Mathematical Analysis and its Contemporary Applications
Volume 6, Issue 3, 2024, 65–77
doi: 10.30495/maca.2024.2043696.1110
ISSN 2716-9898

$(\lambda, \mu)_q$-statistical convergence for double sequences

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Abstract. In this article, we define q-analogue of (λ, μ)-statistical convergence for double sequences. q-analogue of (λ, μ)-statistically Cauchy and pre-Cauchy is also defined. The necessary and sufficient condition for $(\lambda, \mu)_q$-statistically Cauchy is also given.

1. Introduction

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1.$$

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) := \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if

$$t_n(x) \to L \ as \ n \to \infty.$$

If we take $\lambda_n = n$, then the above summability-reduces to $(C, 1)$. We write

2020 *Mathematics Subject Classification.* 40B05, 40H05, 54A20.
Key words and phrases. Statistical convergence, q-analog, de la Valée-Pousin mean
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$$[C,1] := \{x = (x_n) : \exists\ L \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_k - L| = 0\}$$

and

$$[V,\lambda] := \{x = (x_n) : \exists\ L \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |x_k - L| = 0\}$$

for the sets of sequences $x = (x_k)$ which are strongly Cesaro summable and strongly (V,λ)-summable to L i .e. $x_k \longrightarrow [C,1]$ and $x_k \longrightarrow [L,\lambda]$ respectively. The statistical convergence was first introduced by Fast [10] and later on studied by many authors see [5, 11, 12, 21, 22]. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |x_k - L| \ge \varepsilon\}| = 0, \tag{1}$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

Definition 1.1. [16] A sequence $x = (x_n)$ is said to be λ-statistically convergent or S_λ-convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$, and

$$S_\lambda := x : \exists\ L \in R, S_\lambda - \lim x = L.$$

The quantum calculus or a q-calculus is the generalization as well as modifications of the classical calculus. Over the last twenty years, the subject of q-calculus has served as a connection between mathematics and physics. In recent past there is a substantial increase of research in field of q-calculus because of its applications in many fields such as number theory, combinatorics, special functions basic hyper geometric functions mechanics, theory of relativity and other sciences-quantum theory.

The q-statistical convergence is also applied in Approximation theory and Summability also see [20] , q-statistical convergence has been generalized to double q-statistical convergence [4, 15, 19]. q-calculus has been studied by many researchers, for instance see [1, 2, 3, 13]

Definition 1.2. [18] The q-analog of real numbers is given by

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q}, & \text{if } q \in \mathbb{R}^+ - \{1\}\ ; \\ r, & \text{if } q = 1. \end{cases}$$

The formal definition of q-analog is that "Analog of a theorem, identity or expression is a generalization involving a new parameter q that returns the original theorem, identity or expression in the limit as $q \longrightarrow 1$."
This concludes a fact that q-analog of something is not a unique expression as we just need to satisfy the limiting condition of 1 and also that above equation is not a definition but an example of many q-analog of numbers. For example r^q can also be a q-analog of numbers as many others. The prior q-analog is the best one to the real numbers

Definition 1.3. [18] Let $K \subseteq \mathbb{N}$. For $q \geq 1$, the q-density of K is given by

$$\delta_q(K) = \delta_{C_1^q}(K) = \lim_{n\to\infty} \inf(C_1^q \chi_K)_n.$$

Definition 1.4. [18] A number sequence $x = (x_k)$ is said to be q-statistically convergent to L, if for every $\epsilon > 0$, $\delta_q(K) = 0$, where $K = \{k : k \leq n : |x_k - L| \geq \epsilon\}$.

The set of all q-statistically convergent sequences is denoted by S_q.

