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Common fixed point theorem involving contractive conditions of rational type in dislocated quasi-metric space

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Abstract. In this paper, we have proved the existence and uniqueness of a common fixed-point theorem for a finite family of self-mappings involving contractive conditions of Rational type in dislocated quasi-metric spaces by extending and generalizing the result of Jira et al. [10].

1. Introduction

Metric Fixed point theory is one of the most effective research subjects in the development of non-linear analysis. Banach proved a significant result known as the Banach contraction Principle. This principle has been commonly adopted in different directions, varying by the contractive condition or altering the underlying space. Dass and Gupta [7] brought to light the concept of rational type contraction in a metric space. In the same way, some rational type contraction in dq-metric space was introduced in [20].

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This paper aims to prove the existence and uniqueness of a common fixed point of a finite family of compatible mappings in dislocated quasi-metric space introduced by Wilson [23] as a generalization of metric space to our main result.

Hitzler and Seda [8] introduced the concept of complete dislocated quasi-metric space. They also generalized the Banach contraction principle [3] in dislocated metric space. Furthermore, Zeyada et al. [24] introduced the notion of complete dislocated quasi-metric space and established fixed point theorems by generalizing the results of Hitzler and Seda in the same space. Sarwar et al. [20] established some fixed point results for single and a pair of continuous self-mappings in the context of dislocated quasi-metric space which generalize, modify and unify the result of Aage and Salunke in [2]. Isufati [9] proved some fixed point theorem for continuous contractive conditions with rational type expression in the context of dislocated quasi-metric space. Jira et al. [10], proved fixed point results in the setting of dislocated quasi-metric spaces for a pair of self-mappings which generalize the result of Rahman and Sarwar in [19].

In this paper, we prove the existence and uniqueness of a common fixed point theorem for a finite family of self-mappings involving contractive conditions of Rational type in dislocated quasi-metric spaces by extending and generalizing the result of Jira et al. [10].

2. Preliminaries

Definition 2.1. Let X be a non-empty set. A function $\rho : X \times X \longrightarrow R$ be a function, called a distance function provided that for all $\xi, \eta, \theta \in X$,

(1) $\rho(\xi, \eta) \geq 0$,
(2) $\rho(\xi, \eta) = 0$ if and only if $\xi = \eta$,
(3) $\rho(\xi, \eta) = \rho(\eta, \xi)$,
(4) $\rho(\xi, \eta) \leq \rho(\xi, \theta) + \rho(\theta, \eta)$.

If ρ satisfies the conditions (1) to (4), then ρ is called a metric on X. If it satisfies the conditions (1),(2) and (4), then it is called a quasi-metric. If ρ satisfies the conditions (2), (3), and (4), then ρ is called a dislocated metric, and if it satisfies only conditions (2) and (4), then ρ is called a dislocated quasi-metric on X. Non-empty set X together with metric ρ on X is called metric space, and it is given by (X, ρ).

Definition 2.2. [24] Let (X, ρ) be a metric space and $\tau : X \longrightarrow X$ be a self-map. Then τ is said to be a contraction mapping if there exists a constant $k \in [0,1)$ such that $\rho(\tau\xi, \tau\eta) \leq k\rho(\xi, \eta)$, for all $\xi, \eta \in X$.

Definition 2.3. [24] Let (X, ρ) be a metric space. Then, the mapping $\tau : X \longrightarrow X$ is said to be contractive mapping if $\rho(\tau\xi, \tau\eta) < \rho(\xi, \eta)$, for all $\xi, \eta \in X$ with $\xi \neq \eta$.

Definition 2.4. [24] Let (X,ρ) be a dislocated quasi-metric space. A mapping $\tau : X \longrightarrow X$ is called contraction if there exists a constant $k \in [0,1)$ such that $\rho(\tau\xi,\tau\eta) \leq k\rho(\xi,\eta)$, for all $\xi,\eta \in X$.

Definition 2.5. [24] Limit of a convergent sequence in a dislocated quasi-metric space is unique.

Definition 2.6. [24] Let (X,ρ) be a complete dislocated quasi-metric space and $\tau : X \longrightarrow X$ be a contraction. Then τ has a unique fixed point in X.

Definition 2.7. Two self mappings f and g of a non empty set X are said to be commuting if $fg\xi = gf\xi$, for all $\xi \in X$. If $f\xi = g\xi$, for some $\xi \in X$, then ξ is called coincidence point of f and g.

Definition 2.8. [22] Let (X,ρ) be a metric space. Then two self-mappings $f,g : X \longrightarrow X$ if $f\xi = g\xi = \xi$ are called weakly compatible if they commute at their coincidence points.

Definition 2.9. Let X be a non empty set and $\tau : X \longrightarrow X$ be a self-map. For a given $\xi \in X$, we define $\tau_n(x)$ inductively by τ_0 and we call $\tau_n(x)$ is the n^{th} iterate of x under τ.

Theorem 2.1. [10] *Let (X,ρ) be a complete dislocated quasi-metric space and $\tau, f : X \longrightarrow X$ be self-maps satisfying the following condition*

(1) $\tau X \subseteq X$.
(2) *τ and f are weakly compatible and fX is closed subset of X.*
(3) *for all $\xi,\eta \in X$,*

$$\rho(\tau\xi,\tau\eta) \leq a_1^*\varphi\left(\rho(f\xi,f\eta)\right) + a_2^*\varphi\left(\max\left\{\rho(f\xi,f\eta),\rho(f\xi,\tau\xi)\right\}\right) + \frac{a_3^*\varphi\left(\rho(f\xi,f\eta)\left[1+\sqrt{\rho(f\xi,f\eta)\rho(f\xi,\tau\xi)}\right]^2\right)}{\left(1+\rho(f\xi,f\eta)\right)^2},$$

where $a_1^*,a_2^*,a_3^* \geq 0$with $a_1^*+a_2^*+a_3^* < 1$ and φ is a comparison function.Then τ and f have a unique common fixed point if τ and f commute at their coincidence points.

Remark 2.10. [10] For $f = I$ (I is identity on X) form of contractive condition of Theorem 2.1, we get

$$\rho(\tau\xi,\tau\eta) \leq a_1^*\varphi\left(\rho(\xi,\eta)\right) + a_2^*\varphi\left(\max\left\{\rho(\xi,\eta),\rho(\xi,\tau\xi)\right\}\right) + \frac{a_3^*\varphi\left(\rho(\xi,\eta)\left[1+\sqrt{\rho(\xi,\eta)\rho(\xi,\tau\xi)}\right]^2\right)}{\left(1+\rho(\xi,\eta)\right)^2}.$$

Theorem 2.2. [10] *Let (X,ρ) be a complete dislocated quasi-metric space and let $\tau, f : X \longrightarrow X$ be continuous self-maps satisfying the contractive condition of Theorem 2.1. Then f and τ have a unique common fixed point.*

Corollary 2.3. [10] *Let (X,ρ) be a complete dislocated quasi-metric space.Let $\tau : X \longrightarrow X$ be a self-mapping satisfying*

$$\rho(\tau\xi,\tau\eta) \leq a_1^*\varphi\left(\rho(\xi,\eta)\right) + a_2^*\varphi\left(\max\left\{\rho(\xi,\eta),\rho(\xi,\tau\xi)\right\}\right) + \frac{a_3^*\varphi\left(\rho(\xi,\eta)\left[1+\sqrt{\rho(\xi,\eta)\rho(\xi,\tau\xi)}\right]^2\right)}{\left(1+\rho(\xi,\eta)\right)^2}$$

for all $\xi,\eta \in X$, $a_1^*, a_2^*, a_3^* \geq 0$ with $a_1^* + a_2^* + a_3^* < 1$ and φ is a comparison function. Then τ has a unique fixed point.

