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Lie symmetry analysis, power series solutions and conservation laws of time fractional coupled Boussinesq-Whitham-Broer-Kaup equations

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Abstract. In this paper, the Lie symmetry analysis method is applied to time-fractional coupled Boussinesq-Whitham-Broer-Kaup equations, an important physics model. The obtained Lie symmetries are utilized to reduce the system of fractional partial differential equations with Riemann-Liouville fractional derivative to the system of fractional ordinary differential equations with Erdélyi-Kober fractional derivative. Then the power series method is applied to derive explicit power series solutions for the reduced system. In addition, the new conservation theorem and the generalization of Noether operators are developed to construct the conservation laws for the equations studied.

1. Introduction

The coupled Boussinesq-Whitham-Broer-Kaup equations are important mathematical physical equations for describing the physical properties of shallow water waves in the field of fluid dynamics. The equations are firstly derived by Sachs in

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[23] and given by

$$\begin{cases} u_t + v_x + uu_x = 0, \\ v_t + u_{xxx} + (uv)_x = 0, \end{cases} \tag{1}$$

where $u(t,x)$ and $v(t,x)$ are the velocity and the height of the free wave of the fluid in the trough, respectively.

As a generalization of classical calculus, fractional calculus can be traced back to the letter written by L'Hôspital to Leibniz in 1695. Since then, it has gradually gained the attention of mathematicians. Especially in recent decades, it has developed rapidly and been successfully applied in many fields of science and technology [24, 22, 9, 15]. Therefore, it is very important to find the solution of the fractional differential equation. So far, there have been some numerical and analytical methods, such as Adomian decomposition method [3], finite difference method [16], homotopy perturbation method [17], the sub-equation method [38], the variational iteration method [18], Lie symmetry analysis method [7], invariant subspace method [6] and so on. Among them, the Lie symmetry analysis method has received increasing attention.

Lie symmetry analysis method was founded by Norwegian mathematician Sophus Lie at the end of the nineteenth century and then further developed by some other mathematicians, such as Ovsiannikov [21], Olver [20], Ibragimov [10, 11, 12] and so on. As a modern method among many analytic techniques, Lie symmetry analysis has been extended to fractional differential equations (FDEs) by Gazizov et al. [7] in 2007. It was then effectively applied to various models of the FDEs occurring in different areas of applied science (see [4, 5, 8, 19, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]).

In this paper, the Lie symmetry analysis method is extended to the following time-fractional coupled Boussinesq-Whitham-Broer-Kaup equations:

$$\begin{cases} D_t^\alpha u + v_x + uu_x = 0, \\ D_t^\alpha v + u_{xxx} + (uv)_x = 0, \end{cases} \tag{2}$$

with $0 < \alpha < 1$. As we all know, there are many types of definitions for fractional derivatives, such as Riemann-Liouville type, Caputo type, Weyl type and so on. In [1], Emrah Atilgan et al. studied the conformable fractional derivative version of (2) by using the auxiliary equation method. In [2], Shuangqing Chen et al. also studied the conformable fractional derivative version of (2), but they employed the complete discrimination system for the polynomial method.

However, this paper adopts the Riemann-Liouville fractional derivative defined by

$${}_aD_t^\alpha f(t,x) = D_t^n\, {}_aI_t^{n-\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(s,x)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, & n-1 < \alpha < n, n \in \mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N} \end{cases}$$

for $t > a$. We denote the operator ${}_0D_t^\alpha$ as D_t^α throughout this paper.

This paper aims to find all Lie symmetries for Eqs. (2) by using the Lie symmetry analysis method and reducing Eqs. (2) to time-fractional ordinary differential equations. Then explicit power series solutions and conservation laws for Eqs. (2) are obtained. In addition, convergence analysis and numerical simulation are given in this paper for the power series solutions.

This paper is organized as follows. In Section 2, Lie symmetry analysis and reduction of Eqs. (2) are studied. In Section 3, power series solutions, convergence analysis and numerical simulation for Eqs. (2) are presented. The conservation laws of Eqs. (2) are obtained in Sections 4 and the conclusion is given in the last section.

2. Lie symmetry analysis and reduction of Eqs. (2)

Consider time fractional coupled Boussinesq-Whitham-Broer-Kaup equations (2), which are assumed to be invariant under the one-parameter (ϵ) Lie group of continuous point transformations, i.e.,

$$\begin{aligned}
&t^* = t + \epsilon\tau(t,x,u,v) + o(\epsilon), && x^* = x + \epsilon\xi(t,x,u,v) + o(\epsilon),\\
&u^* = u + \epsilon\eta(t,x,u,v) + o(\epsilon), && v^* = v + \epsilon\zeta(t,x,u,v) + o(\epsilon),\\
&D^\alpha_{t^*}u^* = D^\alpha_t u + \epsilon\eta^{\alpha,t} + o(\epsilon), && D^\alpha_{t^*}v^* = D^\alpha_t v + \epsilon\zeta^{\alpha,t} + o(\epsilon),\\
&D_{x^*}u^* = D_x u + \epsilon\eta^{x} + o(\epsilon), && D_{x^*}v^* = D_x v + \epsilon\zeta^{x} + o(\epsilon),\\
&D^2_{x^*}u^* = D^2_x u + \epsilon\eta^{xx} + o(\epsilon), && D^2_{x^*}v^* = D^2_x v + \epsilon\zeta^{xx} + o(\epsilon),\\
&D^3_{x^*}u^* = D^3_x u + \epsilon\eta^{xxx} + o(\epsilon), && D^3_{x^*}v^* = D^3_x v + \epsilon\zeta^{xxx} + o(\epsilon),
\end{aligned} \tag{3}$$

where τ, ξ, η and ζ are infinitesimals and $\eta^{\alpha,t}$, $\zeta^{\alpha,t}$, η^x, ζ^x, η^{xx}, ζ^{xx}, η^{xxx} and ζ^{xxx} are the corresponding prolongations of orders α, 1, 2 and 3, respectively.

