

Common fixed point results for quadruple maps with weak compatibility in complex b-metric spaces with a practical application

A. M. Saddeek* and G. M. Ahmed

ABSTRACT. In this paper, we present common fixed-point theorems for quadruple mappings that satisfy various types of contractions using weakly compatible mappings in complete complex-valued b-metric spaces. Additionally, we provide an illustrative example to demonstrate the main theorems and discuss an application to linear nonhomogeneous systems under suitable conditions.

1. Introduction

In 1922, Banach [3] established a fixed-point theorem for contraction mappings in metric spaces. Subsequently, various authors (see for example [6, 11, 16]) have proven numerous fixed-point theorems, and several generalizations of Banach's theorem have been developed. The concept of a complex-valued metric space was initially introduced in 2011 by Azam et al. [2] as a part of a broader generalization. In 2014, Rao et al. [15] extended this notion to what is now known as the complete complex-valued b-metric space.

2020 *Mathematics Subject Classification.* Primary: 54H25; Secondary: 47H10.

Key words and phrases. Complex-valued b-metric spaces, Weakly compatible maps, Common fixed-point

*Corresponding author



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Numerous authors in [1, 5, 7, 8, 17, 18, 19] have established various fixed-point theorems in both complex-valued metric spaces and complex-valued b-metric spaces, employing different conditions on the mappings.

Recently, Mukheimer [14] demonstrated the uniqueness of a common fixed-point in complete complex-valued b-metric spaces. Similarly, in a very recent paper, Barman et al. [4] established common fixed-point theorems using Hardy and Rogers-type (see [9]) contraction conditions in complete complex-valued b-metric spaces. Their results extend those proven in Ali [1]. In 1996, Jungck [10] introduced the notion of weakly compatible mappings. Since then, various authors have derived numerous compelling fixed-point theorems concerning weakly compatible mappings under different contraction conditions.

This paper aims to establish common fixed-point theorems for quadruple weakly compatible mappings that satisfy various contraction conditions in complete complex-valued b-metric spaces. The results presented herein unify, extend, and improve those found in Barman et.al [4], Ali [1], Marzouki et.al [13], LV and Feng [12], Mukheimer [14], and Banach [3]. Finally, we provide an application to determine the existence and uniqueness of a common solution for two systems of linear non-homogeneous equations with positivity constraints.

2. Preliminaries

Let \mathbb{N} , \mathbb{R} , and \mathbb{C} represent the sets of all positive integers, real numbers, and complex numbers, respectively. For any $c_1, c_2 \in \mathbb{C}$, we write $c_1 \preceq c_2$ if and only if $Re(c_1) \leq Re(c_2)$ and $Im(c_1) \leq Im(c_2)$. Note that for any $c_1, c_2 \in \mathbb{C}$, the relation $c_1 \preceq c_2$ holds if any of the following conditions is satisfied:

- (i) $Re(c_1) = Re(c_2)$ and $Im(c_1) = Im(c_2)$,
- (ii) $Re(c_1) < Re(c_2)$ and $Im(c_1) = Im(c_2)$,
- (iii) $Re(c_1) = Re(c_2)$ and $Im(c_1) < Im(c_2)$,
- (iv) $Re(c_1) < Re(c_2)$ and $Im(c_1) < Im(c_2)$.

We will denote $c_1 \succsim c_2$ if $c_1 \neq c_2$ and at least one of the conditions (ii), (iii), or (iv) is satisfied. Additionally, we will use the notion $c_1 \prec c_2$ if only condition (iv) is satisfied. Consequently:

- (1) If $0 \preceq c_1 \succsim c_2$, then $|c_1| < |c_2|$,
- (2) If $c_1 \preceq c_2$, and $c_2 \prec c_3$, then $c_1 \prec c_3$,
- (3) If $0 \preceq c_1 \preceq c_2$, then $|c_1| \leq |c_2|$,
- (4) If $a, b \in \mathbb{R}$ and $a < b$, then $ac \preceq bc$, for all $c \in \mathbb{C}$.

Definition 2.1. [15] A complex-valued b-metric on a non-empty set \mathbb{M} is a mapping, $\rho : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{C}$, satisfying the following conditions for all $u, v, w \in \mathbb{M}$:

- (i) $0 \preceq \rho(u, v)$,
- (ii) $\rho(u, v) = 0$ if and only if $u = v$,

- (iii) $\rho(u, v) = \rho(v, u)$,
 (iv) $\rho(u, v) \preceq r [\rho(u, w) + \rho(w, v)]$, where $r \geq 1$.