Definition 1.5. [17] A double sequence $x = (x_{st})$ is said to be statistically convergent to the number L
if for every $\epsilon > 0$, $\delta^2(K) = 0$, where

$$K = \{(s,t) : s \leq p \text{ and } t \leq r : |x_{st} - L| \geq \epsilon\}. \tag{2}$$

Definition 1.6. [19] A double sequence $x = (x_{st})$ is said to be q-statistically convergent to L, if for every $\epsilon > 0$, $\delta_q^2(K) = 0$, where

$$K = \{(s,t) : s \leq p \text{ and } t \leq r : |x_{st} - L| \geq \epsilon\}. \tag{3}$$

The set of all q-statistically convergent double sequences is denoted by St_2^q. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be a two non-decreasing sequences of positive numbers, each tending to ∞ such that

$$\lambda_{r+1} \leq \lambda_r + 1\ ,\ \lambda_1 = 1\ ,$$
$$\mu_{s+1} \leq \mu_s + 1\ ,\ \mu_1 = 1.$$

Let $I_r = [r - \lambda_r + 1, r]$, $I_s = [s - \mu_s + 1, s]$ and $I_{r,s} = I_r \times I_s$. The generalized double de la Valée-Pousin mean is defined by

$$t_{r,s}(x) := \frac{1}{\lambda_r \mu_s} \sum_{k\in I_r} \sum_{l\in I_s} x_{k,l}.$$

Definition 1.7. A double sequence $x = (x_{k,l})$ is said to be (V,λ,μ)-summable to a number L if

$$P - \lim_{r,s} t_{r,s}(x_{k,l}) \to L \ as\ r,s \to \infty.$$

If $\lambda_r = r$ and $\mu_s = s$ then (V,λ,μ)-summability reduces to $(C,1,1)$-summability. We write

$$[C,1,1] := \{x = (x_{k,l}) : \exists\ L \in \mathbb{R},\ P - \lim_{r,s\to\infty} \frac{1}{rs}\sum_{k=1}^{r,s} |x_{k,l} - L| = 0\}$$

and

$$[V,\lambda,\mu] := \{x = (x_{k,l}) : \exists\ L \in \mathbb{R},\ P - \lim_{r,s\to\infty} \frac{1}{\lambda_r\mu_s}\sum_{k,l\in I_r\times I_s} |x_{k,l} - L| = 0\}$$

for the sets of double sequences $x = (x_{k,l})$ which are strongly Cesaro summable and strongly (V,λ,μ)-summable L i .e. $x_{k,l} \longrightarrow [C,1,1]$ and $x_{k,l} \longrightarrow [L,\lambda,\mu]$ respectively.

Definition 1.8. A double sequence $x = (x_{k,l})$ is said to be (λ,μ)- statistically convergent or $S_{\lambda,\mu}$convergent to L if for every $\epsilon > 0$

$$P - \lim_{r,s\to\infty} \frac{1}{\lambda_r\mu_s}|\{(k,l) \in I_{r,s} : |x_{kl} - L| \geq \epsilon\}| = 0.$$

i.e., the set $K(\epsilon) = \{(k,l) \in I_r \times I_s : |x_{k,l} - L| \geq \epsilon\}$ has (λ,μ)-density zero.

That is $S_{(\lambda,\mu)} - \lim x = L$ or $x_{kl} \to L(S_{(\lambda,\mu)})$.

2. Main Results

Definition 2.1. We defined a matrix which is the q-analog of a matrix say A which is equivalent to the $[V,\lambda] - summability$ and it is given as follows

$$v_{nk}(q^k) = \begin{cases} \dfrac{q^{k-1}}{[\lambda_n]_q}, & if\ k \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

The summability associated with this matrix is the q-analog of (V,λ)-summability and it is as follows

$$[V,\lambda]_q = \{x = (x_n) : \exists L \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{[\lambda_n]_q}\sum_{k\in I_n} |q^{k-1}x_k - L| = 0\}$$

Definition 2.2. Define a matrix which is the q-analog of a matrix say B which is equivalent to the $[V,\lambda,\mu] - summability$ and it is given as follows

$$v_{rskl}(q) = \begin{cases} \dfrac{q^{k+l-2}}{[\lambda_r]_q[\mu_s]_q}, & if\ k \in I_r\ and\ l \in I_s \\ 0, & \text{otherwise.} \end{cases}$$

The summability associated with this matrix is the q-analog of (V,λ,μ)-summability and it is as follows

$$[V,\lambda,\mu]_q = \{x = (x_{kl}) : \exists\ L \in \mathbb{R}, P - \lim_{r,s\to\infty} \frac{1}{[\lambda_r]_q[\mu_s]_q}\sum_{k,l\in I_{r,s}} |q^{k+l-2}x_{kl} - L| = 0\}$$

Definition 2.3. Let $[\lambda]_q = ([\lambda_r]_q)$ and $[\mu]_q = ([\mu_s]_q)$ be a two non-decreasing sequences of positive numbers tending to ∞ such that

$$[\lambda_{r+1}]_q \leq [\lambda_r]_q + 1, [\lambda_1]_q = 1.$$

$$[\mu_{s+1}]_q \leq [\mu_s]_q + 1, [\mu_1]_q = 1.$$

Such sets are denoted by $\wedge_q^2$.