In support of the following definitions, we are motivated to generalize Theorem 2.1 for a finite family of self-mappings.

Let the set of coincidence point $\mathrm{C}(\tau_1\tau_2...\tau_{n-1},\tau_n)$ and the set of common fixed points $\mathrm{F}(\tau_1\tau_2...\tau_{n-1},\tau_n)$of finite family of self-maps $\tau_1,\tau_2...,\tau_n$ respectively are denoted by $\{\xi \in X : \tau_1\tau_2...\tau_{n-1}\xi = \tau_n\xi\}$ and $\{\xi \in X : \tau_1\tau_2...\tau_{n-1}\xi = \tau_n\xi = x\}$. Then in the sequel, we need to have the following definitions.

Definition 2.11. Finite family of self-maps $\tau_1,\tau_2...,\tau_n$ on a non-empty set X are said to be commuting each other if $\tau_1\tau_2\xi = \tau_2\tau_1\xi, ..., \tau_1\tau_n\xi = \tau_n\tau_1\xi, \tau_2\tau_3\xi = \tau_3\tau_2\xi, ..., \tau_{n-1}\tau_n\xi = \tau_n\tau_{n-1}\xi$ for all $\xi \in X$. That is self-maps $\tau_1,\tau_2...,\tau_n$ on a non-empty set X are said to be commuting each other if

$$\left(\tau_1\tau_2...\tau_{n-1}\right)\tau_n\xi = \tau_n\left(\tau_1\tau_2...\tau_{n-1}\xi\right)$$

for all $\xi \in X$.

Definition 2.12. Finite family of self-maps $\tau_1,\tau_2,...,\tau_n$ of a metric space (X,ρ) are called compatible if

$$\lim_{j\to\infty}\rho\left(\tau_n\tau_1\tau_2...\tau_{n-1}\xi_j, \tau_1\tau_2...\tau_{n-1}\tau_n\xi_j\right) = 0,$$

whenever $\{\xi_j\}_{j=1}^{\infty}$ is a sequence in X such that

$$\lim_{j\to\infty}\tau_n\xi_j = \lim_{j\to\infty}\tau_1\tau_2...\tau_{n-1}\xi_j = t$$

for some $t \in X$.

Definition 2.13. Finite family of self-maps $\tau_1,\tau_2,...,\tau_n$ of a metric space (X,ρ) are called weakly compatible if they commute each other at their coincidence points. That is, $\tau_n u = \tau_1\tau_2...\tau_{n-1}u$, for $u \in X$, then $\tau_1\tau_2...\tau_{n-1}\tau_n u = \tau_u\tau_1\tau_2...\tau_{n-1}u$, for $u \in X$.

Inspired by this, we establish common fixed point theorems for a finite family of self-mappings and show the existence and uniqueness of common fixed points in dislocated quasi-metric spaces involving the contractive condition of rational type by extending Theorem 2.1.

3. Main Result

In this section, we study the existence and uniqueness of a common fixed point theorem for a finite family of self-mappings involving contractive conditions of Rational type in dislocated quasi-metric spaces by extending and generalizing the result of Jira et al. [10].

Theorem 3.1. *Let (X,ρ) be a complete dislocated quasi-metric space and $\tau_1,\tau_2,...,\tau_n : X\longrightarrow X$ be finite family of self-mappings satisfying the following condition*

(1) $\tau_1 X\subseteq \tau_2 X\subseteq\cdots\subseteq\tau_n X$;
(2) $\tau_1,\tau_2,...,\tau_{n-1}$ *and* τ_n *are weakly compatible and* $\tau_n X$ *is closed subset of* X;
(3) *for all* $\xi,\eta\in X$,

$$\begin{aligned}\rho(\tau_1...\tau_{n-1}\xi,\tau_1...\tau_{n-1}\eta) &\le a_1^*\varphi\left(\rho(\tau_n\xi,\tau_n\eta)\right)\\ &+a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi,\tau_n\eta),\rho(\tau_n\eta,\tau_1\tau_2...\tau_{n-1}\eta)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi,\tau_n\eta),\rho(\tau_n\xi,\tau_1\tau_2...\tau_{n-1}\xi)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_n\xi,\tau_1...\tau_{n-1}\xi)\left[1+\sqrt{\rho(\tau_n\xi\tau_n\eta)\rho(\tau_n\eta,\tau_1...\tau_{n-1}\eta}\right]^2}{\left(1+\rho(\tau_n\xi,\tau_n\eta)\right)^2}\right)\end{aligned}$$

where $a_1^*,a_2^*,a_3^*,a_4^*\ge 0$ with $a_1^*+a_2^*+a_3^*<1$ and φ is a comparison function. Then $\tau_1,\tau_2,...,\tau_{n-1}$ and τ_n have a unique common fixed point if $\tau_1,\tau_2,...,\tau_{n-1}$ and τ_n commute each other at their coincidence points.

Proof. Let ξ_0 be an arbitrary element in X, so that $\eta_0=\tau_1\xi_0=\tau_2\xi_1=\cdots=\tau_n\xi_{n-1}$. By condition (1) we have

$$\tau_1\xi_1\in\tau_2X,\tau_2\xi_2\in\tau_3X,...,\tau_{n-1}\xi_{n-1}\in\tau_nX.$$

Then there exists $\xi_n\in X$ such that $\eta_1=\tau_1\xi_1=\tau_2\xi_2=\cdots=\tau_n\xi_n$. Continuing this process we construct a sequence $\{\xi_j\}$ and $\{\eta_j\}$ such that $\eta_j=\tau_1\xi_j=\tau_2\xi_{j+1}=\cdots=\tau_n\xi_{j+(n-1)}$, for $j\in\{0,1,2,...\}$. Now considering two cases we have the following proof.

Case i:

Suppose that $\eta_j=\eta_{j+1}=\cdots=\eta_{j+(n-1)}$, for some $j\in\{0,1,2,...\}$. Then we have $\eta_j=\tau_1\xi_j=\tau_1\xi_{j+1}=\cdots=\tau_1\xi_{j+(n-1)}=\eta_{j+1}=\tau_2\xi_{j+1}=\cdots=\tau_2\xi_{j+(n-1)}=\cdots=\tau_n\xi_{j+(n-1)}=w$,for some $w\in\tau_nX$. Then by the weakly compatibility of $\tau_1,\tau_2,...,\tau_n$, we get

$$
\begin{aligned}
\tau_1 w &= \tau_1\tau_2\xi_{j+(n-1)} = \tau_2\tau_1\xi_{j+(n-1)} \\
&= \tau_2 w = \tau_2\tau_3\xi_{j+(n-1)} = \tau_3\tau_2\xi_{j+(n-1)} \\
&= \tau_3 w = \cdots = \tau_{n-1} w = \tau_{n-1}\tau_n\xi_{j+(n-1)} \\
&= \tau_n\tau_{n-1}\xi_{j+(n-1)} = \tau_n w. \qquad (1)
\end{aligned}
$$

Therefore, $\tau_1\tau_2 w = \tau_2\tau_1 w = \cdots = \tau_{n-1}\tau_n w = \tau_n\tau_{n-1} w$, for $w \in X$ which gives $\tau_1, \tau_2, ..., \tau_n$ commute each other at their coincidence point w and by composition it gives that $\tau_1\tau_2...\tau_{n-1} w = \tau_n w$. Therefore $(\tau_1\tau_2...\tau_{n-1})\,\tau_n w = \tau_n\,(\tau_1\tau_2...\tau_{n-1} w)$, for $w \in X$.