The corresponding group generator is defined by

$$X = \tau(t,x,u,v)\frac{\partial}{\partial t} + \xi(t,x,u,v)\frac{\partial}{\partial x} + \eta(t,x,u,v)\frac{\partial}{\partial u} + \zeta(t,x,u,v)\frac{\partial}{\partial v}. \tag{4}$$

So the prolongation of the above group generator X has the form

$$prX = X + \eta^{\alpha,t}\frac{\partial}{\partial u^\alpha_t} + \zeta^{\alpha,t}\frac{\partial}{\partial v^\alpha_t} + \eta^x\frac{\partial}{\partial u_x} + \zeta^x\frac{\partial}{\partial v_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \zeta^{xx}\frac{\partial}{\partial v_{xx}} + \cdots, \tag{5}$$

where

$$\begin{aligned}
\eta^{\alpha,t} =& D^\alpha_t(\eta) + \tau D^\alpha_t(u_t) - D^\alpha_t(\tau u_t) + \xi D^\alpha_t(u_x) - D^\alpha_t(\xi u_x)\\
=& \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + (\eta_v\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \eta_v}{\partial t^\alpha})\\
&+ \sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}D^{n+1}_t(\tau)]D^{\alpha-n}_t(u) + \mu_1 + \mu_2\\
&+ \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_v}{\partial t^n}D^{\alpha-n}_t(v) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D^n_t(\xi)D^{\alpha-n}_t(u_x),
\end{aligned} \tag{6}$$

with

$$\mu_1=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k},$$

$$\mu_2=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-v)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m v^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial v^k},$$

$$\begin{aligned}\zeta^{\alpha,t}=&\frac{\partial^\alpha\zeta}{\partial t^\alpha}+(\zeta_v-\alpha D_t(\tau))\frac{\partial^\alpha v}{\partial t^\alpha}-v\frac{\partial^\alpha\zeta_v}{\partial t^\alpha}+(\zeta_u\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\zeta_u}{\partial t^\alpha})\\&+\sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^n\zeta_v}{\partial t^n}-\binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(v)+\mu_3+\mu_4\\&+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\zeta_u}{\partial t^n}D_t^{\alpha-n}(u)-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(v_x),\end{aligned}\qquad(7)$$

with

$$\mu_3=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial u^k},$$

$$\mu_4=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-v)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m v^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial v^k},$$

and

$$\eta^x=D_x(\eta)-u_tD_x(\tau)-u_xD_x(\xi),\qquad(8)$$
$$\zeta^x=D_x(\zeta)-v_tD_x(\tau)-v_xD_x(\xi),\qquad(9)$$
$$\eta^{xx}=D_x(\eta^x)-u_{xt}D_x(\tau)-u_{xx}D_x(\xi),\qquad(10)$$
$$\zeta^{xx}=D_x(\zeta^x)-v_{xt}D_x(\tau)-v_{xx}D_x(\xi),\qquad(11)$$
$$\eta^{xxx}=D_x(\eta^{xx})-u_{xxt}D_x(\tau)-u_{xxx}D_x(\xi),\qquad(12)$$
$$\zeta^{xxx}=D_x(\zeta^{xx})-v_{xxt}D_x(\tau)-v_{xxx}D_x(\xi),\qquad(13)$$

where D_t, D_x are the total derivative with respect to t, x respectively.

Remark 2.1. The infinitesimal transformations (3) should conserve the structure of the Riemann-Liouville fractional derivative operator, of which the lower limit in the integral is fixed. Therefore, the manifold $t=0$ should be invariant with respect to such transformations. The invariance condition arrives at

$$\tau(t,x,u,v)|_{t=0}=0.\qquad(14)$$

Remark 2.2. From the expressions of μ_1, μ_2, μ_3 and μ_4, if the infinitesimals η, ζ be linear with respect to the variables u and v, then $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, that is,

$$\frac{\partial^2 \eta}{\partial u^2} = \frac{\partial^2 \eta}{\partial v^2} = \frac{\partial^2 \zeta}{\partial u^2} = \frac{\partial^2 \zeta}{\partial v^2} = 0. \tag{15}$$

The one-parameter Lie symmetry transformations (3) are admitted by the system (2), if the following invariance criterion holds:

$$\begin{cases} prX\big(D_t^\alpha u + v_x + uu_x\big)|_{(2)} = 0, \\ prX\big(D_t^\alpha v + u_{xxx} + (uv)_x\big)|_{(2)} = 0, \end{cases} \tag{16}$$

which can be rewritten as

$$\begin{cases} \big(\eta^{\alpha,t} + \zeta^x + u\eta^x + u_x\eta\big)|_{(2)} = 0, \\ \big(\zeta^{\alpha,t} + \eta^{xxx} + u\zeta^x + v\eta^x + u_x\zeta + v_x\eta\big)|_{(2)} = 0. \end{cases} \tag{17}$$

Putting $\eta^{\alpha,t}$, $\zeta^{\alpha,t}$, η^x, ζ^x and η^{xxx} into (17) and letting coefficients of various derivatives of u and v to be zero, we can obtain the over-determined system of differential equations as follows:

$$\tau_x = \tau_u = \tau_v = \xi_t = \xi_u = \xi_v = \eta_v = 0, \tag{18}$$

$$\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n \in N, \tag{19}$$

$$\binom{\alpha}{n}\frac{\partial^n \zeta_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n \in N, \tag{20}$$

$$\zeta_u + \eta - u(\xi_x - \alpha\tau_t) = 0, \tag{21}$$

$$\zeta_v - \eta_u - (\xi_x - \alpha\tau_t) = 0, \tag{22}$$

$$\eta - u(\xi_x - \alpha\tau_t) = 0, \tag{23}$$

$$\zeta - v(\zeta_v - \eta_u + \xi_x - \alpha\tau_t) = 0, \tag{24}$$

$$\eta_u - \zeta_v - 3\xi_x + \alpha\tau_t = 0, \tag{25}$$

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - v\frac{\partial^\alpha \eta_v}{\partial t^\alpha} + u\eta_x + \zeta_x = 0, \tag{26}$$

$$\frac{\partial^\alpha \zeta}{\partial t^\alpha} - u\frac{\partial^\alpha \zeta_u}{\partial t^\alpha} - v\frac{\partial^\alpha \zeta_v}{\partial t^\alpha} + v\eta_x + u\zeta_x + \eta_{xxx} = 0. \tag{27}$$

Solving these equations altogether, with the conditions (14) and (15), we can obtain infinitesimals as follows:

$$\xi = c_1 x + c_2, \quad \tau = \frac{2c_1}{\alpha}t, \quad \eta = -c_1 u, \quad \zeta = -2c_1 v, \tag{28}$$

where c_1 and c_2 are arbitrary constants. So the system (2) admitted the two-dimension Lie algebra spanned by

$$X_1 = x\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial x}, \tag{29}$$

with $[X_1, X_2] = -X_2$.