Definition 2.4. The q-analog of generalized de la Valée-Pousin mean is given as follows

$$[v_{rs}(x)]_q = \frac{1}{[\lambda_r]_q [\mu_s]_q} \sum_{k \in I_r} \sum_{l \in I_s} q^{k+l-2} x_{k,l}$$

where $I_r = [[r - \lambda_r + 1]_q, [r]_q]$ and $I_s = [[s - \mu_s + 1]_q, [s]_q]$.

Definition 2.5. Let $K \subseteq \mathbb{N}$. For $q \geq 1$, the $(\lambda, \mu)_q$-density of K is given by

$$\delta_{(\lambda,\mu)_q}(K) = P - \lim_{r,s \to \infty} \inf(v_{rskl}(q^{kl})\chi_K)_{rs}.$$

Definition 2.6. A double sequence $x = (x_{kl})$ is said to be $(\lambda, \mu)_q$-statistically convergent to L if for every $\epsilon > 0$, $\delta_{(\lambda,\mu)_q}(K) = 0$, where $K = \{k \in I_r \text{and } l \in I_s : |x_{kl} - L| \geq \epsilon\}| = 0$.

Such set is denoted by $S_{(\lambda,\mu)_q}$

Proposition 2.1. *If M and N are two subsets of $\mathbb{N}$ such that $M \subset N$, then $\delta_{(\lambda,\mu)_q}(M) \leq \delta_{(\lambda,\mu)_q}(N)$.*

Proposition 2.2. *If $M \subseteq \mathbb{N}$, then $\delta_{(\lambda,\mu)_q}(M) + \delta_{(\lambda,\mu)_q}(M^c) = 1$.*

Proposition 2.3. *The q-analogs given in Definitions 2.2, 2.4, 2.5 and 2.6 is valid.*

Proof: When $q \to 1$, $[\lambda_r]_q \to \lambda_r$,$[\mu_s]_q \to \lambda_s$ and $q^{k+l-2} \to 1$. Thus we get

$$v_{rskl}(q^{kl}) \to v_{rskl} = \begin{cases} \dfrac{1}{\lambda_r \mu_s}, & if\ k \in I_r\ and\ l \in I_s, \\ 0, & otherwise. \end{cases}$$

v_{rskl}-summability is actually equivalent to $[V, \lambda, \mu]$-summability. We get

$$\lim_{q \to 1} v_{rskl}(q^{kl}) = v_{rskl}. \tag{4}$$

Also,

$$\lim_{q\to 1}[v_{rs}(x)]_q = \lim_{q\to 1}\frac{1}{[\lambda_r]_q[\mu_s]_q}\sum_{\substack{k\in I_r\\ l\in I_s}} q^{k+l-2}x_{kl}$$
$$= \lim_{q\to 1}\frac{1}{\lambda_r\mu_s}\sum_{\substack{k\in I_r\\ l\in I_s}} x_{kl}$$
$$= v_{rs}(x). \tag{5}$$

Next,

$$\lim_{q\to 1}\delta_{(\lambda,\mu)_q}(K) = \lim_{q\to 1}\{\lim_{r,s\to\infty}\inf(v_{rskl}(q^{kl})\chi_K)_{rs}\}$$
$$= \lim_{r,s\to\infty}\inf(v_{rskl}\chi_K)_{rs}$$
$$= \delta_{(\lambda,\mu)}(K). \tag{6}$$

Equations (4), (5) and (6) justifies Definitions 2.2, 2.4 and 2.5. Since $(\lambda,\mu)_q$-density reduces to (λ,μ)-density when $q\to 1$, from the definition it is clear that $(\lambda,\mu)_q$-statistical convergent will become (λ,μ)-statistical convergence. Thus Definition 2.6 is also justified.