Claim 1: $\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}w) = 0$. By using condition (3) of Theorem 3.1 given the above, we have

$$
\begin{aligned}
&\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}w) \\
\leq &a_1^*\varphi\left(\rho(\tau_n w, \tau_n w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w, \tau_n w)\rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w}\right]^2}{\left(1+\rho(\tau_n w, \tau_n w)\right)^2}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\rho(\tau_1 w, \tau_1 w) \leq &a_1^*\varphi\left(\rho(\tau_n w, \tau_n w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n w), \rho(\tau_n w, \tau_1 w)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n w), \rho(\tau_n w, \tau_1 w)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1 w)\left[1+\sqrt{\rho(\tau_n w, \tau_n w)\rho(\tau_n w, \tau_1 w}\right]^2}{\left(1+\rho(\tau_n w, \tau_n w)\right)^2}\right) \\
= &a_1^*\varphi\left(\rho(\tau_1 w, \tau_1 w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_1 w, \tau_1 w), \rho(\tau_1 w, \tau_1 w)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_1 w, \tau_1 w), \rho(\tau_1 w, \tau_1 w)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\tau_1 w, \tau_1 w)\left[1+\sqrt{\rho(\tau_1 w, \tau_1 w)\rho(\tau_1 w, \tau_1 w}\right]^2}{\left(1+\rho(\tau_1 w, \tau_1 w)\right)^2}\right).
\end{aligned}
$$

Because $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$
\begin{aligned}
\rho(\tau_1 w, \tau_1 w) \leq &a_1^*\rho(\tau_1 w, \tau_1 w) + a_2^*\rho(\tau_1 w, \tau_1 w) + a_3^*\rho(\tau_1 w, \tau_1 w) + a_4^*\rho(\tau_1 w, \tau_1 w) \\
\leq &\left(a_1^* + a_2^* + a_3^* + a_4^*\right)\rho(\tau_1 w, \tau_1 w).
\end{aligned}
$$

From $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, we have

$$\begin{aligned}
&\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}w)\\
\leq & a_1^*\varphi\left(\rho(\tau_n w, \tau_n w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w, \tau_n w)\rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w}\right]^2}{\left(1+\rho(\tau_n w, \tau_n w)\right)^2}\right)
\end{aligned}$$

is satisfied if

$$\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}w) = 0. \tag{2}$$

<u>Claim 2:</u> $\tau_1\tau_2...\tau_{n-1}w = w$.

$$\begin{aligned}
\rho(\tau_1\tau_2...\tau_{n-1}w, w) =&\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)})\\
=&\rho(\tau_1 w, \tau_1\xi_{j+(n-1)})\\
\leq& a_1^*\varphi\left(\rho(\tau_n w, \tau_n\xi_{j+(n-1)})\right)\\
&+ a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-1)}), \rho(\tau_n\xi_{j+(n-1)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)})\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-1)}), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\theta\left[1+\vartheta\right]^2}{\left(1+\rho(\tau_n w, \tau_n\xi_{j+(n-1)})\right)^2}\right)\\
=&a_1^*\varphi\left(\rho(\tau_n w, \tau_n\xi_{j+(n-1)})\right)\\
&+ a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-1)}), \rho(\tau_n\xi_{j+(n-1)}, \tau_1\xi_{j+(n-1)})\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-1)}), \rho(\tau_n w, \tau_1 w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1 w)\left[1+\varpi\right]^2}{\left(1+\rho(\tau_n w, \tau_n\xi_{j+(n-1)})\right)^2}\right)\\
=&a_1^*\varphi\left(\rho(\tau_1 w, w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_1 w, w), \rho(w, w)\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_1 w, w), \rho(\tau_1 w, \tau_1 w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\rho(\tau_1 w, \tau_1 w)\left[1+\sqrt{\rho(\tau_1 w, w)\rho(w, w}\right]^2}{\left(1+\rho(\tau_1 w, w)\right)^2}\right)
\end{aligned}$$

where,

$$\theta = \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)$$

,

$$\vartheta = \sqrt{\rho(\tau_n w, \tau_n\xi_{j+(n-1)})\rho(\tau_n\xi_{j+(n-1)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)}}$$

and

$$\varpi=\sqrt{\rho(\tau_n w,\tau_n\xi_{j+(n-1)})\rho(\tau_n\xi_{j+(n-1)},\tau_1\xi_{j+(n-1)}}.$$

The fact $\rho(\tau_1 w,\tau_1 w)=0$, implies that

$$\rho(\tau_1 w,w)\leq a_1^*\varphi\left(\rho(\tau_1 w,w)\right)+a_2^*\varphi\left(\rho(\tau_1 w,w)\right).$$

Moreover, $\varphi\left(t\right)\leq t$, for all $t\geq 0$, implies that

$$\begin{aligned}\rho(\tau_1 w,w)\leq&a_1^*\rho(\tau_1 w,w)+a_2^*\rho(\tau_1 w,w)\\ \leq&\left(a_1^*+a_2^*\right)\rho(\tau_1 w,w).\end{aligned}$$

Since $0\leq a_1^*+a_2^*+a_3^*+a_4^*<1$, we have

$$\begin{aligned}&\rho(\tau_1\tau_2...\tau_{n-1}w,w)\\ &\leq a_1^*\varphi\left(\rho(\tau_n w,\tau_n\xi_{j+(n-1)})\right)\\ &+a_2^*\varphi\left(\max\left\{\rho(\tau_n w,\tau_n\xi_{j+(n-1)}),\rho(\tau_n\xi_{j+(n-1)},\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)})\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n w,\tau_n\xi_{j+(n-1)}),\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w,\tau_n\xi_{j+(n-1)})\rho(\tau_n\xi_{j+(n-1)},\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)})}\right]^2}{\left(1+\rho(\tau_n w,\tau_n\xi_{j+(n-1)})\right)^2}\right)\end{aligned}$$

is possible if

$$\rho(\tau_1\tau_2...\tau_{n-1}w,w)=\rho(\tau_1 w,w)=0.\tag{3}$$

Similarly,

$$\begin{aligned}\rho(w,\tau_1\tau_2...\tau_{n-1}w)=&\rho(\tau_1\xi_{j+(n-1)},\tau_1 w)\\ \leq&a_1^*\varphi\left(\rho(\tau_n\xi_{j+(n-1)},\tau_n w)\right)\\ &+a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi_{j+(n-1)},\tau_n w),\rho(\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)},\tau_n\xi_{j+(n-1)})\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi_{j+(n-1)},\tau_n w),\rho(\tau_1\tau_2...\tau_{n-1}w,\tau_n w)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\theta\left[1+\vartheta\right]^2}{\left(1+\rho(\tau_n\xi_{j+(n-1)},\tau_n w)\right)^2}\right)\\ \leq&a_1^*\varphi\left(\rho(w,\tau_1 w)\right)+a_2^*\varphi\left(\max\left\{\rho(w,\tau_1 w),\rho(w,w)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(w,\tau_1 w),\rho(\tau_1 w,\tau_1 w)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_1 w,\tau_1 w)\left[1+\sqrt{\rho(w,\tau_1 w)\rho(w,w}\right]^2}{\left(1+\rho(w,\tau_1 w)\right)^2}\right)\end{aligned}$$

where,

$$\rho(\tau_1\tau_2...\tau_{n-1}w,\tau_n w)$$

and

$$\sqrt{\rho(\tau_n\xi_{j+(n-1)},\tau_n w)\rho(\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)},\tau_n\xi_{j+(n-1)}}.$$