Subsequently, the pair (\mathbb{M}, ρ) is denoted as a complex-valued b-metric with a coefficient $r \geq 1$.

Definition 2.2. [15] Let (\mathbb{M}, ρ) be a complex-valued b-metric space, and $\{u_n\}$ be a sequence in \mathbb{M} with $u \in \mathbb{M}$. Then:

- (i) The sequence $\{u_n\}$ is said to converge to u if $\lim_{n \rightarrow \infty} \rho(u_n, u) = 0$.
 (ii) The sequence $\{u_n\}$ is a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \rho(u_m, u_n) = 0$.
 (iii) The metric space (\mathbb{M}, ρ) is considered a complete complex-valued b-metric space when every Cauchy sequence in \mathbb{M} converges.

Remark 2.3. If $r = 1$ in the Definition 2.1, then the pair (\mathbb{M}, ρ) is referred as complex-valued metric space (see [2]).

The following lemmas attributed to [2], play a crucial role in our analysis.

Lemma 2.1. Let (\mathbb{M}, ρ) be a complex-valued b-metric space, and consider a sequence $\{u_n\}$ in \mathbb{M} . The sequence $\{u_n\}$ converges to u if and only if

$$\lim_{n \rightarrow \infty} |\rho(u_n, u)| = 0.$$

Lemma 2.2. Let (\mathbb{M}, ρ) be a complex-valued b-metric space, and let $\{u_n\}$ be a sequence in \mathbb{M} . The sequence $\{u_n\}$ is a Cauchy sequence if and only if

$$\lim_{n, m \rightarrow \infty} |(\rho(u_n, u_{n+m}))| = 0.$$

We conclude this section with the following necessary definitions:

Definition 2.4. [10] Let S and T be two self-maps on a set \mathbb{M} . The mappings S and T are said to be weakly compatible if $STu = TSu$ where $Su = Tu$ for some $u \in \mathbb{M}$.

Definition 2.5. [20] The max function corresponding to the partial order relation \preceq is defined as follows for all $c_1, c_2, c_3 \in \mathbb{C}$:

- (i) $\max\{c_1, c_2\} = c_2$ if and only if $c_1 \preceq c_2$;
 (ii) If $c_1 \preceq \max\{c_2, c_3\}$, then either $c_1 \preceq c_2$ or $c_1 \preceq c_3$;
 (iii) $\max\{c_1, c_2\} = c_2$ if and only if either $c_1 \preceq c_2$ or $|c_1| \leq |c_2|$.

3. Rational-Type Common Fixed-Point

In this section, we establish a common fixed-point theorem for quadruple self-maps that satisfy a rational-type contraction with weak compatibility in a complete complex-valued b-metric space. We discuss the consequences of this theorem.

Theorem 3.1. *Let (\mathbb{M}, ρ) be a complete complex-valued b -metric space with a coefficient $r \geq 1$. Consider self-maps F, G, S , and $T : \mathbb{M} \rightarrow \mathbb{M}$ that satisfy the following condition:*

$$\begin{aligned} \rho(Fu, Gv) \preceq & k_1 \rho(Sv, Tu) + k_2 \max \left\{ \rho(Sv, Tu), \frac{\rho(Tu, Fu)\rho(Sv, Gv)}{1 + \rho(Fu, Gv)} \right\} \\ & + k_3 \min \{ \rho(Tu, Gv), \rho(Sv, Fu) \}, \end{aligned} \quad (1)$$

for all $u, v \in \mathbb{M}$, where

$$r(k_1 + k_2) + r^2 k_3 < 1, \quad k_i \geq 0, \quad i = 1, 2, 3. \quad (2)$$

If the following conditions hold:

- (i) $F\mathbb{M} \subseteq S\mathbb{M}$ and $G\mathbb{M} \subseteq T\mathbb{M}$,
- (ii) Either $S\mathbb{M}$ or $T\mathbb{M}$ is a complete subspace of \mathbb{M} ,
- (iii) $\{F, T\}$ and $\{G, S\}$ are weakly compatible.

Then there exists a unique common fixed-point for F, G, S and T in \mathbb{M} .