Remark 2.7. (i): If $[\lambda_r]_q = [r]_q$ and $[\mu_s]_q = [s]_q$, the $(\lambda,\mu)_q$-statistical convergence would become q-statistical convergence.
(ii): If $\lambda_r = r, \mu_s = s$ and $q\to 1$, the $(\lambda,\mu)_q$-statistical convergence would become statistical convergence.

Theorem 2.4. *Let $x = (x_{kl})$ be a real double sequence. Then x is $(\lambda,\mu)_q$-statistically convergent to L if and only if there exists a subset $K = \{(k,l)\} \subseteq \mathbb{N}\times\mathbb{N},\ k,l = 1,2,........,$ such that $\delta_{(\lambda,\mu)_q}(K) = 1$ and $\lim_{\substack{k,l\to\infty\\ (k,l)\in K}} x_{kl} = L$.*

Proof. Let $x = (x_{kl})$ to be $(\lambda,\mu)_q$-statistically convergent to L . For $n = 1,2,.....$ we assume

$$K_n = \{(k,l)\in\mathbb{N}\times\mathbb{N} : |x_{kl}-L| \geq \frac{1}{n}\}, M_n = \{(k,l)\in\mathbb{N}\times\mathbb{N} : |x_{kl}-L| \leq \frac{1}{n}\}.$$

Then

$$\delta_{(\lambda,\mu)_q}(K_n) = P-\lim_{r,s}\sum_{k,l\in K_n}\frac{q^{k+l-2}}{[\lambda_r]_q[\mu_s]_q} = 0.$$

It is clear that $U_1 \supset U_2 \supset ... \supset U_i \supset U_{i+1} \supset ...$ and

$$\delta_{(\lambda,\mu)_q}(U_n) = P-\lim_{r,s}\sum_{k,l\in U_n}\frac{q^{k+l-2}}{[\lambda_r]_q[\mu_s]_q} = 1. \tag{7}$$

Now we will prove the convergence of (x_{kl}) to the limit L in U_n. In contradiction let (x_{kl}) does not have the limit L . Then there must exists an $\epsilon > 0$ such that

$|x_{kl} - L| \geq \epsilon$ for infinitely many $(k,l)'s$. Let $U_\epsilon = \{(k,l) : |x_{kl} - L| < \epsilon\}$, and $\epsilon > \frac{1}{n}(n = 1, 2, ...)$.Then

$$\delta_{(\lambda,\mu)_q}(U_\epsilon) = P - \lim_{r,s} \sum_{k,l \in U_\epsilon} \frac{q^{k+l-2}}{[\lambda_r]_q[\mu_s]_q} = 0.$$

also, $U_n \subset U_\epsilon$. Hence

$$\delta_{(\lambda,\mu)_q}(U_n) = P - \lim_{r,s} \sum_{k,l \in U_n} \frac{q^{k+l-2}}{[\lambda_r]_q[\mu_s]_q} = 0.$$

$\delta_{(\lambda,\mu)_q}(U_n) = 0$, which is a contradiction to our assumption, therefore x_{kl} is convergent to L.

Conversely, let there exists $K = \{(k,l)\} \subseteq \mathbb{N} \times \mathbb{N}$, $k,l = 1, 2, ...,$ such that $\delta_{(\lambda,\mu)_q}(K) = 1$ and $\lim_{k,l \to \infty (k,l) \in K} x_{kl} = L$. Then $\exists\ n_0 \in \mathbb{N}$ such that $|x_{kl} - L| < \epsilon, \forall\ k,l \geq n_0$. Now,

$$K_\epsilon = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \epsilon\} \subseteq \mathbb{N} \times \mathbb{N} - \{(k,l) > n_0\}.$$

Thus

$$\delta_{(\lambda,\mu)_q}(K_\epsilon) = P - \lim_{r,s} \sum_{k,l \in K_\epsilon} \frac{q^{k+l-2}}{[\lambda_r]_q[\mu_s]_q} = 0.$$

This concludes the proof of the theorem. □

Let $m_{(\lambda,\mu)_q}$ denote the set of all $(\lambda,\mu)_q$ -statistically bounded double sequence of real numbers.