By $\rho(w,w)=0$, we have

$$\rho(w,\tau_1 w) \leq a_1^*\varphi\left(\rho(w,\tau_1 w)\right) + a_2^*\varphi\left(\rho(w,\tau_1 w)\right).$$

Since $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}\rho(w,\tau_1 w) \leq & a_1^*\rho(w,\tau_1 w) + a_2^*\rho(w,\tau_1 w)\\ & \leq \left(a_1^* + a_2^*\right)\rho(w,\tau_1 w).\end{aligned}$$

Then from $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, we have

$$\begin{aligned}\rho(w,\tau_1\tau_2...\tau_{n-1}w) =& \rho(\tau_1\xi_{j+(n-1)},\tau_1 w)\\ \leq & a_1^*\varphi\left(\rho(\tau_n\xi_{j+(n-1)},\tau_n w)\right)\\ & + a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi_{j+(n-1)},\tau_n w), \rho(\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)},\tau_n\xi_{j+(n-1)})\right\}\right)\\ & + a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi_{j+(n-1)},\tau_n w), \rho(\tau_1\tau_2...\tau_{n-1}w,\tau_n w)\right\}\right)\\ & + a_4^*\varphi\left(\frac{[1+\vartheta]^2}{\left(1+\rho(\tau_n\xi_{j+(n-1)},\tau_n w)\right)^2}\right)\end{aligned}$$

where,

$$\vartheta = \sqrt{\rho(\tau_n\xi_{j+(n-1)},\tau_n w)\rho(\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)},\tau_n\xi_{j+(n-1)})}$$

and

$$\theta = \rho(\tau_1\tau_2...\tau_{n-1}w,\tau_n w)$$

is possible if

$$\rho(w,\tau_1\tau_2...\tau_{n-1}w) = 0. \tag{4}$$

Now, from equations (3) and (4), we obtain that $\tau_1\tau_2...\tau_{n-1}w = w$. By equation (1), we obtain $\tau_1 w = \tau_2 w = ... = \tau_n w = w$. Therefore w is a common fixed point of $\tau_1, \tau_2, ..., \tau_n$. Next, we show the uniqueness of w. Suppose that w and z are two distinct fixed points of $\tau_1, \tau_2, ..., \tau_n$. This means that

$$\tau_1 w = \tau_2 w = ... = \tau_n w = w, \qquad \text{and} \qquad \tau_1 z = \tau_2 z = ... = \tau_n z = z.$$

By using condition (3) of the above Theorem 3.1, we have the following:

$$\begin{aligned}\rho(w,z) =&\rho(\tau_1\tau_2...\tau_{n-1}w,\tau_1\tau_2...\tau_{n-1}z)\\ \leq& a_1^*\varphi\left(\rho(\tau_n w,\tau_n z)\right)+a_2^*\varphi\left(\max\left\{\rho(\tau_n w,\tau_n z),\rho(\tau_n z,\tau_1\tau_2...\tau_{n-1}z)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n w,\tau_n z),\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w,\tau_n z)\rho(\tau_n z,\tau_1\tau_2...\tau_{n-1}z}\right]^2}{\left(1+\rho(\tau_n w,\tau_n z)\right)^2}\right)\\ \leq& a_1^*\varphi\left(\rho(w,z)\right)+a_2^*\varphi\left(\max\left\{\rho(w,z),\rho(z,z)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(w,z),\rho(w,w)\right\}\right)+a_4^*\varphi\left(\frac{\rho(w,w)\left[1+\sqrt{\rho(w,z)\rho(z,z}\right]^2}{\left(1+\rho(w,z)\right)^2}\right).\end{aligned}$$

Since $\rho(w,w)=0$,

$$\rho(w,z)\leq a_1^*\varphi\left(\rho(w,z)\right)+a_2^*\varphi\left(\rho(w,z)\right).$$

Then $\varphi(t)\leq t$, for all $t\geq 0$, implies that

$$\begin{aligned}\rho(w,z)\leq& a_1^*\rho(w,z)+a_2^*\rho(w,z)\\ \leq&\left(a_1^*+a_2^*\right)\rho(w,z).\end{aligned}$$

Since $0\leq a_1^*+a_2^*+a_3^*+a_4^*<1$, we have

$$\begin{aligned}\rho(w,z) =&\rho(\tau_1\tau_2...\tau_{n-1}w,\tau_1\tau_2...\tau_{n-1}z)\\ \leq& a_1^*\varphi\left(\rho(\tau_n w,\tau_n z)\right)+a_2^*\varphi\left(\max\left\{\rho(\tau_n w,\tau_n z),\rho(\tau_n z,\tau_1\tau_2...\tau_{n-1}z)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n w,\tau_n z),\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w,\tau_n z)\rho(\tau_n z,\tau_1\tau_2...\tau_{n-1}z}\right]^2}{\left(1+\rho(\tau_n w,\tau_n z)\right)^2}\right)\end{aligned}$$

is possible if

$$\rho(w,z)=0. \tag{5}$$

Similarly,

$$\begin{aligned}
\rho(z,w) =& \rho(\tau_1\tau_2...\tau_{n-1}z, \tau_1\tau_2...\tau_{n-1}w) \\
\leq& a_1^*\varphi\left(\rho(\tau_n z, \tau_n w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n z, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n z, \tau_n w), \rho(\tau_n z, \tau_1\tau_2...\tau_{n-1}z)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n z, \tau_1\tau_2...\tau_{n-1}z)\left[1+\sqrt{\rho(\tau_n z, \tau_n w)\rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w}\right]^2}{\left(1+\rho(\tau_n z, \tau_n w)\right)^2}\right) \\
\leq& a_1^*\varphi\left(\rho(z,w)\right) + a_2^*\varphi\left(\max\left\{\rho(z,w), \rho(w,w)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(z,w), \rho(z,z)\right\}\right) + a_4^*\varphi\left(\frac{\rho(z,z)\left[1+\sqrt{\rho(z,w)\rho(w,w}\right]^2}{\left(1+\rho(z,w)\right)^2}\right).
\end{aligned}$$

Then $\rho(z,z)=0$, implies that

$$\rho(z,w) \leq a_1^*\varphi\left(\rho(z,w)\right) + a_2^*\varphi\left(\rho(z,w)\right).$$

Since $\varphi\left(t\right) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}
\rho(z,w) \leq& a_1^*\rho(z,w) + a_2^*\rho(z,w) \\
\leq& \left(a_1^* + a_2^*\right)\rho(z,w).
\end{aligned}$$

Since $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, we have

$$\begin{aligned}
\rho(z,w) =& \rho(\tau_1\tau_2...\tau_{n-1}z, \tau_1\tau_2...\tau_{n-1}w) \\
\leq& a_1^*\varphi\left(\rho(\tau_n z, \tau_n w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n z, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n z, \tau_n w), \rho(\tau_n z, \tau_1\tau_2...\tau_{n-1}z)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n z, \tau_1\tau_2...\tau_{n-1}z)\left[1+\sqrt{\rho(\tau_n z, \tau_n w)\rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w}\right]^2}{\left(1+\rho(\tau_n z, \tau_n w)\right)^2}\right)
\end{aligned}$$

is possible if

$$\rho(z,w) = 0. \tag{6}$$

Now, from equations (5) and (6), we have that $w = z$. Hence, w is a unique common fixed point of $\tau_1, \tau_2, \ldots, \tau_n$.