The characteristic equation corresponding to the group generator X_1 is

$$\frac{\mathrm{d}x}{x} = \frac{\alpha \mathrm{d}t}{2t} = \frac{\mathrm{d}u}{-u} = \frac{\mathrm{d}v}{-2v}, \tag{30}$$

from which, we obtain the similarity variables $xt^{-\frac{\alpha}{2}}$, $ut^{\frac{\alpha}{2}}$ and vt^{α}. So we get the invariant solutions of the system (2) as follows:

$$u(t,x) = t^{-\frac{\alpha}{2}} f(\omega), \quad v(t,x) = t^{-\alpha} g(\omega), \tag{31}$$

with $\omega = xt^{-\frac{\alpha}{2}}$.

Theorem 2.1. *The similarity transformations $u(t,x) = t^{-\frac{\alpha}{2}} f(\omega)$, $v(t,x) = t^{-\alpha} g(\omega)$ with the similarity variable $\omega = xt^{-\frac{\alpha}{2}}$ reduce the system (2) to the system of fractional ordinary differential equations given by*

$$\begin{cases} (\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha} f)(\omega) + f(\omega) f'(\omega) + g'(\omega) = 0, \\ \\ (\mathcal{P}_{\frac{2}{\alpha}}^{1-2\alpha,\alpha} g)(\omega) + f(\omega) g'(\omega) + g(\omega) f'(\omega) + f'''(\omega) = 0, \end{cases} \tag{32}$$

where $(\mathcal{P}_{\delta}^{\iota,\kappa})$ is the left-hand Erdélyi-Kober fractional differential operator defined by

$$(\mathcal{P}_{\delta}^{\iota,\kappa}\psi)(\omega) := \prod_{j=0}^{m-1} (\iota + j - \frac{1}{\delta}\omega \frac{\mathrm{d}}{\mathrm{d}\omega})(\mathcal{K}_{\delta}^{\iota+\kappa, m-\kappa}\psi)(\omega), \quad \omega > 0,\ \delta > 0,\ \kappa > 0,$$

$$m = \begin{cases} [\kappa] + 1, & if\ \kappa \notin \mathbb{N}, \\ \kappa, & if\ \kappa \in \mathbb{N}, \end{cases}$$

where

$$(\mathcal{K}_{\delta}^{\iota,\kappa}\psi)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)} \int_1^{\infty} (s-1)^{\kappa-1} s^{-(\iota+\kappa)} \psi(\omega s^{\frac{1}{\delta}}) \mathrm{d}s, & if\ \kappa > 0, \\ \psi(\omega), & if\ \kappa = 0, \end{cases}$$

is the left-hand Erdélyi-Kober fractional integral operator.

Proof. For $0 < \alpha < 1$, the Riemann-Liouville time fractional derivative of $u(t,x)$ can be obtained as follows:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} (t^{-\frac{\alpha}{2}} f(\omega)) = \frac{\partial}{\partial t} \Big[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\frac{-\alpha}{2}} f(xs^{-\frac{\alpha}{2}}) \mathrm{d}s \Big].$$

Assuming $r = \frac{t}{s}$, we have

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial}{\partial t} \Big[\frac{t^{1-\frac{3\alpha}{2}}}{\Gamma(1-\alpha)} \int_1^{\infty} (r-1)^{-\alpha} r^{\frac{3\alpha}{2}-2} f(\omega r^{\frac{\alpha}{2}}) \mathrm{d}r \Big] = \frac{\partial}{\partial t} \Big[t^{1-\frac{3\alpha}{2}} (\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2}, 1-\alpha} f)(\omega) \Big].$$

Because of $\omega = xt^{-\frac{\alpha}{2}}$, the following relation holds:

$$t\frac{\partial}{\partial t}\psi(\omega) = tx(-\frac{\alpha}{2}) t^{-\frac{\alpha}{2}-1} \psi'(\omega) = -\frac{\alpha}{2} \omega \frac{\mathrm{d}}{\mathrm{d}\omega} \psi(\omega).$$

Hence, we arrive at

$$\frac{\partial}{\partial t}\Big[t^{1-\frac{3\alpha}{2}}(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},1-\alpha}f)(\omega)\Big]=t^{-\frac{3\alpha}{2}}\Big[(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},1-\alpha}f)(\omega)\Big]=t^{-\frac{3\alpha}{2}}(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}f)(\omega).$$

Similarly, the Riemann-Liouville time fractional derivative of $v(t,x)$ is

$$\frac{\partial^\alpha v}{\partial t^\alpha}=t^{-2\alpha}(\mathcal{P}_{\frac{2}{\alpha}}^{1-2\alpha,\alpha}g)(\omega).$$

Meanwhile,

$$v_x+uu_x=t^{-\frac{3\alpha}{2}}(g'(\omega)+f(\omega)f'(\omega)),$$
$$u_{xxx}+(uv)_x=t^{-2\alpha}(f'''(\omega)+f(\omega)g'(\omega)+g(\omega)f'(\omega)).$$