Theorem 2.5. *The set $m_{(\lambda,\mu)_q}$ is closed linear subspace of the normed space l^2_∞.*

Proof. Let $x^{(kl)} = (x^{(kl)}_{rs}) \in m_{(\lambda,\mu)_q}$ and $x^{(kl)} \longrightarrow (x_{rs}) \in l^2_\infty$. Since $x^{(kl)} \in m_{(\lambda,\mu)_q}$, there exist real numbers C_{kl} such that

$$st_{(\lambda,\mu)_q} - \lim_{r,s} x^{(kl)}_{rs} = C_{kl}\ (k,l = 1, 2, ...).$$

As $x_{(kl)} \longrightarrow X$, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|x^{(mn)} - x^{(kl)}| \leq \frac{\epsilon}{3}\ \forall\ m \geq k \geq n_0, n \geq l \geq n_0.$$

where $|.|$ denote the norm in l^2_∞. According to Theorem 2.4, there exists subsets U and V of $\mathbb{N} \times \mathbb{N}$ such that $\delta_{(\lambda,\mu)_q}(U) = \delta_{(\lambda,\mu)_q}(V) = 1$ and

$$\lim_{\substack{r,s \longrightarrow \infty \\ (r,s) \in U}} x_{rs} = C_{kl}, \lim_{\substack{r,s \longrightarrow \infty \\ (r,s) \in V}} x_{rs} = C_{mn}.$$

Also, $\delta_{(\lambda,\mu)_q}(U\cap V)=1$, i.e.,it is a infinite set, we can choose $(u,v)\in U\cap V$such that

$$|x^{(mn)}_{u,v}-C_{mn}|\leq\frac{\epsilon}{3}\text{ and }|x^{(kl)}_{u,v}-C_{kl}|\leq\frac{\epsilon}{3}$$

For each $m\geq l\geq n_0$ and $n\geq k\geq n_o$, we have

$$\begin{aligned}|C_{mn}-C_{kl}|&\leq|C_{mn}-x^{(mn)}_{uv}|+|x^{(mn)}_{uv}-x^{(kl)}_{uv}|+|x^{(kl)}_{uv}-C_{kl}|\\&<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\&=\epsilon.\end{aligned}$$

Thus (C_{kl}) is a Cauchy sequence of real numbers and hence convergent. Let $\lim\limits_{k,l}C_{kl}=C$. We shall prove the $(\lambda,\mu)_q$-statistical convergence of x to C. Now $x^{(kl)}$ is converging to $x=(x_{rs})$. Thus $\forall\ \epsilon>0,\exists\ N_1(\epsilon)$ such that

$$|x^{(kl)}_{rs}-x_{rs}|<\frac{\epsilon}{3}\ \forall(r,s)\geq N_1(\epsilon).$$

Also,$C_{(kl)}\to C$. Thus $\forall\epsilon>0,\exists\ N_2(\epsilon)$ such that

$$|C_{rs}-C|<\frac{\epsilon}{3}\ \forall(r,s)\geq N_2(\epsilon).$$

Further, $x^{(kl)}$ is a $(\lambda,\mu)_q$-statistical convergent to C_{kl}. So there exists $A=\{r,s\}\subset\mathbb{N}\times\mathbb{N}$ such that $\delta_{(\lambda,\mu)_q}(A)=1$ and for each $\epsilon>0,\exists\ N_3(\epsilon)$such that

$$|x^{(kl)}_{rs}-C_{kl}|<\frac{\epsilon}{3}\ \forall(r,s)\geq N_3(\epsilon),(r,s)\in A.$$

Let max $\{N_1(\epsilon),N_2(\epsilon),N_3(\epsilon)\}=N_4(\epsilon)$.Then for a given $\epsilon>0$ and for all $(r,s)\geq N_4(\epsilon),(r,s)\in A$

$$\begin{aligned}|x_{rs}-C|&\leq|x_{rs}-x^{(kl)}_{rs}|+|x^{(kl)}_{rs}-C_{rs}|+|C_{rs}-C|\\&=\epsilon.\end{aligned}$$