PROOF. Let u_0 be an arbitrary point in \mathbb{M} , and using condition (i), we define the sequences $\{u_n\}$ and $\{v_n\}$ in \mathbb{M} as follows:

$$v_{2n} = Su_{2n+1} = Fu_{2n} \quad \text{and} \quad v_{2n+1} = Tu_{2n+2} = Gu_{2n+1} \quad \text{for all } n = 0, 1, 2, \dots \quad (3)$$

If $v_{2n} = v_{2n+1}$, for some n , then u_{2n+1} is a coincident point of S and T . Similarly, if $v_{2n+1} = v_{2n+2}$, for some n , then $Fu_{2n+2} = Tu_{2n+2}$, implying that u_{2n+2} is a coincident point of F and T . Now, let's assume that $v_{2n} \neq v_{2n+1}$, for all n . Then, by condition (1), we have:

$$\begin{aligned} \rho(v_{2n}, v_{2n+1}) &= \rho(Fu_{2n}, Gu_{2n+1}) \\ &\preceq k_1 \rho(Su_{2n+1}, Tu_{2n}) + k_2 \max \left\{ \rho(Su_{2n+1}, Tu_{2n}), \frac{\rho(Tu_{2n}, Fu_{2n})\rho(Su_{2n+1}, Gu_{2n+1})}{1 + \rho(Fu_{2n}, Gu_{2n+1})} \right\} \\ &\quad + k_3 \min \{ \rho(Tu_{2n}, Gu_{2n+1}), \rho(Su_{2n+1}, Fu_{2n}) \} \\ &= k_1 \rho(v_{2n}, v_{2n-1}) + k_2 \max \left\{ \rho(v_{2n}, v_{2n-1}), \frac{\rho(v_{2n-1}, v_{2n})\rho(v_{2n}, v_{2n+1})}{1 + \rho(v_{2n}, v_{2n+1})} \right\} \\ &\quad + k_3 \min \{ \rho(v_{2n-1}, v_{2n+1}), \rho(v_{2n}, v_{2n}) \} \\ &\preceq (k_1 + k_2) \rho(v_{2n-1}, v_{2n}). \end{aligned}$$

Similarly,

$$\rho(v_{2n-1}, v_{2n}) \preceq (k_1 + k_2) \rho(v_{2n-2}, v_{2n-1}).$$

Thus, for any $n \in \mathbb{N}$,

$$\begin{aligned}
|\rho(v_n, v_{n+1})| &\leq (k_1 + k_2)|\rho(v_{n-1}, v_n)| \\
&\leq (k_1 + k_2)^2|\rho(v_{n-2}, v_{n-1})| \\
&\leq \dots \\
&\leq (k_1 + k_2)^n|\rho(v_0, v_1)|.
\end{aligned} \tag{4}$$

Now, considering $n, m \in \mathbb{N}$ where $m > n$, we have:

$$\begin{aligned}
|\rho(v_n, v_{n+m})| &\leq r[|\rho(v_n, v_{n+1})| + |\rho(v_{n+1}, v_{n+m})|] \\
&\leq r|\rho(v_n, v_{n+1})| + r^2|\rho(v_{n+1}, v_{n+2})| + r^2|\rho(v_{n+2}, v_{n+m})| \\
&\leq r|\rho(v_n, v_{n+1})| + r^2|\rho(v_{n+1}, v_{n+2})| + r^3|\rho(v_{n+2}, v_{n+3})| \\
&\quad + r^3|\rho(v_{n+3}, v_{n+m})| \leq \dots \\
&\leq r|\rho(v_n, v_{n+1})| + r^2|\rho(v_{n+1}, v_{n+2})| + r^3|\rho(v_{n+2}, v_{n+3})| + \dots \\
&\quad + r^{m-1}|\rho(v_{n+m-2}, v_{n+m-1})| + r^m|\rho(v_{n+m-1}, v_{n+m})|.
\end{aligned} \tag{5}$$