Case-ii:

Suppose that $\eta_j \neq \eta_{j+1} \neq \cdots \neq \eta_{j+(n-1)}$, for each $j \in \{0, 1, 2, ...\}$. Then,

$$\begin{aligned}\rho(\eta_j, \eta_{j+1}) &= \rho(\tau_1\xi_j, \tau_1\xi_{j+1}) \\ &= \rho(\tau_1\xi_{j+1}, \tau_1\xi_{j+2}) = \cdots = \rho(\tau_1\xi_{j+(n-2)}, \tau_1\xi_{j+(n-1)}) \\ &= \rho(\eta_{j+1}, \eta_{j+2}) = \rho(\tau_2\xi_{j+1}, \tau_2\xi_{j+2}) \\ &= \cdots = \rho(\tau_2\xi_{j+(n-2)}, \tau_2\xi_{j+(n-1)}) = \cdots = \rho(\eta_{j+(n-2)}, \eta_{j+(n-1)}) \\ &= \rho(\tau_{n-1}\xi_{j+(n-2)}, \tau_{n-1}\xi_{j+(n-1)}). \end{aligned} \tag{7}$$

Then

$$\begin{aligned}&\rho(\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-2)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)}) \\ \leq& a_1^*\varphi\left(\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)})\right) \\ &+ a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)}), \rho(\tau_n\xi_{j+(n-1)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)})\right\}\right) \\ &+ a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)}), \rho(\tau_n\xi_{j+(n-2)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-2)})\right\}\right) \\ &+ a_4^*\varphi\left(\frac{\theta\left[1+\vartheta\right]^2}{\left(1+\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)})\right)^2}\right)\end{aligned}$$

where

$$\theta = \rho(\tau_n\xi_{j+(n-2)}, \tau_1\tau_2..\tau_{n-1}\xi_{j+(n-2)}) \tag{8}$$

and

$$\vartheta = \sqrt{\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)})\rho(\tau_n\xi_{j+(n-1)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)}}. \tag{9}$$

By equation (7), we have,

$$\rho(\eta_{j+(n-2)}, \eta_{j+(n-1)}) = \rho(\tau_1\xi_{j+(n-2)}, \tau_1\xi_{j+(n-1)}).$$

Then

$$\begin{aligned}&\rho(\tau_1\xi_{j+(n-2)}, \tau_1\xi_{j+(n-1)}) \\ \leq& a_1^*\varphi\left(\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)})\right) \\ &+ a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)}), \rho(\tau_n\xi_{j+(n-1)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-1)})\right\}\right) \\ &+ a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)}), \rho(\tau_n\xi_{j+(n-2)}, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-2)})\right\}\right) \\ &+ a_4^*\varphi\left(\frac{\theta\left[1+\vartheta\right]^2}{\left(1+\rho(\tau_n\xi_{j+(n-2)}, \tau_n\xi_{j+(n-1)})\right)^2}\right),\end{aligned}$$

where θ and ϑ are as in (8) and (9). Also, we have

$$\begin{aligned}
\rho(\eta_{j+(n-2)}, \eta_{j+(n-1)}) =& \rho(\tau_1\xi_{j+(n-2)}, \tau_1\xi_{j+(n-1)}) \\
=& \rho(\tau_1\xi_j, \tau_1\xi_{j+1}) \\
=& \rho(\eta_j, \eta_{j+1}) \\
\leq& a_1^*\varphi\left(\rho(\eta_{j-1}, \eta_j)\right) + a_2^*\varphi\left(\max\left\{\rho(\eta_{j-1}, \eta_j), \rho(\eta_j, \eta_j)\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\eta_{j-1}, \eta_j), \rho(\eta_{j-1}, \eta_j)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\eta_{j-1}, \eta_j)\left[1 + \sqrt{\rho(\eta_{j-1}, \eta_j)\rho(\eta_{j-1}, \eta_j}\right]^2}{\left(1 + \rho(\eta_{j-1}, \eta_j)\right)^2}\right).
\end{aligned}$$

Since $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}
\rho(\eta_j, \eta_{j+1}) \leq& a_1^*\rho(\eta_{j-1}, \eta_j) + a_2^*\rho(\eta_{j-1}, \eta_j) + a_3^*\rho(\eta_{j-1}, \eta_j) + a_4^*\rho(\eta_{j-1}, \eta_j) \\
\leq& \left(a_1^* + a_2^* + a_3^* + a_4^*\right)\rho(\eta_{j-1}, \eta_j).
\end{aligned}$$

Let $p = a_1^* + a_2^* + a_3^* + a_4^*$. Then,

$$\rho(\eta_j, \eta_{j+1}) \leq p\rho(\eta_{j-1}, \eta_j). \tag{10}$$

Since $0 \leq p < 1$, we obtain $\rho(\eta_{j-1}, \eta_j) \leq p\rho(\eta_{j-2}, \eta_{j-1})$. Then,

$$\begin{aligned}
\rho(\eta_j, \eta_{j+1}) &\leq p\rho(\eta_{j-1}, \eta_j) \\
&\leq p^2\rho(\eta_{j-2}, \eta_{j-1}).
\end{aligned}$$

If we continue this process, we get that $\rho(\eta_j, \eta_{j+1} \leq p^j\rho(\eta_0, \eta_1)$. Since $0 \leq p < 1$, we have $\lim_{j\to\infty} p^j\rho(\eta_0, \eta_1) = 0$. Thus,

$$\lim_{j\to\infty} \rho(\eta_j, \eta_{j+1}) = 0. \tag{11}$$

Similarly, we can easily show that,

$$\lim_{j\to\infty} \rho(\eta_{j+1}, \eta_j) = 0. \tag{12}$$

Now, we show that $\{\eta_j\}$ is a Cauchy sequence in X. Let $m, j \in \mathbb{N}$ with $m > j$, applying triangular inequality

$$\begin{aligned}
\rho(\eta_j, \eta_m) &\leq \rho(\eta_j, \eta_{j+1}) + \rho(\eta_{j+1}, \eta_m) \\
&\leq \rho(\eta_j, \eta_{j+1}) + \rho(\eta_{j+1}, \eta_{j+2}) + \cdots + \rho(\eta_{m-1}, \eta_m) \\
&\leq p^j\rho(\eta_0, \eta_1) + p^{j+1}\rho(\eta_0, \eta_1) + \cdots + p^{m-1}\rho(\eta_0, \eta_1) \\
&\leq p^j\left(1 + p + \cdots + p^{m-j-1}\right)\rho(\eta_0, \eta_1) \\
&\leq \frac{p^j}{1-p}\rho(\eta_0, \eta_1).
\end{aligned}$$

Since $0 \leq p < 1$, $\lim_{p\to 0} \frac{p^j}{1-p}\rho(\eta_0,\eta_1) = 0$. This implies,

$$\lim_{j,m\to\infty} \rho(\eta_j,\eta_m) = 0. \tag{13}$$

Let $m, j \in \mathbb{N}$ with $m < j$, applying triangular inequality

$$\begin{aligned}
\rho(\eta_m,\eta_j) &\leq \rho(\eta_m,\eta_{m+1}) + \rho(\eta_{m+1},\eta_j) \\
&\leq \rho(\eta_m,\eta_{m+1}) + \rho(\eta_{m+1},\eta_{m+2}) + \cdots + \rho(\eta_{j-1},\eta_j) \\
&\leq p^m\rho(\eta_0,\eta_1) + p^{m+1}\rho(\eta_0,\eta_1) + \cdots + p^{j-1}\rho(\eta_0,\eta_1) \\
&\leq p^m\left(1 + p + \cdots + p^{j-m-1}\right)\rho(\eta_0,\eta_1) \\
&\leq \frac{p^m}{1-p}\rho(\eta_0,\eta_1).
\end{aligned}$$