Thus $x=(x_rs)$ is $(\lambda,\mu)_q$-statistically convergent to C as $\delta_{(\lambda,\mu)_q}(A)=1$. Hence $x\in m_{(\lambda,\mu)_q}$ □

Theorem 2.6. *The set $m_{(\lambda,\mu)_q}$ is nowhere dense in l^2_∞.*

Proof. Every closed linear subspace of a linear normed space, which is different from the space itself , is nowhere dense in it. Using theorem 2.5, it suffices to proof that $m_{(\lambda,\mu)_q}\neq l^2_\infty$. Consider a sequence $x=(x_{mn})$ such that

$$x_{mn}=\begin{cases}1, & \text{if } m \text{ and } n \text{ are even;}\\0, & \text{otherwise.}\end{cases}\tag{8}$$

Clearly x is not a $(\lambda,\mu)_q$-statistically convergent sequence but $x\in l^2_\infty(E)$. Hence $m_{(\lambda,\mu)_q}\neq l^2_\infty$. □

3. $(\lambda,\mu)_q$-Statistically Cauchy Sequence

Definition 3.1. Let $x = (x_{rs})$ be a real double sequence . It will $(\lambda,\mu)_q$-statistically Cauchy if for a given $\epsilon > 0$, $\exists\, \mathcal{U} = \mathcal{U}(\epsilon)$ and $\mathcal{V} = \mathcal{V}(\epsilon)$ such that $\forall\, r, j \geq \mathcal{U}, s, k \geq \mathcal{V}, \delta_{(\lambda,\mu)_q}(K) = 0$, where

$$K = \{(r,s) : r \leq j \text{ and } s \leq k : |x_{rs} - x_{jk}| \geq \epsilon\}.$$

Theorem 3.1. *A real double sequence $x = (x_{rs})$ is $(\lambda,\mu)_q$-statistically convergent if and only if it is $(\lambda,\mu)_q$-statistically Cauchy.*

Proof. Consider the double sequence $x = (x_{rs})$ to be $(\lambda,\mu)_q$-statistically convergent to $\mathcal{L}$. For every $\epsilon > 0$,

$$\delta_{(\lambda,\mu)_q}(K_1) = P - \lim_{j,k} \sum_{r,s\in K_1} \frac{q^{r+s-2}}{[\lambda_j]_q[\mu_k]_q} = 0,$$

where $K_1 = \{(r,s) : r \leq j \text{ and } s \leq k : |x_{rs} - \mathcal{L}| \geq \epsilon\}$. We choose $\mathcal{U}$ and $\mathcal{V}$ so that $|x_{\mathcal{U}\mathcal{V}} - \mathcal{L}| \geq \epsilon$. Now let

$$K_2 = \{(r,s) : r \leq j \text{ and } s \leq k : |x_{rs} - x_{\mathcal{U}\mathcal{V}}| \geq \epsilon\}$$

$$K_3 = \{(r,s) : r = \mathcal{U} \leq k \text{ and } s = \mathcal{V} \leq j : |x_{\mathcal{U}\mathcal{V}} - \mathcal{L}| \geq \epsilon\}$$

Then $K_2 \subseteq K_1 + K_3$. Thus

$$\begin{aligned}\delta_{(\lambda,\mu)_q}(K_2) &= P - \lim_{j,k} \sum_{r,s\in K_2} \frac{q^{r+s-2}}{[\lambda_j]_q[\mu_k]_q}\\ &\leq P - \lim_{j,k} \frac{1}{[\lambda_j]_q[\mu_k]_q}\Big(\sum_{r,s\in K_1} q^{r+s-2} + \sum_{r,s\in K_2} q^{r+s-2}\Big).\\ &= \delta_{(\lambda,\mu)_q}(K_1) + \delta_{(\lambda,\mu)_q}(K_2)\\ &= 0.\end{aligned}$$

Therefore x is $(\lambda,\mu)_q$-statistically Cauchy.