This completes the proof. □

3. Power series solutions, convergence analysis and numerical simulation

Next we use the power series method to derive the power series solutions of the reduced equations (32). Let us assume that the solutions have the following form:

$$f(\omega)=\sum_{k=0}^{\infty}a_k\omega^k,\quad g(\omega)=\sum_{k=0}^{\infty}b_k\omega^k,\tag{33}$$

where a_k and b_k are determined later. Then

$$f'(\omega)=\sum_{k=0}^{\infty}(k+1)a_{k+1}\omega^k,\quad g'(\omega)=\sum_{k=0}^{\infty}(k+1)b_{k+1}\omega^k,\tag{34}$$

$$f''(\omega)=\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\omega^k,\quad g''(\omega)=\sum_{k=0}^{\infty}(k+2)(k+1)b_{k+2}\omega^k,\tag{35}$$

$$f'''(\omega)=\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)a_{k+3}\omega^k,\quad g'''(\omega)=\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)b_{k+3}\omega^k.\tag{36}$$

From the definition of the left-hand Erdélyi-Kober fractional integral operator, we get

$$\begin{aligned}(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}f)(\omega)=&\frac{1}{\Gamma(m-\alpha)}\int_1^\infty(s-1)^{m-\alpha-1}s^{-(1-\frac{3\alpha}{2}+m)}f(\omega s^{\frac{\alpha}{2}})\mathrm{d}s\\=&\frac{1}{\Gamma(m-\alpha)}\int_1^\infty(s-1)^{m-\alpha-1}s^{-(1-\frac{3\alpha}{2}+m)}\sum_{k=0}^{\infty}(a_k\omega^k s^{\frac{k\alpha}{2}})\mathrm{d}s\\=&\sum_{k=0}^{\infty}a_k\omega^k\Big[\frac{1}{\Gamma(m-\alpha)}\int_1^\infty(s-1)^{m-\alpha-1}s^{-(1-\frac{3\alpha}{2}-\frac{k\alpha}{2}+m)}\mathrm{d}s\Big].\end{aligned}$$

Because of Beta function $B(p,q)=\int_0^1x^{p-1}(1-x)^{q-1}\mathrm{d}s$, assuming $t=\frac{1}{x}$, we have

$$B(p,q)=\int_0^1x^{p-1}(1-x)^{q-1}\mathrm{d}x=\int_1^\infty(t-1)^{q-1}t^{-(p+q)}\mathrm{d}t.$$

So

$$(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}f)(\omega)=\sum_{k=0}^{\infty}a_k\omega^k\frac{B(1-\frac{\alpha}{2}-\frac{k\alpha}{2},m-\alpha)}{\Gamma(m-\alpha)}=\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{k\alpha}{2})}{\Gamma(1-\frac{3\alpha}{2}-\frac{k\alpha}{2}+m)}a_k\omega^k.$$

With $m=[\alpha]+1=1$, we get

$$\begin{aligned}(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}f)(\omega)=&\prod_{j=0}^{m-1}(1-\frac{3\alpha}{2}+j-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}f)(\omega)\\ =&(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{k\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}-\frac{k\alpha}{2})}a_k\omega^k)\\ =&\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{k\alpha}{2})}{\Gamma(1-\frac{3\alpha}{2}-\frac{k\alpha}{2})}a_k\omega^k.\end{aligned}\tag{37}$$

Similarly,

$$(\mathcal{P}_{\frac{2}{\alpha}}^{1-2\alpha,\alpha}g)(\omega)=\sum_{k=0}^{\infty}\frac{\Gamma(1-\alpha-\frac{k\alpha}{2})}{\Gamma(1-2\alpha-\frac{k\alpha}{2})}b_k\omega^k.\tag{38}$$

Substituting (33)-(38) into the system (32) arrives at the following equations:

$$\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k\omega^k=-\sum_{k=0}^{\infty}\Big[\sum_{m+n=k}(n+1)a_ma_{n+1}\omega^k+(k+1)b_{k+1}\Big]\omega^k,$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+2)\alpha}{2})}{\Gamma(1-\frac{(k+4)\alpha}{2})}b_k\omega^k=-\sum_{k=0}^{\infty}\Big[\sum_{m+n=k}(n+1)(a_mb_{n+1}+b_ma_{n+1})+(k+3)(k+2)(k+1)a_{k+3}\Big]\omega^k.$$

Equating the coefficients of different powers of ω arrives at the following system:

$$\begin{cases}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k=-(k+1)b_{k+1}-\sum_{m+n=k}(n+1)a_ma_{n+1},\\ \frac{\Gamma(1-\frac{(k+2)\alpha}{2})}{\Gamma(1-\frac{(k+4)\alpha}{2})}b_k=-(k+3)(k+2)(k+1)a_{k+3}-\sum_{m+n=k}(n+1)(a_mb_{n+1}+b_ma_{n+1}),\end{cases}\tag{39}$$

from which, we can obtain the explicit expressions of a_k and b_k. For $k=0$, we have

$$\begin{cases}b_1=-\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1-\frac{3\alpha}{2})}a_0-a_0a_1,\\ a_3=-\frac{1}{6}\Big[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}b_0+b_0a_1+a_0b_1\Big].\end{cases}\tag{40}$$

For $k=1$, we have

$$\begin{cases}b_2=-\frac{1}{2}\Big[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}a_1+a_1^2+2a_0a_2\Big],\\ a_4=-\frac{1}{24}\Big[\frac{\Gamma(1-\frac{3\alpha}{2})}{\Gamma(1-\frac{5\alpha}{2})}b_1+2(b_0a_2+a_1b_1+a_0b_2)\Big].\end{cases}\tag{41}$$

For $k=2$, we have

$$\begin{cases} b_3 = -\frac{1}{3}\Big[\frac{\Gamma(1-\frac{3\alpha}{2})}{\Gamma(1-\frac{5\alpha}{2})}a_2 + 3(a_1a_2 + a_0a_3)\Big], \\ a_5 = -\frac{1}{60}\Big[\frac{\Gamma(1-2\alpha)}{\Gamma(1-3\alpha)}b_2 + 3(a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3)\Big]. \end{cases} \tag{42}$$

