A}$ instead of positive real numbers.

Definition 2.3. Let $\mathcal{H}$ be a Hilbert C^*-module. A sequence $\{h_i\}_{i\in I}$ in $M(\mathcal{H})$ is called a *standard* $*$-frame of multipliers in $\mathcal{H}$ if for each $x \in \mathcal{H}$, the series $\sum_{i\in I}\langle x, h_i\rangle_{M(\mathcal{H})}\langle h_i, x\rangle_{M(\mathcal{H})}$ is convergent in $\mathcal{A}$ and there exist two strictly nonzero elements C and D in $\mathcal{A}$ such that

$$C\langle x, x\rangle_{\mathcal{H}}C^* \le \sum_{i\in I}\langle x, h_i\rangle_{M(\mathcal{H})}\langle h_i, x\rangle_{M(\mathcal{H})} \le D\langle x, x\rangle_{\mathcal{H}}D^*,$$

for all $x \in \mathcal{H}$. The elements C and D are called $*$-frame bounds for $\{h_i\}_{i\in I}$. The frame of multipliers is called tight if $C = D$ and called a Parseval or normalized if $C = D = 1$. If in the above we only need to have the upper bound, then $\{h_i\}_{i\in I}$ is called a $*$-Bessel sequence.

3. The $*$- woven frame of multipliers

In this section, the woven frame and $*$-woven frame of multipliers are introduced and an example of these frames is presented.

Let the sequence $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers in $\mathcal{H}$ with lower and upper $*$-frame bounds C and D, respectively. We can define a linear map $T : \mathcal{H} \longrightarrow l_2(\mathcal{A})$ by $T(x) = \{\langle x, h_i\rangle_{M(\mathcal{H})}\}_{i\in I}$ which is called the *pre-$*$-frame operator* or *$*$-frame transform* for $\{h_i\}_{i\in I}$. The pre-$*$-frame operator T is invertible and $\|T\| \leq \|D\|$ and also its adjoint operator $T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} h_i a_i$ is called the *synthesis operator*. We remind that the frame operator $Sx = T^*Tx = \sum_{i\in I} h_i\langle h_i, x\rangle_{M(\mathcal{H})}$ is bounded, positive, adjointable and invertible. For each $x \in \mathcal{H}$, the inequality $\|C^{-1}\|^2 \leq \|S\| \leq \|D\|^2$ holds, and moreover the reconstruction formula $x = \sum_{i\in I}\langle x, S^{-1}x_i\rangle x_i$ holds $x \in \mathcal{H}$.

Definition 3.1. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. Then, two standard frames of multipliers $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ for $\mathcal{H}$ are called *woven* if there exist constants $0 < A \leq B < \infty$ such that for every $\sigma \subset I$, the family $\{h_i\}_{i\in\sigma} \bigcup \{t_i\}_{i\in\sigma^c}$ is a standard frame of multipliers for $\mathcal{H}$ with lower and upper bounds A and B, respectively.

Definition 3.2. For C^*-algebra $\mathcal{A}$ and Hilbert $\mathcal{A}$-module $\mathcal{H}$, two standard frames of multipliers $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ for $\mathcal{H}$ are called the *weakly woven* if for every $\sigma \subset I$, the family $\{h_i\}_{i\in\sigma} \bigcup \{t_i\}_{i\in\sigma^c}$ is a frame of multipliers for $\mathcal{H}$, and each $\{h_i\}_{i\in\sigma} \bigcup \{t_i\}_{i\in\sigma^c}$ is said to be a *weaving*.

It is clear that if $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ are the woven frame of multipliers in $\mathcal{H}$, then they are a weakly woven frame of multipliers in $\mathcal{H}$.