By using (4) in the right hand side of (5), we find that

$$\begin{aligned}
|\rho(v_n, v_{n+m})| &\leq r(k_1 + k_2)^n|\rho(v_0, v_1)| + r^2(k_1 + k_2)^{n+1}|\rho(v_0, v_1)| \\
&\quad + r^3(k_1 + k_2)^{n+2}|\rho(v_0, v_1)| + \dots \\
&\quad + r^{m-1}(k_1 + k_2)^{n+m-2}|\rho(v_0, v_1)| + r^m(k_1 + k_2)^{n+m-1}|\rho(v_0, v_1)| \\
&= r(k_1 + k_2)^n[1 + r(k_1 + k_2) + (r(k_1 + k_2))^2 \\
&\quad + \dots + (r(k_1 + k_2))^{m-2} + (r(k_1 + k_2))^{m-1}]|\rho(v_0, v_1)| \\
&= r(k_1 + k_2)^n \frac{1 - (r(k_1 + k_2))^m}{1 - r(k_1 + k_2)} |\rho(v_0, v_1)| \\
&\leq \frac{r}{1 - r(k_1 + k_2)} (k_1 + k_2)^n |\rho(v_0, v_1)|, \text{ since } r(k_1 + k_2) < 1.
\end{aligned}$$

Combining this with $\lim_{n \rightarrow \infty} (k_1 + k_2)^n = 0$, we can deduce that

$$\lim_{m, n \rightarrow \infty} |\rho(v_n, v_{n+m})| = 0.$$

Therefore, according to Lemma 2.2, the sequences $\{u_n\}$ is a Cauchy sequence in (\mathbb{M}, ρ) . If $S\mathbb{M}$ is a complete subspace of \mathbb{M} and $F\mathbb{M} \subseteq S\mathbb{M}$, there exists $v^* \in S\mathbb{M}$ such that $\lim_{n \rightarrow \infty} \rho(v_{2n}, v^*) = 0$. Consequently, we can find $u^* \in \mathbb{M}$ such that $v^* = Su^*$. We assert that $v^* = Gu^*$. The inequality,

$$\begin{aligned}
|\rho(Gu^*, v_{2n+1})| &\leq r[|\rho(Gu^*, v_{2n})| + |\rho(v_{2n}, v_{2n+1})|] \\
&\leq r[r[|\rho(Gu^*, v^*)| + |\rho(v^*, v_{2n})|] + r|\rho(v_{2n}, v_{2n+1})|],
\end{aligned}$$

implies that the sequence $\{\rho(Gu^*, v_{2n+1})\}$ is bounded, and similarly $\{\rho(Gu^*, v_{2n})\}$ is bounded. Thus, there exists a strictly increasing function $\eta : \mathbb{N} \rightarrow \mathbb{N}$ such that

$\{\rho(Gu^*, v_{2\eta(n)})\}$ and $\{\rho(Gu^*, v_{2\eta(n)+1})\}$ are convergent. Now, using (1) and (3), we obtain

$$\begin{aligned}
\rho(Su^*, Gu^*) &\preceq r[\rho(Su^*, v_{2\eta(n)}) + \rho(v_{2\eta(n)}, Gu^*)] \\
&= r\rho(Su^*, v_{2\eta(n)}) + r(\rho(Fu_{2\eta(n)}, Gu^*)) \\
&\preceq r\rho(Su^*, v_{2\eta(n)}) + rk_1\rho(Su^*, Tu_{2\eta(n)}) \\
&\quad + rk_2 \max \left\{ \rho(Su^*, Tu_{2\eta(n)}), \frac{\rho(Tu_{2\eta(n)}, Fu_{2\eta(n)})\rho(Su^*, Gu^*)}{1 + \rho(Fu_{2\eta(n)}, Gu^*)} \right\} \\
&\quad + rk_3 \min \{ \rho(Tu_{2\eta(n)}, Gu^*), \rho(Su^*, Fu_{2\eta(n)}) \} \\
&= r\rho(v^*, v_{2\eta(n)}) + rk_1\rho(v^*, v_{2\eta(n)-1}) \\
&\quad + rk_2 \max \left\{ \rho(v^*, v_{2\eta(n)-1}), \frac{\rho(v_{2\eta(n)-1}, v_{2\eta(n)})\rho(v^*, Gu^*)}{1 + \rho(v_{2\eta(n)}, Gu^*)} \right\} \\
&\quad + rk_3 \min \{ \rho(v_{2\eta(n)-1}, Gu^*), \rho(v^*, v_{2\eta(n)}) \} \\
&\preceq r\rho(v^*, v_{2\eta(n)}) + rk_1\rho(v^*, v_{2\eta(n)-1}) \\
&\quad + rk_2 \max \left\{ \rho(v^*, v_{2\eta(n)-1}), \frac{\rho(v_{2\eta(n)-1}, v_{2\eta(n)})\rho(v^*, Gu^*)}{1 + \rho(v_{2\eta(n)}, Gu^*)} \right\} \\
&\quad + rk_3 \min \{ r[\rho(v_{2\eta(n)-1}, v^*) + \rho(v^*, Gu^*)]\rho(v^*, v_{2\eta(n)}) \}.
\end{aligned}$$

Letting $n \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} |\rho(Su^*, Gu^*)| = 0$. Hence $|\rho(Su^*, Gu^*)| = 0$, implying $v^* = Su^* = Gu^*$.