For $k>2$, we have

$$\begin{cases} b_{k+1} = \frac{-1}{(k+1)}\Big[\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k + \sum_{m+n=k}(n+1)a_ma_{n+1}\Big], \\ a_{k+3} = \frac{-1}{(k+3)(k+2)(k+1)}\Big[\frac{\Gamma(1-\frac{(k+2)\alpha}{2})}{\Gamma(1-\frac{(k+4)\alpha}{2})}b_k + \sum_{m+n=k}(n+1)(a_mb_{n+1} + b_ma_{n+1})\Big]. \end{cases} \tag{43}$$

In what follows, the convergence analysis of the power series solutions for the system (32) will be presented. From (40)-(43), we get

$$\begin{cases} |b_{k+1}| \le 2(|a_k| + \sum_{m+n=k}|a_m||a_{n+1}|), \\ |a_{k+3}| \le |b_k| + \sum_{m+n=k}(|a_m||b_{n+1}| + |b_m||a_{n+1}|). \end{cases} \tag{44}$$

Assuming

$$C(\theta) = \sum_{k=0}^{\infty} c_k\theta^k, \quad D(\theta) = \sum_{k=0}^{\infty} d_k\theta^k, \tag{45}$$

where $c_0 = |a_0|$, $d_0 = |b_0|$, $c_1 = |a_1|$, $c_2 = |a_2|$ and

$$\begin{cases} d_{k+1} = 2(c_k + \sum_{m+n=k} c_mc_{n+1}), \ k = 0,1,2\cdots, \\ c_{k+3} = d_k + \sum_{m+n=k}(c_md_{n+1} + d_mc_{n+1}), \ k = 0,1,2\cdots, \end{cases} \tag{46}$$

that is, $|a_k| \le c_k$, $|b_k| \le d_k$, $k = 0,1,2\cdots$, we can get

$$\begin{aligned} C(\theta) &= c_0 + c_1\theta + c_2\theta^2 + \sum_{k=0}^{\infty} c_{k+3}\theta^{k+3} \\ &= c_0 + c_1\theta + c_2\theta^2 + \sum_{k=0}^{\infty}\big(d_k + \sum_{m+n=k}(c_md_{n+1} + d_mc_{n+1})\big)\theta^{k+3} \\ &= c_0 + c_1\theta + c_2\theta^2 + \Big[D(\theta) + \sum_{k=0}^{\infty}\sum_{m+n=k}(c_md_{n+1} + d_mc_{n+1})\theta^k\Big]\theta^3, \end{aligned} \tag{47}$$

$$\begin{aligned} D(\theta) &= d_0 + \sum_{k=0}^{\infty} d_{k+1}\theta^{k+1} = d_0 + 2\sum_{k=0}^{\infty}\Big[c_k + \sum_{m+n=k} c_mc_{n+1}\Big]\theta^{k+1} \\ &= d_0 + 2\Big[C(\theta) + \sum_{k=0}^{\infty}\sum_{m+n=k} c_mc_{n+1}\theta^k\Big]\theta. \end{aligned} \tag{48}$$

Consider the implicit functional system with respect to the independent variable θ,

$$H(\theta, C, D) = C - c_0 - c_1\theta - c_2\theta^2 - \Big[D(\theta) + \sum_{k=0}^{\infty}\sum_{m+n=k}(c_md_{n+1} + d_mc_{n+1})\theta^k\Big]\theta^3, \tag{49}$$

$$H(\theta, C, D) = C - c_0 - c_1\theta - c_2\theta^2 - \Big[D(\theta) + \sum_{k=0}^{\infty} \sum_{m+n=k} (c_m d_{n+1} + d_m c_{n+1})\theta^k\Big]\theta^3, \quad (50)$$

$$I(\theta, C, D) = D - d_0 - 2\Big[C(\theta) + \sum_{k=0}^{\infty} \sum_{m+n=k} c_m c_{n+1}\theta^k\Big]\theta, \quad (51)$$

from which, H and I are analytic in the neighborhood of point $(0, c_0, d_0)$ and $H(0, c_0, d_0) = 0$, $I(0, c_0, d_0) = 0$. The Jacobian determinant is

$$J = \frac{\partial(H, I)}{\partial(C, D)} \neq 0. \quad (52)$$

Then the two series $C = C(\theta)$ and $D = D(\theta)$ are analytic in the neighborhood of $(0, c_0, d_0)$ with positive radius by implicit function theorem, that is, the series $f(\omega)$ and $g(\omega)$ are convergent in the neighborhood of $(0, c_0, d_0)$.

Therefore, the power series solutions of time fractional coupled Boussinesq-Whitham-Broer-Kaup equations (2) have the form

$$u(t, x) = t^{-\frac{\alpha}{2}} f(\omega) = \sum_{k=0}^{\infty} a_k x^k t^{-\frac{(k+1)\alpha}{2}}, \quad v(t, x) = t^{-\alpha} g(\omega) = \sum_{k=0}^{\infty} b_k x^k t^{-\frac{(k+2)\alpha}{2}}, \quad (53)$$

where a_k and b_k are defined by (40)-(43) with arbitrary initial conditions $a_0 = f(0)$, $b_0 = g(0)$, $a_1 = f'(0)$ and $a_2 = \frac{1}{2} f''(0)$.

In Figs. 1-3, we illustrate the power series solutions (53) with different parameter values. For the given initial conditions $a_0 = b_0 = a_1 = a_2 = 1$, these figures show that the difference of fractional order α affects the changes of the velocity $u(t, x)$ and the height $v(t, x)$ of the free wave surface. Therefore, time fractional coupled Boussinesq-Whitham-Broer-Kaup equations (2) can better reflect the continuous change trend of the real situation compared with the classical equations (1).