Proposition 3.1. [13] *Suppose that $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ are Bessel sequences of multipliers in $\mathcal{H}$ with Bessel bounds B_1 and B_2, respectively. Then, for any subset σ of I, the family $\{h_i\}_{i\in\sigma} \bigcup \{t_i\}_{i\in\sigma^c}$ is a Bessel sequence of multipliers with Bessel bound $B_1 + B_2$.*

Example 3.3. Let $\mathcal{A}$ be a C^*-algebra and $l^2(\mathcal{A})$ be a Hilbert $\mathcal{A}$-module with inner product $\langle\{a_m\}_m, \{a_m\}_m\rangle = \sum_{m\in I} a_m a_m^*$. Consider the mapping $h_i : \mathcal{A} \to l^2(\mathcal{A})$ defined by $h_i(a) = (\overbrace{0, \ldots, 0}^{i-1-\text{times}}, a, 0, \ldots)$. We have $h_i^*(\{a_m\}_m) = a_i$. Moreover, for each $n \in N$, define the families $\{s_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ in $M(l^2(\mathcal{A}))$ via

$$\{s_i\}_{i\in I} = \{h_1, nh_1, h_2, \frac{n}{2}h_2, h_3, \frac{n}{3}h_3, \ldots\}$$

and

$$\{t_i\}_{i\in I} = \{h_1, nh_1, h_2, \frac{n}{2}h_2, h_3, \frac{n}{2}h_3, h_4, \frac{n}{3}h_4, h_5, \frac{n}{3}h_5, \ldots\}.$$

Then, for each $\{a_m\}_m \in l^2(\mathcal{A})$, the sequences $\{s_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ are frames of multipliers in $l^2(\mathcal{A})$ with lower and upper bounds 1 and $(1 + n)$, respectively. In addition, for any subset σ of I, $\{s_i\}_{i\in\sigma}\bigcup\{t_i\}_{i\in\sigma^c}$ is a frames of multipliers in $l^2(\mathcal{A})$ with lower and upper bounds 1 and $(1+n)$, respectively.

Definition 3.4. For C^*-algebra $\mathcal{A}$ and Hilbert $\mathcal{A}$-module $\mathcal{H}$, two standard $*$-frames of multipliers $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ for $\mathcal{H}$ are *woven* if there exist two strictly nonzero elements C and D in $\mathcal{A}$ such that for every $\sigma\subset I$, the family $\{h_i\}_{i\in\sigma}\bigcup\{t_i\}_{i\in\sigma^c}$ is a standard $*$-frame of multipliers for $\mathcal{H}$, with bounds C, D. In this case, we say $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ are (C,D)-$*$-woven frame of multipliers in $\mathcal{H}$.

Theorem 3.2. *Given $j\in\{1,2\}$. Let $\{h_{ji}\}_{i\in I}$ be the (C,D)-$*$- woven frame of multipliers in $\mathcal{H}$. If T is a surjective element in $L(\mathcal{H})$, then $\{Th_{ji}\}_{i\in I}\,(j=1,2)$ are also $(C\|(TT^*)^{-1}\|^{\frac{-1}{2}}, D\|T\|)$-$*$-woven in $\mathcal{H}$.*

Proof. By our assumption and Lemma 2.1 for every $\sigma\subset I$ and every $x\in\mathcal{H}$ we have

$$\begin{aligned}
&\sum_{i\in\sigma}\langle x,Th_{1i}\rangle\langle Th_{1i},x\rangle_{M(\mathcal{H})}+\sum_{i\in\sigma^c}\langle x,Th_{2i}\rangle\langle Th_{2i},x\rangle_{M(\mathcal{H})}\\
=\;&\sum_{i\in\sigma}\langle T^*x,h_{1i}\rangle\langle T^*h_{1i},x\rangle_{M(\mathcal{H})}+\sum_{i\in\sigma^c}\langle T^*x,h_{2i}\rangle\langle h_{2i},T^*x\rangle_{M(\mathcal{H})}\\
\le\;& D\langle T^*x,T^*x\rangle_{\mathcal{H}}D^*\\
\le\;& D\|T\|^2\langle x,x\rangle_{\mathcal{H}}D^*\\
=\;&(D\|T\|)\langle x,x\rangle_{\mathcal{H}}(D\|T\|)^*.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\sum_{i\in\sigma}\langle x,Th_{1i}\rangle\langle Th_{1i},x\rangle_{M(\mathcal{H})}+\sum_{i\in\sigma^c}\langle x,Th_{2i}\rangle\langle Th_{2i},x\rangle_{M(\mathcal{H})}\\
=\;&\sum_{i\in\sigma}\langle T^*x,h_{1i}\rangle\langle T^*h_{1i},x\rangle_{M(\mathcal{H})}+\sum_{i\in\sigma^c}\langle T^*x,h_{2i}\rangle\langle h_{2i},T^*x\rangle_{M(\mathcal{H})}\\
\ge\;& C\langle T^*x,T^*x\rangle_{\mathcal{H}}C^*\\
\ge\;& C\|(TT^*)^{-1}\|^{-1}\langle x,x\rangle_{\mathcal{H}}C^*\\
=\;&(C\|(TT^*)^{-1}\|^{\frac{-1}{2}})\langle x,x\rangle_{\mathcal{H}}(C\|(TT^*)^{-1}\|^{\frac{-1}{2}})^*.
\end{aligned}$$