Since $G\mathbb{M} \subseteq T\mathbb{M}$, there exists a point $w \in \mathbb{M}$ such that $v^* = Tw$. Furthermore, using (1), we deduce that:

$$\begin{aligned}
\rho(Fw, v^*) &= \rho(Fw, Gu^*) \\
&\preceq k_1\rho(Su^*, Tw) + k_2 \max \left\{ \rho(Su^*, Tw), \frac{\rho(Tw, Fw)\rho(Su^*, Gu^*)}{1 + \rho(Fw, Gu^*)} \right\} \\
&\quad + k_3 \min \{ \rho(Tw, Gu^*), \rho(Su^*, Fw) \} \\
&= 0.
\end{aligned}$$

Hence, $|\rho(Fw, v^*)| = 0$, which implies that $v^* = Fw$. Consequently, this leads to $v^* = Fw = Tw$; hence, $v^* = Su^* = Gu^* = Fw = Tw$. Because $\{G, S\}$ is weakly compatible, it follows that $GSu^* = SGu^*$, thereby implying $Gv^* = Sv^*$.

Now, let us prove that v^* is indeed a fixed-point of G . If $Gv^* \neq v^*$, then according to (1):

$$\begin{aligned} \rho(v^*, Gv^*) &= \rho(Fw, Gv^*) \\ &\leq k_1\rho(Sv^*, Tw) + k_2 \max \left\{ \rho(Sv^*, Tw), \frac{\rho(Tw, Fw)\rho(Sv^*, Gv^*)}{1 + \rho(Fw, Gv^*)} \right\} \\ &\quad + k_3 \min \{ \rho(Tw, Gv^*), \rho(Sv^*, Fw) \} \\ &= (k_1 + k_2 + k_3)\rho(v^*, Gv^*) \\ &< \rho(v^*, Gv^*). \end{aligned}$$

This implies that $|\rho(v^*, Gv^*)| < |\rho(v^*, Gv^*)|$, which is a contradiction. Therefore, $v^* = Gv^*$. Hence, $v^* = Gv^* = Sv^*$.

Similarly, due to the weak compatibility of $\{F, T\}$, we observe that $v^* = Fv^* = Tv^*$. Indeed, since

$$\begin{aligned} \rho(Fv^*, v^*) &= \rho(Fv^*, Gv^*) \\ &\leq k_1\rho(Sv^*, Tv^*) + k_2 \max \left\{ \rho(Sv^*, Tv^*), \frac{\rho(Tv^*, Fv^*)\rho(Sv^*, Gv^*)}{1 + \rho(Fv^*, Gv^*)} \right\} \\ &\quad + k_3 \min \{ \rho(Tv^*, Gv^*), \rho(Sv^*, Fv^*) \} \\ &= (k_1 + k_2 + k_3)\rho(v^*, Fv^*). \end{aligned}$$

This implies that $|\rho(v^*, Fv^*)| < |\rho(v^*, Fv^*)|$, a contradiction. Thus, $v^* = Fv^* = Gv^* = Sv^* = Tv^*$, and v^* is a common fixed-point of F, G, S , and T .

In a similar manner, it is evident that v^* is the common fixed-point of F, G, S , and T wherever TMI is complete. Finally, to establish the uniqueness of v^* , let us suppose that v_1^* and v_2^* ; $v_1^* \neq v_2^*$ are common fixed-points of F, G, S and T . Using (1), we obtain

$$\begin{aligned} \rho(v_1^*, v_2^*) &= \rho(Fv_1^*, Gv_2^*) \\ &\leq k_1\rho(Sv_2^*, Tv_1^*) + k_2 \max \left\{ \rho(Sv_2^*, Tv_1^*), \frac{\rho(Tv_1^*, Fv_1^*)\rho(Sv_2^*, Gv_2^*)}{1 + \rho(Fv_1^*, Gv_2^*)} \right\} \\ &\quad + k_3 \min \{ \rho(Tv_1^*, Gv_2^*), \rho(Sv_2^*, Fv_1^*) \} \\ &= (k_1 + k_2 + k_3)\rho(v_1^*, v_2^*) \\ &< \rho(v_1^*, v_2^*). \end{aligned}$$