4. Conservation laws of Eqs. (2)

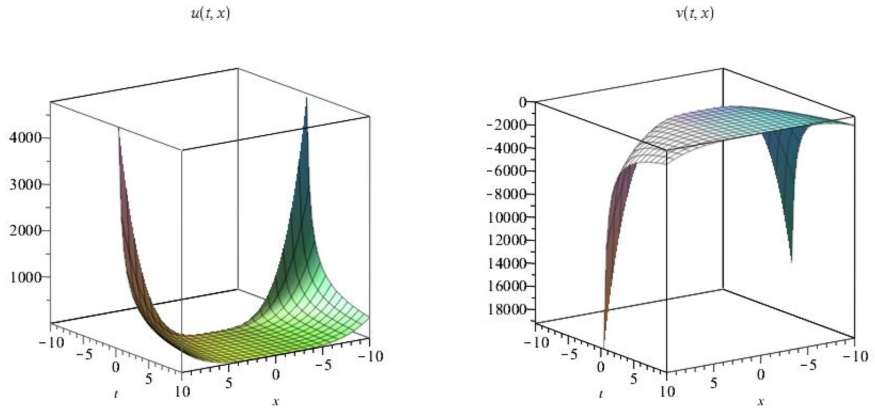
In this section, we construct conservation laws of Eqs. (2) by using the generalization of the Noether operators and the new conservation theorem [**13, 14**].

The system (2) is denoted as

$$\begin{cases} F_1 = D_t^\alpha u + v_x + u u_x = 0, \\ F_2 = D_t^\alpha v + u_{xxx} + (uv)_x = 0, \end{cases} \quad (54)$$

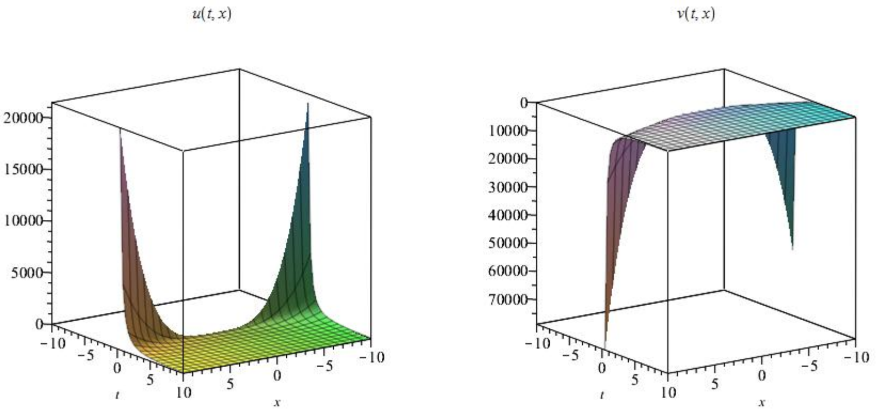
of which the formal Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= p(t, x) F_1 + q(t, x) F_2 \\ &= p(t, x)\big(D_t^\alpha u + v_x + u u_x\big) + q(t, x)\big(D_t^\alpha v + u_{xxx} + (uv)_x\big), \end{aligned} \quad (55)$$



(a) (b)

Figure 1. Numerical simulation of the power series solutions (53) with $a_0 = b_0 = a_1 = a_2 = 1$, $b_1 = -1.688365219$, $a_3 = 0.01719507718$, $b_2 = -1.792597378$, $a_4 = 0.2381049175$, $b_3 = -1.165779206$, $b_4 = -0.7563022914$ and $\alpha = 0.3$.

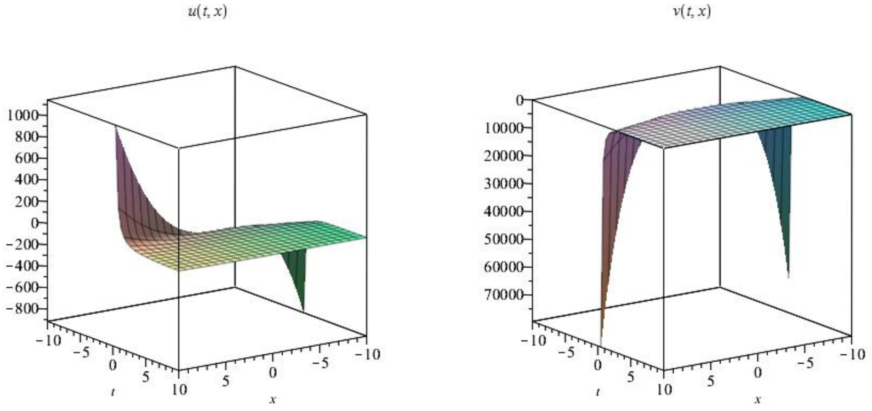


(a) (b)

Figure 2. Numerical simulation of the power series solutions (53) with $a_0 = b_0 = a_1 = a_2 = 1$, $b_1 = -0.9328716540$, $a_3 = 0.1227355057$, $b_2 = -1.098229310$, $a_4 = 0.4490858001$, $b_3 = -2.545190408$, $b_4 = -1.060062402$ and $\alpha = 0.7$.

where $p(t, x)$ and $q(t, x)$ are new dependent variables. The Euler-Lagrange operators are

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha u)} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \tag{56}$$



(a) (b)

FIGURE 3. Numerical simulation of the power series solutions (53) with $a_0 = b_0 = a_1 = a_2 = 1$, $b_1 = -0.5915327160$, $a_3 = 0.2082259476$, $b_2 = -0.6710885150$, $a_4 = -0.00298349118$, $b_3 = -0.8718983963$, $b_4 = -0.6852929807$ and $\alpha = 0.9$.

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha v)} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v_{i_1 \cdots i_s}}, \tag{57}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α. It is defined by the right-sided of Caputo fractional derivative, i.e.,

$$(D_t^\alpha)^* f(t,x) \equiv {}_t^c D_T^\alpha f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{1}{(t-s)^{\alpha-n+1}} \frac{\partial^n}{\partial s^n} f(s,x) \mathrm{d}s, & n-1 < \alpha < n, n \in \mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N}. \end{cases}$$

The system of adjoint equations to (54) is given by

$$\begin{cases} F_1^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^\alpha)^* p - u p_x - v q_x - q_{xxx} = 0, \\ \\ F_2^* = \frac{\delta \mathcal{L}}{\delta v} = (D_t^\alpha)^* q - p - u q_x = 0. \end{cases} \tag{58}$$

Next we use the above adjoint equations and the new conservation theorem to construct conservation laws of Eqs. (2). From the classical definition of the conservation laws, a vector $C = (C^t, C^x)$ is called a conserved vector for the governing equation if it satisfies the conservation equation $[D_t C^t + D_x C^x]_{F_1, F_2 = 0} = 0$. By using Noether theorem the components of conserved vector can be obtained.