Conversely consider the sequence x as $(\lambda,\mu)_q$-statistically Cauchy. Suppose that x is not $(\lambda,\mu)_q$-statistically convergent. Thus there exist $\mathcal{U}$ and $\mathcal{V}$ such that

$$\delta_{(\lambda,\mu)_q}(K_2) = P - \lim_{j,k} \sum_{r,s\in K_2} \frac{q^{r+s-2}}{[\lambda_j]_q[\mu_k]_q} = 0.$$

Hence by Proposition 2.1

$$\delta_{(\lambda,\mu)_q}(K_2^c) = \delta_{(\lambda,\mu)_q}(\{(r,s) : r \leq j \text{ and } s \leq k : |x_{rs} - x_{\mathcal{U}\mathcal{V}}| \geq \epsilon\}) = 1.$$

Let $|x_{rs} - \mathcal{L}| < \epsilon$.Then $|x_{rs} - x_{\mathcal{U}\mathcal{V}}| < 2|x_{rs} - \mathcal{L}| < \epsilon$. As x is not $(\lambda,\mu)_q$-statistical convergent,

$$\delta_{(\lambda,\mu)_q}(K_1) = P - \lim_{j,k} \sum_{r,s\in K_1} \frac{q^{r+s-2}}{[\lambda_j]_q[\mu_k]_q} = 1.$$

Therefore,

$$\delta_{(\lambda,\mu)_q}(K_1)^c = \delta_{(\lambda,\mu)_q}\{(r,s) : r \leq j \text{ and } s \leq k : |x_{rs} - \mathcal{L}| < \epsilon\} = 0.$$

Thus

$$\delta_{(\lambda,\mu)_q}\{(r,s) : r \leq j \text{ and } s \leq k : |x_{rs} - x_{jk}| < \epsilon\} = 0.$$

So, $\delta_{(\lambda,\mu)_q}(K_2) = 1$, we reached at a contradiction. Hence x is $(\lambda,\mu)_q$-statistically convergent. □

4. $(\lambda,\mu)_q$-Statistically Pre-Cauchy Sequence

Statistically pre-Cauchy sequence was first introduced by J. Connor et al. [6] . Further the concept was generalized to double sequence and consequently its q-analog was defined by M. Mursaleen et al. [19]

Proposition 4.1. *The necessary and sufficient condition for double sequence $x = (x_{rs})$ to be$(\lambda,\mu)_q$-Statistically convergent is that there exists $\mathcal{U} \subset \mathbb{N}$ such that its $(\lambda,\mu)_q$-density is 1 and for a given $\epsilon > 0$, there is a $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{U} \setminus \mathcal{V}$ is finite and*

$$\mathcal{V} \times \mathcal{V} \subset \{(r,s,j,k) : |x_{rs} - x_{jk}| \geq \epsilon\}. \tag{9}$$

Definition 4.1. Let $x = (x_{rs})$ be a real double sequence. It is said to be $(\lambda,\mu)_q$-statistically pre-Cauchy if for a given $\epsilon > 0$,

$$\lim_{m,n} \sum_{r,s} \frac{1}{[\lambda_m]_q^2[\mu_n]_q^2} |\{(r,s,j,k) : r,j \leq m \text{ and } s,j \leq n : |x_{rs} - x_{jk}| \geq \epsilon\}| = 0.$$

Theorem 4.2. *If the double sequence $x = (x_{rs})$ is $(\lambda,\mu)_q$-statistically convergent. Then it is also $(\lambda,\mu)_q$-statistically pre-Cauchy.*

Proof. Let $x = (x_{rs})$ be a $(\lambda,\mu)_q$-statistically convergent sequence and let the set $\mathcal{U}$ is same as in Proposition 4.1. For a given $\epsilon > 0$, select a subset $\mathcal{V}$ of $\mathcal{U}$ so that $\mathcal{U} \setminus \mathcal{V}$ is finite and

$$\mathcal{V} \times \mathcal{V} \subset \{(r,s,j,k) : |x_{rs} - x_{jk}| \geq \epsilon\}.$$

For each $(m,n) \in \mathbb{N} \times \mathbb{N}$,

$$\begin{aligned}
[\delta^{mn}_{(\lambda,\mu)_q}(\mathcal{V})]^2 &= \lim_{m,n} \frac{1}{[\lambda_m]_q^2[\mu_n]_q^2} \sum_{r,j \leq m} \sum_{s,k \leq n} (q^{r+s-2})^2 \chi_{\mathcal{V}\times\mathcal{V}} \\
&\leq \lim_{m,n} \frac{q^{r+s-2}}{[\lambda_m]_q^2[\mu_n]_q^2} |\{(r,s,j,k) : r,j \leq m \text{ and } s,j \leq n : |x_{rs} - x_{jk}| \geq \epsilon\}|.
\end{aligned}$$