Then $0 \leq p < 1$, implies that $\lim_{p\to 0} \frac{p^m}{1-p}\rho(\eta_0,\eta_1) = 0$. This implies,

$$\lim_{j,m\to\infty} \rho(\eta_m,\eta_j) = 0. \tag{14}$$

From equations (13) and (14), we get

$$\lim_{m,j\to\infty} \rho(\eta_m,\eta_j) = \lim_{j,m\to\infty} \rho(\eta_j,\eta_m) = 0.$$

Thus, $\{\eta_j\}$ is a Cauchy sequence in X, for each $j \in \{0,1,2,...\}$. Because X is complete, there exists $q \in X$ such that $\lim_{j\to\infty} \eta_j = q$. Thus,

$$\lim_{j\to\infty} \tau_1\xi_j = \lim_{j\to\infty} \tau_2\xi_{j+1} = \cdots = \lim_{j\to\infty} \tau_n\xi_{j+(n-1)} = q$$

from which, we have

$$\lim_{j\to\infty} \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-2)} = \lim_{j\to\infty} \tau_n\xi_{j+(n-1)} = q.$$

Since $\tau_n X$ is a closed subset of X, there exists $w \in \tau_n X$ such that

$$q = \tau_n X.$$

Now, we show that $\tau_1\tau_2...\tau_{n-1}w = q$.

$$\begin{aligned}
&\rho(\tau_1...\tau_{n-1}w, \tau_1...\tau_{n-1}\xi_{j+(n-2)}) \\
\leq &a_1^*\varphi\left(\rho(\tau_n w, \tau_n\xi_{j+(n-2)})\right) \\
&+ a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-2)}), \rho(\tau_n\xi_{j+(n-2)}, \tau_1...\tau_{n-1}\xi_{j+(n-2)})\right\}\right) \\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-2)}), \rho(\tau_n w, \tau_1...\tau_{n-1}w)\right\}\right) \\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1...\tau_{n-1}w)\left[1 + \sqrt{\rho(\tau_n w, \tau_n\xi_{j+(n-2)})\rho(\tau_n\xi_{j+(n-2)}, \tau_1...\tau_{n-1}\xi_{j+(n-2)})}\right]^2}{\left(1 + \rho(\tau_n w, \tau_n\xi_{j+(n-2)})\right)^2}\right).
\end{aligned}$$

By equation (7), we have

$$\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}\xi_{j+(n-2)}) = \rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\xi_{j+(n-2)}) = \rho(\tau_1\tau_2...\tau_{n-1}w, q).$$

Then,

$$\begin{aligned}
&\rho(\tau_1\tau_2...\tau_{n-1}w, q)\\
=&\rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\xi_{j+(n-2)})\\
\leq& a_1^*\varphi\left(\rho(\tau_n w, \tau_n\xi_{j+(n-2)})\right)\\
&+ a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-2)}), \rho(\tau_n\xi_{j+(n-2)}, \tau_1...\tau_{n-1}\xi_{j+(n-2)})\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n\xi_{j+(n-2)}), \rho(\tau_n w, \tau_1...\tau_{n-1}w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w, \tau_n\xi_{j+(n-2)})\rho(\tau_n\xi_{j+(n-2)}, \tau_1...\tau_{n-1}\xi_{j+(n-2)})}\right]^2}{\left(1+\rho(\tau_n w, \tau_n\xi_{j+(n-2)})\right)^2}\right).
\end{aligned}$$

Since $\tau_1 w = \tau_2 w = ... = \tau_n w$, we obtain that

$$\begin{aligned}
\rho(\tau_1 w, q) \leq& a_1^*\varphi\left(\rho(\tau_1 w, \tau_n\xi_{j+(n-2)})\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_1 w, \tau_n\xi_{j+(n-2)}), \rho(\tau_n\xi_{j+(n-2)}, q)\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_1 w, \tau_n\xi_{j+(n-2)}), \rho(\tau_n w, \tau_1 w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\rho(\tau_1 w, \tau_1 w)\left[1+\sqrt{\rho(\tau_1 w, \tau_n\xi_{j+(n-2)})\rho(\tau_n\xi_{j+(n-2)}, q}\right]^2}{\left(1+\rho(\tau_1 w, \tau_n\xi_{j+(n-2)})\right)^2}\right).
\end{aligned}$$

Letting $j \to \infty$, we get $\lim_{j\to\infty} \tau_n\xi_{j+(n-2)} = q$. Hence,

$$\begin{aligned}
\rho(\tau_1 w, q) \leq& a_1^*\varphi\left(\rho(\tau_1 w, q)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_1 w, q), \rho(q, q)\right\}\right)\\
&+ a_3^*\varphi\left(\min\left\{\rho(\tau_1 w, q), \rho(\tau_1 w, \tau_1 w)\right\}\right)\\
&+ a_4^*\varphi\left(\frac{\rho(\tau_1 w, \tau_1 w)\left[1+\sqrt{\rho(\tau_1 w, q)\rho(q, q}\right]^2}{\left(1+\rho(\tau_1 w, q)\right)^2}\right).
\end{aligned}$$

Then $\rho(\tau_1 w, \tau_1 w) = 0$, implies that

$$\rho(\tau_1\tau_2...\tau_{n-1}w, q) \leq a_1^*\varphi\left(\rho(\tau_1 w, q)\right) + a_2^*\varphi\left(\rho(\tau_1 w, q)\right).$$

Then from $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}
\rho(\tau_1\tau_2...\tau_{n-1}w, q) \leq& a_1^*\rho(\tau_1 w, q) + a_2^*\rho(\tau_1 w, q)\\
\leq& \left(a_1^* + a_2^*\right)\rho(\tau_1 w, q).
\end{aligned}$$

Since $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, the given inequality is satisfied if

$$\rho(\tau_1\tau_2...\tau_{n-1}w, q) = 0. \tag{15}$$

Similarly,

$$\begin{aligned}&\rho(q,\tau_1\tau_2...\tau_{n-1}w)\\ \leq& a_1^*\varphi\left(\rho(\tau_n\xi_{j+(n-2)},\tau_n w)\right)+a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi_{j+(n-2)},\tau_n w),\rho(\tau_n w,\tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi_{j+(n-2)},\tau_n w),\rho(\tau_n\xi_{j+(n-2)},\tau_1...\tau_{n-1}\xi_{j+(n-2)})\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_n\xi_{j+(n-2)},\tau_1...\tau_{n-1}\xi_{j+(n-2)})\left[1+\sqrt{\rho(\tau_n\xi_{j+(n-2)},\tau_n w)\rho(\tau_n w,\tau_1...\tau_{n-1}w}\right]^2}{\left(1+\rho(\tau_n\xi_{j+(n-2)},\tau_n w)\right)^2}\right).\end{aligned}$$