Plugging the relations above, we see that the family $\{Th_{ji}\}_{i\in I}$ $(j=1,2)$ are the $*$-woven frame of multipliers in $\mathcal{H}$. □

Corollary 3.3. *Given $j\in\{1,2\}$. Let $\{h_{ji}\}_{i\in I}$ in $M(\mathcal{A})$ be the (C,D)-$*$-woven frame of multipliers in $\mathcal{H}$. If T is a surjective multiplier in $M(\mathcal{H})$, then $\{Th_{ji}\}_{i\in I}\,(j=1,2)$ are also $(C\|(TT^*)^{-1}\|^{\frac{-1}{2}}, D\|T\|)$-$*$-woven in $\mathcal{H}$.*

In the following theorem, we construct a $*$-woven frame by means of an invertible element in $L(\mathcal{H})$.

Theorem 3.4. *For $j = 1, 2$, suppose that $\{h_{ji}\}_{i\in I}$ are the sequence in $M(\mathcal{H})$. If there exists an invertible map $V \in L(\mathcal{H})$ such that $\{Vh_{ji}\}_{i\in I}$ is a (C, D)-$*$-woven frame of multipliers in $\mathcal{H}$, then $\{h_{ji}\}_{i\in I}\,(j = 1, 2)$ are the $(C\|V^*\|, D\|(V^*)^{-1})$-$*$-woven frame of multipliers in $\mathcal{H}$.*

Proof. Since V is an invertible element in $L(\mathcal{H})$, for any $x \in \mathcal{H}$ we have

$$\|V^*\|^2\langle x, x\rangle_{\mathcal{H}} \le \langle (V^*)^{-1}x, (V^*)^{-1}x\rangle_{\mathcal{H}} \le \|(V^*)^{-1}\|^2\langle x, x\rangle_{\mathcal{H}}. \tag{1}$$

On the other hand, for any subset σ of I, we get

$$\begin{aligned} C\langle (V^*)^{-1}x, (V^*)^{-1}x\rangle_{\mathcal{H}}C^* &\le \sum_{i\in\sigma}\langle (V^*)^{-1}x, Vh_{1i}\rangle_{M(\mathcal{H})}\langle Vh_{1i}, (V^*)^{-1}x\rangle_{M(\mathcal{H})} \\ &\quad+\sum_{i\in\sigma^c}\langle (V^*)^{-1}x, Vh_{2i}\rangle_{M(\mathcal{H})}\langle Vh_{2i}, (V^*)^{-1}x\rangle_{M(\mathcal{H})} \\ &= \sum_{i\in\sigma}\langle x, V^{-1}Vh_{1i}\rangle\langle V^{-1}Vh_{1i}, x\rangle_{M(\mathcal{H})} \\ &\quad+\sum_{i\in\sigma^c}\langle x, V^{-1}Vh_{2i}\rangle\langle V^{-1}Vh_{2i}, x\rangle_{M(\mathcal{H})} \\ &= \sum_{i\in\sigma}\langle x, h_{1i}\rangle\langle h_{1i}, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c}\langle x, h_{2i}\rangle\langle h_{2i}, x\rangle_{M(\mathcal{H})}. \end{aligned}$$

Now (1) and the above relation necessitate that

$$(C\|V^*\|)\langle x, x\rangle_{\mathcal{H}}(C\|V^*\|)^* \le \sum_{i\in\sigma}\langle x, h_{1i}\rangle\langle h_{1i}, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c}\langle x, h_{2i}\rangle\langle h_{2i}, x\rangle_{M(\mathcal{H})}.$$