Firstly, from the fundamental operator identity, i.e.,

$$prX + D_t \tau \cdot \mathcal{I} + D_x \xi \cdot \mathcal{I} = W^u \cdot \frac{\delta}{\delta u} + W^v \cdot \frac{\delta}{\delta v} + D_t \mathcal{N}^t + D_x \mathcal{N}^x, \tag{59}$$

where prX is mentioned in (5), $\mathcal{I}$ is the identity operator and $W^u = \eta - \tau u_t - \xi u_x$, $W^v = \zeta - \tau v_t - \xi v_x$ are the characteristics for group generator X, we can get the

Noether operators as follows:

$$\begin{aligned}\mathcal{N}^t =&\tau\mathcal{I}+\sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W^u)D_t^k\frac{\partial}{\partial(D_t^\alpha u)}-(-1)^n J(W^u, D_t^n\frac{\partial}{\partial(D_t^\alpha u)})\\ &+\sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W^v)D_t^k\frac{\partial}{\partial(D_t^\alpha v)}-(-1)^n J(W^v, D_t^n\frac{\partial}{\partial(D_t^\alpha v)}),\end{aligned} \tag{60}$$

$$\begin{aligned}\mathcal{N}^x =&\xi\mathcal{I}+W^u\big(\frac{\partial}{\partial u_x}-D_x\frac{\partial}{\partial u_{xx}}+D_x^2\frac{\partial}{\partial u_{xxx}}\big)+W^v\big(\frac{\partial}{\partial v_x}-D_x\frac{\partial}{\partial v_{xx}}+D_x^2\frac{\partial}{\partial v_{xxx}}\big)\\ &+D_x(W^u)\big(\frac{\partial}{\partial u_{xx}}-D_x\frac{\partial}{\partial u_{xxx}}\big)+D_x(W^v)\big(\frac{\partial}{\partial v_{xx}}-D_x\frac{\partial}{\partial v_{xxx}}\big)\\ &+D_x^2(W^u)\frac{\partial}{\partial u_{xxx}}+D_x^2(W^v)\frac{\partial}{\partial v_{xxx}},\end{aligned} \tag{61}$$

where $n=[\alpha]+1$ and J is given by

$$J(f,g)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau,x)g(\theta,x)}{(\theta-\tau)^{\alpha+1-n}}\mathrm{d}\theta\mathrm{d}\tau. \tag{62}$$

The components of conserved vector are defined by $C^t=\mathcal{N}^t\mathcal{L}$, $C^x=\mathcal{N}^x\mathcal{L}$.

Case 1: $X_1=x\frac{\partial}{\partial x}+\frac{2}{\alpha}t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-2v\frac{\partial}{\partial v}$

The characteristics of X_1 are

$$W^u=-u-\frac{2}{\alpha}tu_t-xu_x,\quad W^v=-2v-\frac{2}{\alpha}tv_t-xv_x. \tag{63}$$

Therefore, for $0<\alpha<1$,

$$\begin{aligned}C^t =&pD_t^{\alpha-1}(W^u)+J(W^u,p_t)+qD_t^{\alpha-1}(W^v)+J(W^v,q_t)\\ =&-pD_t^{\alpha-1}(u+\frac{2}{\alpha}tu_t+xu_x)-J(u+\frac{2}{\alpha}tu_t+xu_x,p_t)\\ &-qD_t^{\alpha-1}(2v+\frac{2}{\alpha}tv_t+xv_x)-J(2v+\frac{2}{\alpha}tv_t+xv_x,q_t),\end{aligned} \tag{64}$$

$$\begin{aligned}C^x =&(up+vq+q_{xx})W^u+(p+uq)W^v-q_xD_x(W^u)+qD_x^2(W^u)\\ =&-(up+vq+q_{xx})(u+\frac{2}{\alpha}tu_t+xu_x)-(p+uq)(2v+\frac{2}{\alpha}tv_t+xv_x)\\ &+q_x(2u_x+\frac{2}{\alpha}tu_{xt}+xu_{xx})-q(3u_{xx}+\frac{2}{\alpha}tu_{xxt}+xu_{xxx}).\end{aligned} \tag{65}$$

Case 2: $X_2=\frac{\partial}{\partial x}$

The characteristics of X_2 are

$$W^u=-u_x,\quad W^v=-v_x. \tag{66}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned} C^t =& pD_t^{\alpha-1}(W^u) + J(W^u, p_t) + qD_t^{\alpha-1}(W^v) + J(W^v, q_t) \\ =& -pD_t^{\alpha-1}(u_x) - J(u_x, p_t) - qD_t^{\alpha-1}(v_x) - J(v_x, q_t), \end{aligned} \tag{67}$$

$$\begin{aligned} C^x =& (up + vq + q_{xx})W^u + (p + uq)W^v - q_x D_x(W^u) + qD_x^2(W^u) \\ =& -(up + vq + q_{xx})u_x - (p + uq)v_x + q_x u_{xx} - qu_{xxx}. \end{aligned} \tag{68}$$

5. Conclusion

In this paper, we can see that Lie symmetry analysis method is effective for studying an important model in mathematical physics, namely time fractional coupled Boussinesq-Whitham-Broer-Kaup equations as follows:

$$\begin{cases} D_t^{\alpha} u + v_x + uu_x = 0, \\ D_t^{\alpha} v + u_{xxx} + (uv)_x = 0. \end{cases}$$

We obtain all Lie symmetries for the equation when $0 < \alpha < 1$, and reduce the corresponding system of fractional partial differential equations to the system of fractional ordinary differential equations. Furthermore, we derive the power series solutions for the reduced systems and obtain the conservation laws for every Lie symmetry. The paper shows that Lie symmetry analysis method and the power series method provide the direct and powerful mathematical tools to further study other fractional differential equations in mathematical physics.