Then $\tau_1\tau_2...\tau_{n-1}w=\tau_n w$ and $\tau_1\tau_2...\tau_{n-1}\xi_{j+(n-2)}=\tau_1\xi_{j+(n-2)}$, imply

$$\begin{aligned}&\rho(q,\tau_1...\tau_{n-1}w)\\ \leq& a_1^*\varphi\left(\rho(\tau_n\xi_{j+(n-2)},\tau_1 w)\right)+a_2^*\varphi\left(\max\left\{\rho(\tau_n\xi_{j+(n-2)},\tau_1 w),\rho(\tau_1 w,\tau_1 w)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(\tau_n\xi_{j+(n-2)},\tau_1 w),\rho(\tau_n\xi_{j+(n-2)},\tau_1\xi_{j+(n-2)})\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(\tau_n\xi_{j+(n-2)},\tau_1\xi_{j+(n-2)})\left[1+\sqrt{\rho(\tau_n\xi_{j+(n-2)},\tau_1 w)\rho(\tau_1 w,\tau_1 w}\right]^2}{\left(1+\rho(\tau_n\xi_{j+(n-2)},\tau_1 w)\right)^2}\right).\end{aligned}$$

Letting $j\to\infty$, we get $\lim_{j\to\infty}\tau_n\xi_{j+(n-2)}=\lim_{j\to\infty}\tau_1\xi_{j+(n-2)}=q$. Then, we have

$$\begin{aligned}\rho(q,\tau_1\tau_2...\tau_{n-1}w)\leq& a_1^*\varphi\left(\rho(q,\tau_1 w)\right)+a_2^*\varphi\left(\max\left\{\rho(q,\tau_1 w),\rho(\tau_1 w,\tau_1 w)\right\}\right)\\ &+a_3^*\varphi\left(\min\left\{\rho(q,\tau_1 w),\rho(q,q)\right\}\right)\\ &+a_4^*\varphi\left(\frac{\rho(q,q)\left[1+\sqrt{\rho(q,\tau_1 w)\rho(\tau_1 w,\tau_1 w}\right]^2}{\left(1+\rho(q,\tau_1 w)\right)^2}\right).\end{aligned}$$

Thne $\rho(q,q)=0$, implies that

$$\rho(q,\tau_1\tau_2...\tau_{n-1}w)\leq a_1^*\varphi\left(\rho(q,\tau_1 w)\right)+a_2^*\varphi\left(\rho(q,\tau_1 w)\right).$$

From $\varphi(t)\leq t$, for all $t\geq 0$, we obtain

$$\begin{aligned}\rho(q,\tau_1\tau_2...\tau_{n-1}w)\leq& a_1^*\rho(q,\tau_1 w)+a_2^*\rho(q,\tau_1 w)\\ \leq&\left(a_1^*+a_2^*\right)\rho(q,\tau_1 w).\end{aligned}$$

Since $0\leq a_1^*+a_2^*+a_3^*+a_4^*<1$, the given inequality is satisfied if

$$\rho(q,\tau_1\tau_2...\tau_{n-1}w)=0. \tag{16}$$

Using equations (15) and (16), we have $q=\tau_1\tau_2...\tau_{n-1}w$. Then $q=\tau_1 w=\tau_2 w=...=\tau_n w$ from which we have

$$q=\tau_1\tau_2...\tau_{n-1}w=\tau_n w.$$

By the weakly compatibility of $\tau_1,\tau_2,...,\tau_n$, we have

$$\left(\tau_1\tau_2...\tau_{n-1}\right)\tau_n w=\tau_n\left(\tau_1\tau_2...\tau_{n-1}w\right).$$

Then,

$$(\tau_1\tau_2...\tau_{n-1})\,\tau_n w = (\tau_1\tau_2...\tau_{n-1})\,q = \tau_n\,(\tau_1\tau_2...\tau_{n-1}w) = \tau_n q.$$

Thus q is a coincidence point of $\tau_1, \tau_2, ..., \tau_n$. Consider

$$\begin{aligned}\rho(\tau_1...\tau_{n-1}q, q) =& \rho(\tau_1\tau_2...\tau_{n-1}q, \tau_1\tau_2...\tau_{n-1}w)\\ =& \rho(\tau_1\tau_2...\tau_{n-1}q, \tau_1 w)\\ \leq& a_1^*\varphi\left(\rho(\tau_n q, \tau_n w)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n q, \tau_n w), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\ &+ a_3^*\varphi\left(\min\left\{\rho(\tau_n q, \tau_n w), \rho(\tau_n q, \tau_1\tau_2...\tau_{n-1}q)\right\}\right)\\ &+ a_4^*\varphi\left(\frac{\rho(\tau_n q, \tau_1...\tau_{n-1}q)\left[1+\sqrt{\rho(\tau_n q, \tau_n w)\rho(\tau_n w, \tau_1...\tau_{n-1}w}\right]^2}{\left(1+\rho(\tau_n q, \tau_n w)\right)^2}\right).\end{aligned}$$

From $\tau_1\tau_2...\tau_{n-1}q = \tau_n q$ and $\tau_n w = q$,

$$\begin{aligned}\rho(\tau_1\tau_2...\tau_{n-1}q, q) \leq& a_1^*\varphi\left(\rho(\tau_1 q, q)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_1 q, q), \rho(q, q)\right\}\right)\\ &+ a_3^*\varphi\left(\min\left\{\rho(\tau_1 q, q), \rho(\tau_1 q, \tau_1 q)\right\}\right)\\ &+ a_4^*\varphi\left(\frac{\rho(\tau_1 q, \tau_1 q)\left[1+\sqrt{\rho(\tau_1 q, q)\rho(q, q}\right]^2}{\left(1+\rho(\tau_1 q, q)\right)^2}\right).\end{aligned}$$

From $\rho(q, q) = 0$, we have

$$\rho(\tau_1\tau_2...\tau_{n-1}q, q) \leq a_1^*\varphi\left(\rho(\tau_1 q, q)\right) + a_2^*\varphi\left(\rho(\tau_1 q, q)\right).$$

Since $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}\rho(\tau_1\tau_2...\tau_{n-1}q, q) \leq& a_1^*\rho(\tau_1 q, q) + a_2^*\rho(\tau_1 q, q)\\ \leq& \left(a_1^* + a_2^*\right)\rho(\tau_1 q, q).\end{aligned}$$

Since $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, the given inequality is satisfied if

$$\rho(\tau_1\tau_2...\tau_{n-1}q, q) = 0. \tag{17}$$

Similarly,

$$\begin{aligned}\rho(q, \tau_1...\tau_{n-1}q) =& \rho(\tau_1\tau_2...\tau_{n-1}w, \tau_1\tau_2...\tau_{n-1}q)\\ =& \rho(\tau_1 w, \tau_1\tau_2...\tau_{n-1}q)\\ \leq& a_1^*\varphi\left(\rho(\tau_n w, \tau_n q)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n w, \tau_n q), \rho(\tau_n q, \tau_1\tau_2...\tau_{n-1}q)\right\}\right)\\ &+ a_3^*\varphi\left(\min\left\{\rho(\tau_n w, \tau_n q), \rho(\tau_n w, \tau_1\tau_2...\tau_{n-1}w)\right\}\right)\\ &+ a_4^*\varphi\left(\frac{\rho(\tau_n w, \tau_1...\tau_{n-1}w)\left[1+\sqrt{\rho(\tau_n w, \tau_n q)\rho(\tau_n q, \tau_1...\tau_{n-1}q}\right]^2}{\left(1+\rho(\tau_n w, \tau_n q)\right)^2}\right).\end{aligned}$$