In a similar way, we obtain

$$\sum_{i\in\sigma}\langle x, h_{1i}\rangle\langle h_{1i}, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c}\langle x, h_{2i}\rangle\langle h_{2i}, x\rangle_{M(\mathcal{H})} \le (D\|(V^*)^{-1})\|\langle x, x\rangle_{\mathcal{H}}(D\|(V^*)^{-1})^*.$$

Therefore, $\{h_{ji}\}_{i\in I}$ are the $(C\|V^*\|, D\|(V^*)^{-1})$-$*$-woven frame of multipliers in $\mathcal{H}$. □

The images of $*$-woven frames are again the $*$-woven frames under a co-isometry map as follows.

Proposition 3.5. *Let $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ be the (C, D)-$*$-woven frames of multipliers in $\mathcal{H}$. If $V : \mathcal{H} \longrightarrow \mathcal{K}$ is a co-isometry map, then $\{Vh_i\}_{i\in I}$ and $\{Vt_i\}_{i\in I}$ are the (C, D)-$*$-woven frames of multipliers in $\mathcal{K}$.*

Proof. For every subset σ of I and each $y \in \mathcal{K}$, being the co-isometrically of $\mathcal{K}$ implies that

$$\begin{aligned} C\langle y, y\rangle_{\mathcal{K}}C^* &= C\langle V^*y, V^*y\rangle_{\mathcal{H}}C^* \\ &\le \sum_{i\in\sigma}\langle V^*y, h_{1i}\rangle\langle h_{1i}, V^*y\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c}\langle V^*y, t_{2i}\rangle\langle t_{2i}, V^*y\rangle_{M(\mathcal{H})} \\ &\le D\langle V^*y, V^*y\rangle_{\mathcal{H}}D^* = D\langle y, y\rangle_{\mathcal{H}}D^*. \end{aligned}$$

This shows that

$$\begin{aligned} C\langle y,y\rangle_{\mathcal{K}}C^* &\leq \sum_{i\in\sigma}\langle y,Vh_{1i}\rangle\langle Vh_{1i},y\rangle_{M(\mathcal{K})}+\sum_{i\in\sigma^c}\langle y,Vt_{2i}\rangle\langle Vt_{2i},y\rangle_{M(\mathcal{K})} \\ &\leq D\langle y,y\rangle_{\mathcal{K}}D^*. \end{aligned}$$

Therefore, $\{Vh_i\}_{i\in I}$ and $\{Vt_i\}_{i\in I}$ are the (C,D)-$*$-woven frames of multipliers in $\mathcal{K}$. ☐

We recall that the centralizer of an algebra $\mathcal{A}$ is the set $Z(\mathcal{A})=\{a\in\mathcal{A}: ab=ba, \forall b\in\mathcal{A}\}$. In the next result, we are going to construct a new woven frame of multipliers in Hilbert C^*-module $\mathcal{A}$.

Theorem 3.6. *Let $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ be the (C,D)-$*$-woven frames of multipliers in $\mathcal{H}$ with $C,D\in Z(\mathcal{A})$. Suppose that x is an element in $\mathcal{H}$ such that $\langle x,x\rangle_{\mathcal{H}}$ is an invertible element in the center of $\mathcal{A}$. Then, $\{\langle h_i,x\rangle_{M(\mathcal{H})}\}_{i\in I}$ and $\{\langle t_i,x\rangle_{M(\mathcal{H})}\}_{i\in I}$ are the $*$-woven frames in $\mathcal{A}$.*

Proof. For every subset σ of I and $a\in\mathcal{A}$, by the definition of $*$-woven, we have

$$\begin{aligned} aC\langle x,x\rangle_{\mathcal{H}}C^*a^* &\leq a\Big(\sum_{i\in\sigma}\langle x,h_i\rangle\langle h_i,x\rangle_{M(\mathcal{H})}\sum_{i\in\sigma^c}\langle x,t_i\rangle\langle t_i,x\rangle_{M(\mathcal{H})}\Big)a^* \\ &\leq aD\langle x,x\rangle_{\mathcal{H}}D^*a^*, \end{aligned}$$