References

[1] E. Atilgan, M. Senol, A. Kurt, and O. Tasbozan, *New wave solutions of time-fractional coupled Boussinesq-Whitham-Broer-Kaup equation as a model of water waves*, China Ocean Eng., **33** (2019), 477–483.

[2] S. Q. Chen, M. H. Li, B. Guan, Y. Li, Y. Wang, X Lin, and T. Liu, *Abundant variant wave patterns by coupled Boussinesq-Whitham-Broer-Kaup equations*, Chinese J. Phys., **78** (2022), 485–494.

[3] V. Daftardar-Gejji and H. Jafari, *Adomian decomposition: a tool for solving a system of fractional differential equations*, J. Math. Anal. Appl., **301** (2005), 508–518.

[4] E. H. El Kinani and A. Ouhadan, *Lie symmetry analysis of some time fractional partial differential equations*, Int. J. Mod. Phys. Conf. Ser., **38** (2015), 1560075.

[5] Y. Q. Feng and J. C. Yu, *Lie symmetry analysis of fractional ordinary differential equation with neutral delay*, AIMS Mathematics, **6** (2021), 3592–3605.

[6] R. K. Gazizov and A. A. Kasatkin, *Construction of exact solutions for fractional order differential equations by the invariant subspace method*, Comput. Math. Appl., **66** (2013), 576–584.

[7] R. K. Gazizov, A. A. Kasatkin, and S. Y. Lukashchuk, *Continuous transformation groups of fractional differential equations*, Vestnik USATU, **9** (2007), 125–135.

[8] R. K. Gazizov, A. A. Kasatkin, and S. Y. Lukashchuk, *Symmetry properties of fractional diffusion equations*, Phys. Scr., **T136** (2009), 014016.

[9] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[10] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 1, CRC Press, 1993.

[11] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 2, CRC Press, 1994.

[12] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume 3, CRC Press, 1995.

[13] N. H. Ibragimov, *A new conservation theorem*, J. Math. Anal. Appl., **333** (2007), 311–328.

[14] N. H. Ibragimov, *Nonlinear self-adjointness and conservation laws*, J. Phys. A-Math. Theor., **44** (2011), 432002.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, New York, 2006.

[16] M. M. Meerschaert, H. P. Scheffler, and C. Tadjeran, *Finite difference methods for two-dimensional fractional dispersion equation*, J. Comput. Phys., **211** (2006), 249–261.

[17] S. Momani and Z. Odibat, *Homotopy perturbation method for nonlinear partial differential equations of fractional order*, Phys. Lett. A, **365** (2007), 345–350.

[18] S. Momani and Z. Odibat, *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos Solitons Fractals, **31** (2007), 1248–1255.

[19] A. M. Nass, *Symmetry analysis of space-time fractional Poisson equation with a delay*, Quaest. Math., **42** (2019), 1221–1235.

[20] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Heidelberg: Springer, 1986.

[21] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[22] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[23] R. L. Sachs, *On the integrable variant of the Boussinesq system: Painlevé property rational solutions, a related many-body system, and equivalence with the AKNS hierarchy*, Physica D, **30** (1988), 1–27.

[24] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

[25] M. Yourdkhany and M. Nadjafikhah, *Symmetries, similarity invariant solution, conservation laws and exact solutions of the time-fractional Harmonic Oscillator equation*, J. Geom. Phys., **153** (2020), 103661.

[26] J. C. Yu, *Lie symmetry analysis of time fractional Burgers equation, Korteweg-de Vries equation and generalized reaction-diffusion equation with delays*, Int. J. Geom. Meth. Mod. Phys., **19** (2022), 2250219.

[27] J. C. Yu, *Lie symmetry, exact solutions and conservation laws of time fractional Black–Scholes equation derived by the fractional Brownian motion*, J. Appl. Anal., **30** (2024), 137–145.

[28] J. C. Yu and Y. Q. Feng, *Lie symmetry analysis and exact solutions of space-time fractional cubic Schrödinger equation*, Int. J. Geom. Meth. Mod. Phys., **19** (2022), 2250077.

[29] J. C. Yu and Y. Q. Feng, *Lie symmetry, exact solutions and conservation laws of some fractional partial differential equations*, J. Appl. Anal. Comput., **13** (2023), 1872–1889.

[30] J. C. Yu and Y. Q. Feng, *Group classification of time fractional Black-Scholes equation with time-dependent coefficients*, Fract. Calc. Appl. Anal., **27** (2024), 2335–2358.

[31] J. C. Yu and Y. Q. Feng, *Lie symmetry analysis, power series solutions and conservation laws of (2+1)-dimensional time fractional modified Bogoyavlenskii–Schiff equations*, J. Nonlinear Math. Phys., **31** (2024), 27.

[32] J. C. Yu and Y. Q. Feng, *Lie symmetries, exact solutions and conservation laws of time fractional Boussinesq-Burgers system in ocean waves*, Commun. Theor. Phys., **76** (2024), 125002.

[33] J. C. Yu and Y. Q. Feng, *On the generalized time fractional reaction–diffusion equation: Lie symmetries, exact solutions and conservation laws*, Chaos Solitons Fractals, **182** (2024), 114855.

[34] J. C. Yu and Y. Q. Feng, *Symmetry analysis, optimal system, conservation laws and exact solutions of time fractional diffusion-type equation*, Int. J. Geom. Meth. Mod. Phys., Online Ready, (2024), 2450286.

[35] J. C. Yu, Y. Q. Feng, and X. J. Wang, *Lie symmetry analysis and exact solutions of time fractional Black-Scholes equation*, Int. J. Financ. Eng., **9** (2022), 2250023.

[36] Z. Y. Zhang, *Symmetry determination and nonlinearization of a nonlinear time-fractional partial differential equation*, Proc. R. Soc. A, **476** (2020), 20190564.

[37] Z. Y. Zhang and G. F. Li, *Lie symmetry analysis and exact solutions of the time-fractional biological population model*, Phys. A, **540** (2020), 123134.

[38] S. Zhang and H. Q. Zhang, *Fractional sub-equation method and its applications to nonlinear fractional PDEs*, Phys. Lett. A, **375** (2011), 1069–1073.

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