Since $P - \lim_{m,n} \delta^{m,n}_{(\lambda,\mu)}(\mathcal{V}) = 1$, it follows that

$$\lim_{m,n} \frac{q^{r+s-2}}{[\lambda_m]_q^2[\mu_n]_q^2} |\{(r,s,j,k) : r,j \leq m \text{ and } s,j \leq n : |x_{rs} - x_{jk}| \geq \epsilon\}| = 1$$

which shows that the sequence x is $(\lambda,\mu)_q$-statistically pre-Cauchy. □

Theorem 4.3. *A bounded double sequence $x=(x_{rs})$ is $(\lambda,\mu)_q$-statistically pre-Cauchy if and only if*

$$\lim_{m,n}\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}\sum_{r,j\le m}\sum_{s,k\le n}q^{r+s-2}|x_{rs}-x_{jk}|=0. \tag{10}$$

Proof. Let

$$\lim_{m,n}\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}\sum_{r,j\le m}\sum_{s,k\le n}q^{r+s-2}|x_{rs}-x_{jk}|=0$$

For each $\epsilon>0$ and $n\in\mathbb{N}$, we have

$$\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}\sum_{r,j\le m}\sum_{s,k\le n}q^{r+s-2}|x_{rs}-x_{jk}|\ge\epsilon\left(\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}|\{(r,s,j,k):|x_{rs}-x_{jk}|\ge\epsilon\}|\right)\ge 0. \tag{11}$$

Using squeeze theorem of limits, we get that x is $(\lambda,\mu)_q$-statistically pre-Cauchy.

Conversely let x is $(\lambda,\mu)_q$-statistically pre-Cauchy and $\sup_{m,n}|x_{m,n}|=\mathcal{B}<\infty$. For a given $\epsilon>0$ and for every $m,n\in\mathbb{N}$,

$$\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}\sum_{r,j\le m}\sum_{s,k\le n}q^{r+s-2}|x_{rs}-x_{jk}|\le\frac{\epsilon}{2}+2\mathcal{B}\left(\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}|\{(r,s,j,k):|x_{rs}-x_{jk}|\ge\frac{\epsilon}{2}\}|\right). \tag{12}$$

Using the fact that x is $(\lambda,\mu)_q$-statistically pre-Cauchy, we can have $\mathcal{V}\in\mathbb{N}$ such that for all $n>\mathcal{N}$

$$\frac{\epsilon}{2}+2\mathcal{V}\left(\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}|\{(r,s,j,k):|x_{rs}-x_{jk}|\ge\frac{\epsilon}{2}\}|\right)<\epsilon. \tag{13}$$

From equation 12 and 13, we get

$$\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}\sum_{r,j\le m}\sum_{s,k\le n}q^{r+s-2}|x_{rs}-x_{jk}|=0. \tag{14}$$

Conversely, let $t=(t_{r,s})=r^2s^2$ be a double sequence, define the sequence x by $x_{r,s}=r^4s^4$ if $x_{r,s}=t_{r,s}$ and $x_{r,s}=1$, otherwise. We will show that converse does not hold for x. Since $\delta^2_{(\lambda,\mu)_q}\{r,s:x_{r,s}\ne 1\}=0$, x is $(\lambda,\mu)_q$-statistically convergent to 1. From the previous result it will also be $(\lambda,\mu)_q$-statistically pre-Cauchy. Now

$$q^{r+s-2}\frac{1}{t_{r,s}^2}|x_{t_{r,s}-1}-x_{t_{r,s}}|=q^{r+s-2}\frac{1}{r^4s^4}|1-r^4s^4|\to 1$$

as $r,s\longrightarrow\infty$. Thus

$$\limsup_{m,n}\frac{1}{[\lambda_m]_q^2[\mu_n]_q^2}\sum_{r,j\le m}\sum_{s,k\le n}q^{r+s-2}|x_{rs}-x_{jk}|\ge 1.$$

□

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Received : *October* 2024
Accepted : *November* 2024