and also

$$\begin{aligned} a\Big(\sum_{i\in\sigma}\langle x,h_i\rangle\langle h_i,x\rangle_{M(\mathcal{H})}+\sum_{i\in\sigma^c}\langle x,t_i\rangle\langle t_i,x\rangle_{M(\mathcal{H})}\Big) &= \sum_{i\in\sigma}\langle a,\langle h_i,x\rangle_{M(\mathcal{H})}\rangle_{\mathcal{A}}\langle\langle x,h_i\rangle_{M(\mathcal{H})},a\rangle_{\mathcal{A}} \\ &\quad+\sum_{i\in\sigma^c}\langle a,\langle t_i,x\rangle_{M(\mathcal{H})}\rangle_{\mathcal{A}}\langle\langle x,t_i\rangle_{M(\mathcal{H})},a\rangle_{\mathcal{A}}. \end{aligned}$$

Since $\sqrt{\langle x,x\rangle_{\mathcal{H}}}$ and C,D are in the center of $\mathcal{A}$ ([2, Theorem 6.2.10]), the following inequalities are valid for each $a\in\mathcal{A}$,

$$\begin{aligned} \Big(C\sqrt{\langle x,x\rangle_{\mathcal{H}}}\big)\langle a,a\rangle_{\mathcal{A}}(C\sqrt{\langle x,x\rangle_{\mathcal{H}}}\Big)^* &\leq \sum_{i\in\sigma}\langle a,\langle h_i,x\rangle_{M(\mathcal{H})}\rangle_{\mathcal{A}}\langle\langle x,h_i\rangle_{M(\mathcal{H})},a\rangle_{\mathcal{A}} \\ &\quad+\sum_{i\in\sigma^c}\langle a,\langle t_i,x\rangle_{M(\mathcal{H})}\rangle_{\mathcal{A}}\langle\langle x,t_i\rangle_{M(\mathcal{H})},a\rangle_{\mathcal{A}} \\ &\leq \Big(D\sqrt{\langle x,x\rangle_{\mathcal{H}}}\big)\langle a,a\rangle_{\mathcal{A}}(D\sqrt{\langle x,x\rangle_{\mathcal{H}}}\Big)^*. \end{aligned}$$

The last inequality shows that $\{\langle h_i,x\rangle_{M(\mathcal{H})}\}_{i\in I}$ and $\{\langle t_i,x\rangle_{M(\mathcal{H})}\}_{i\in I}$ are the $*$-woven frames in $\mathcal{A}$ and the proof is now completed. ☐

Theorem 3.7. *Let $\{h_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ be the (C,D)-$*$-woven frames of multipliers in $\mathcal{H}$ with lower and upper bounds A and B, respectively. Suppose that $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are sequences in $Z(\mathcal{A})$ such that there exist two positive elements C and*

D in $Z(\mathcal{A})$ for which $0 < C \leq a_i^* a_i \leq D$ and $0 < C \leq b_i^* b_i \leq D$ for all $i \in I$. Then, $\{a_i h_i\}_{i\in I}$ and $\{b_i t_i\}_{i\in I}$ are $*$-woven frames of multipliers in $\mathcal{H}$ with lower and upper bounds AC and BD, respectively.

Proof. For every subset σ of I and $x \in \mathcal{H}$ we have

$$\begin{aligned}\textstyle\sum_{i\in\sigma} & \langle x, a_i h_i\rangle_{M(\mathcal{H})}\langle a_i h_i, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c}\langle x, b_i t_i\rangle_{M(\mathcal{H})}\langle b_i t_i, x\rangle_{M(\mathcal{H})} \\ = \quad & \sum_{i\in\sigma} a_i^*\langle x, h_i\rangle_{M(\mathcal{H})} a_i \langle h_i, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c} b_i^*\langle x, t_i\rangle_{M(\mathcal{H})} b_i \langle t_i, x\rangle_{M(\mathcal{H})} \\ = \quad & \sum_{i\in\sigma} a_i^* a_i \langle x, h_i\rangle_{M(\mathcal{H})}\langle h_i, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c} b_i^* b_i \langle x, t_i\rangle_{M(\mathcal{H})}\langle t_i, x\rangle_{M(\mathcal{H})}.\end{aligned}$$