Since $\tau_1\tau_2...\tau_{n-1}w = \tau_n w = q$ and $\tau_n q = \tau_1 q$, we have

$$\begin{aligned}\rho(q, \tau_1\tau_2...\tau_{n-1}q) \leq & a_1^*\varphi\left(\rho(\tau_1 w, \tau_1 q)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_1 w, \tau_1 q), \rho(\tau_1 q, \tau_1 q)\right\}\right) \\ & + a_3^*\varphi\left(\min\left\{\rho(\tau_1 w, \tau_1 q), \rho(\tau_1 w, q)\right\}\right) \\ & + a_4^*\varphi\left(\frac{\rho(\tau_1 w, q)\left[1+\sqrt{\rho(\tau_1 w, \tau_1 q)\rho(\tau_1 q, \tau_1 q}\right]^2}{\left(1+\rho(\tau_1 w, \tau_1 q)\right)^2}\right)\end{aligned}$$

$$\begin{aligned}\rho(q, \tau_1\tau_2...\tau_{n-1}q) \leq & a_1^*\varphi\left(\rho(q, \tau_1 q)\right) + a_2^*\varphi\left(\max\left\{\rho(q, \tau_1 q), \rho(\tau_1 q, \tau_1 q)\right\}\right) \\ & + a_3^*\varphi\left(\min\left\{\rho(q, \tau_1 q), \rho(q, q)\right\}\right) \\ & + a_4^*\varphi\left(\frac{\rho(q, q)\left[1+\sqrt{\rho(q, \tau_1 q)\rho(\tau_1 q, \tau_1 q}\right]^2}{\left(1+\rho(q, \tau_1 q)\right)^2}\right).\end{aligned}$$

Since $\rho(q, q) = 0$,

$$\rho(q, \tau_1\tau_2...\tau_{n-1}q) \leq a_1^*\varphi\left(\rho(q, \tau_1 q)\right) + a_2^*\varphi\left(\rho(q, \tau_1 q)\right).$$

Since $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}\rho(q, \tau_1\tau_2...\tau_{n-1}q) \leq & a_1^*\rho(q, \tau_1 q) + a_2^*\rho(q, \tau_1 q) \\ & \leq \left(a_1^* + a_2^*\right)\rho(q, \tau_1 q).\end{aligned}$$

Since $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, the given inequality is satisfied if

$$\rho(q, \tau_1\tau_2...\tau_{n-1}q) = 0. \tag{18}$$

Using equations (17) and (18), we have $q = \tau_1\tau_2...\tau_{n-1}q$. Thus, $q = \tau_1\tau_2...\tau_{n-1}q = \tau_n q$. Therefore, q is a common fixed point of $\tau_1\tau_2...\tau_{n-1}$ and τ_n in X.

Next we show that q is unique in X. Let r be another common fixed point of $\tau_1\tau_2...\tau_{n-1}$ and τ_n in X. So, $\tau_1 r = \tau_2 r = ... = \tau_{n-1}r = \tau_n r = r$, that is, $\tau_1\tau_2...\tau_{n-1}r = \tau_n r = r$. Consider,

$$\begin{aligned}\rho(q, r) = & \rho(\tau_1\tau_2...\tau_{n-1}q, \tau_1\tau_2...\tau_{n-1}r) \\ \leq & a_1^*\varphi\left(\rho(\tau_n q, \tau_n r)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n q, \tau_n r), \rho(\tau_n r, \tau_1\tau_2...\tau_{n-1}r)\right\}\right) \\ & + a_3^*\varphi\left(\min\left\{\rho(\tau_n q, \tau_n r), \rho(\tau_n q, \tau_1\tau_2...\tau_{n-1}q)\right\}\right) \\ & + a_4^*\varphi\left(\frac{\rho(\tau_n q, \tau_1\tau_2...\tau_{n-1}q)\left[1+\sqrt{\rho(\tau_n q, \tau_n r)\rho(\tau_n r, \tau_1\tau_2...\tau_{n-1}r}\right]^2}{\left(1+\rho(\tau_n q, \tau_n r)\right)^2}\right).\end{aligned}$$

Since $\tau_1\tau_2...\tau_{n-1}q = \tau_n q = q$ and $\tau_n r = r$,

$$\begin{aligned}\rho(q,r) \leq & a_1^*\varphi\left(\rho(q,r)\right) + a_2^*\varphi\left(\max\left\{\rho(q,r),\rho(r,r)\right\}\right) \\ & + a_3^*\varphi\left(\min\left\{\rho(q,r),\rho(q,q)\right\}\right) \\ & + a_4^*\varphi\left(\frac{\rho(q,q)\left[1+\sqrt{\rho(q,r)\rho(r,r}\right]^2}{\left(1+\rho(q,r)\right)^2}\right).\end{aligned}$$

From $\rho(q,q) = 0$, we have

$$\rho(q,r) \leq a_1^*\varphi\left(\rho(q,r)\right) + a_2^*\varphi\left(\rho(q,r)\right).$$

Since $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}\rho(q,r) \leq & a_1^*\rho(q,r) + a_2^*\rho(q,r) \\ & \leq \left(a_1^* + a_2^*\right)\rho(q,r).\end{aligned}$$

Since $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, the given inequality is satisfied if

$$\rho(q,r) = 0. \tag{19}$$

Similarly,

$$\begin{aligned}\rho(r,q) = & \rho(\tau_1\tau_2...\tau_{n-1}r, \tau_1\tau_2...\tau_{n-1}q) \\ \leq & a_1^*\varphi\left(\rho(\tau_n r,\tau_n q)\right) + a_2^*\varphi\left(\max\left\{\rho(\tau_n r,\tau_n q),\rho(\tau_n q,\tau_1\tau_2...\tau_{n-1}q)\right\}\right) \\ & + a_3^*\varphi\left(\min\left\{\rho(\tau_n r,\tau_n q),\rho(\tau_n r,\tau_1\tau_2...\tau_{n-1}r)\right\}\right) \\ & + a_4^*\varphi\left(\frac{\rho(\tau_n r,\tau_1\tau_2...\tau_{n-1}r)\left[1+\sqrt{\rho(\tau_n r,\tau_n q)\rho(\tau_n q,\tau_1\tau_2...\tau_{n-1}q}\right]^2}{\left(1+\rho(\tau_n r,\tau_n q)\right)^2}\right).\end{aligned}$$

Since $\tau_1\tau_2...\tau_{n-1}r = \tau_n r = r$ and $\tau_n q = q$,

$$\begin{aligned}\rho(r,q) \leq & a_1^*\varphi\left(\rho(r,q)\right) + a_2^*\varphi\left(\max\left\{\rho(r,q),\rho(q,q)\right\}\right) + a_3^*\varphi\left(\min\left\{\rho(r,q),\rho(r,r)\right\}\right) \\ & + a_4^*\varphi\left(\frac{\rho(r,r)\left[1+\sqrt{\rho(r,q)\rho(q,q}\right]^2}{\left(1+\rho(r,q)\right)^2}\right).\end{aligned}$$

Since $\rho(q,q) = 0$, we have

$$\rho(r,q) \leq a_1^*\varphi\left(\rho(r,q)\right) + a_2^*\varphi\left(\rho(r,q)\right).$$

Since $\varphi(t) \leq t$, for all $t \geq 0$, we obtain

$$\begin{aligned}\rho(r,q) \leq & a_1^*\rho(r,q) + a_2^*\rho(r,q) \\ & \leq \left(a_1^* + a_2^*\right)\rho(r,q).\end{aligned}$$

Since $0 \leq a_1^* + a_2^* + a_3^* + a_4^* < 1$, the given inequality is satisfied if

$$\rho(r, q) = 0. \tag{20}$$

From equations (19) and (20), we conclude that $r = q$. So, q is a unique common fixed point of $\tau_1, \tau_2, ..., \tau_n$ in X. □

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