It follows the above that

$$\begin{aligned}(AC^{\frac{1}{2}})\langle x, x\rangle(AC^{\frac{1}{2}})^* \quad & \leq \quad \sum_{i\in\sigma}\langle x, a_i h_i\rangle_{M(\mathcal{H})}\langle a_i h_i, x\rangle_{M(\mathcal{H})} + \sum_{i\in\sigma^c}\langle x, b_i t_i\rangle_{M(\mathcal{H})}\langle b_i t_i, x\rangle_{M(\mathcal{H})} \\ & \leq \quad (BD^{\frac{1}{2}})\langle x, x\rangle(BD^{\frac{1}{2}})^*.\end{aligned}$$

Therefore, two frames $\{a_i h_i\}_{i\in I}$ and $\{b_i t_i\}_{i\in I}$ are $*$-woven frames of multipliers in $\mathcal{H}$ with lower and upper bounds $AC^{\frac{1}{2}}, BD^{\frac{1}{2}}$, respectively. □

References

[1] A. Alijani and M. Dehghan, *$*$-frames in Hilbert C^*-modules*, U. P. B. Sci. Bull. Series A., **73**(4) (2011), 89–106.

[2] B. Aupetit, *A Prime on Spectral Theory*, Springer-Verlag, 1991.

[3] T. Bemrose, P. G. Casazza, K. Grochenig, M. C. Lammers, and R. G. Lynch, *Weaving frames*, Oper. Matrices., **10**(4) (2016), 1093–1116.

[4] P. G. Casazza and R. G. Lynch, *Weaving properties of Hilbert space frames*, Int. Conf. Sampl. Theory Appl. (SampTA), IEEE, 2015, pp. 110–114

[5] P. G. Casazza, D. Freeman, and R. G. Lynch, *Weaving Schauder frames*, J. Approx. Theory, **211** (2016), 42–60.

[6] O. Christensen, *An Introduction to Frames and Riesz Bases*, Brikhouser, 2016.

[7] J. B. Conway, *A Course in Operator Theory*, AMS, 2000.

[8] I. Daubechies, A. Grassman, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986), 1271–1283.

[9] F. R. Davidson, *C^*-algebra by example*, Fields Instut., Monog, 1996.

[10] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Proc. Amer. Math. Soc., **72** (1952), 341–366.

[11] M. Frank and D. R. Larson, *A module frame concept for Hilbert C^*-modules*, The Functional and Harmonic Analysis of Wavelets and Frames, AMS Special Session on the Functional and Harmonic Analysis of Wavelets, San Antonio, Texas, 1999, pp. 207–247.

[12] D. Gabor, *Theory of communications*, J. Electr. Eng. (London), **49**(3) (1946), 429–457.

[13] F. Ghobadzadeh, A. Najati, G. A. Anastassiou, and C. Park, *Woven frames in Hilbert C^*-modules*, J. Comput. Anal. Appl., **25**(7) (2018), 1220–1232.

[14] W. Jing, *Frames in Hilbert C^*-modules*, Ph.D. Thesis, University of Central Florida, 2006.

[15] A. Khosravi and B. Khosravi, *Frames and bases in tensor products of Hilbert spaces and Hilbert C^*-modules*, Proc. Indian Acod. Sci., **117**(1) (2007), 1–12.

[16] E. C. Lance, *Hilbert C^*-modules, A Toolkit for Operator Algebraists*, University of Leeds, Cambridge University Press, 1995.

[17] B. Magajna, *Hilbert C^*-modules in which all closed submodules are complemented*, Proc. Amer. Math. Soc., **125**(3) (1997), 849-852.

[18] V. M. Manuilov, *Adjointability of operators on Hilbert C^*-modules*, Acta Math. Univ. Comenianae, **65**(2) (1996), 161–169.

[19] G. J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, San Diego, California, 1990.

[20] A. Nejati and A. Rahimi, *Generalized frames in Hilbert spaces*, Bull. Iran. Math. Soc., **35**(1) (2009), 97–109.

[21] N. E. Wegge Olsen, *K-Theory and C^*-Algebras, A Friendly Approach*, Oxford University Press, Oxford, England, 1993.

[22] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl., **322** (2006), 437–452.

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