That is, $|\rho(v_1^*, v_2^*)| = 0$. Therefore, $v_1^* = v_2^*$. This concludes the proof of our theorem. \square

Remark 3.1. If $r > 1$ in condition (2) of Theorem 3.1, the following inequalities hold:

$$k_1 + k_2 + rk_3 < 1, \quad r(k_1 + k_2) < 1, \quad rk_1 < 1, \quad rk_2 < 1, \quad r^3k_1 < 1, \quad \text{and} \quad \sum_{i=1}^3 k_i < 1.$$

Following Remark 3.1, we can derive the, subsequent corollaries from Theorem 3.1.

Corollary 3.2. [4] *Let (\mathbb{M}, ρ) be a complex-valued b-metric space with constant $r > 1$. Let $F, G : \mathbb{M} \rightarrow \mathbb{M}$ satisfy:*

$$\begin{aligned} \rho(Fu, Gv) \leq & k_1 \rho(u, v) + k_2 \max \left\{ \rho(u, v), \frac{\rho(u, Fu)\rho(v, Gv)}{1 + \rho(Fu, Gv)} \right\} \\ & + k_3 \min \{ \rho(u, Gv), \rho(v, Fu) \} \end{aligned}$$

for all $u, v \in \mathbb{M}$, where $k_i \in [0, 1)$, $i = 1, 2, 3$ satisfy $(k_1 + k_2) + rk_3 < 1$. If \mathbb{M} is a complete space, then F and G possess a unique common fixed point.

Remark 3.2. If (\mathbb{M}, ρ) is a complex-valued b-metric space with constant $r > 1$, Corollary 3.2 implies Corollary 2 in [4] when setting $F = G$. Furthermore, Corollary 3 in [4] is deduced by choosing $F = G$ and setting $k_3 = 0$. Moreover, Theorem 1 in [1] follows when both $F=G$ and $k_1 = k_3 = 0$.

Corollary 3.3. [13, Theorem 3.1] *Suppose (\mathbb{M}, ρ) is a complex-valued b-metric space with a constant $r > 1$, and let $F, G : \mathbb{M} \rightarrow \mathbb{M}$ satisfy the condition:*

$$\rho(Fu, Gv) \leq \frac{1}{r^3} \rho(u, v), \quad \text{for all } u, v \in \mathbb{M}.$$

If \mathbb{M} is a complete space, then F and G possess a unique common fixed-point.

Corollary 3.4. [5, Theorem 2.1] *Suppose that (\mathbb{M}, ρ) is a complex-valued b-metric space with a constant $r > 1$. Let $F : \mathbb{M} \rightarrow \mathbb{M}$ be a mapping satisfying:*

$$\rho(Fu, Fv) \leq k_1 \rho(u, v), \quad \text{for all } u, v \in \mathbb{M},$$

$k_1 \in (0, 1)$ and $rk_1 < 1$. If \mathbb{M} is a complete space, then F has a unique fixed-point.

This result signifies a Banach fixed-point theorem in a complete complex-valued b-metric space.

4. Hardy and Rogers-Type Common Fixed-Point Theorem

In this section, we establish a common fixed-point theorem for quadruple self-maps satisfying a Hardy and Rogers-type contraction with weak compatibility in a complete complex-valued b-metric space. We discuss the consequences of this result.

Theorem 4.1. *Let $F, G, S,$ and T be self-mappings of a complete complex-valued b-metric space (\mathbb{M}, ρ) with a coefficient $r \geq 1$. Suppose that for all $u, v \in \mathbb{M}$,*

$$\rho(Fu, Gv) \leq k_1 \rho(Sv, Tu) + k_2 \rho(Fu, Tu) + k_3 \rho(Gv, Sv) + k_4 \rho(Tu, Gv) + k_5 \rho(Sv, Fu) \quad (6)$$

where,

$$\sum_{i=1}^5 k_i < \alpha, \quad r\alpha < 1, \quad \alpha \in (0, 1) \quad \text{and} \quad r(k_3 + rk_4) < 1. \quad (7)$$

If the conditions (i) – (iii) of Theorem 3.1 hold, then there exists a unique common fixed-point of F , G , S , and T in \mathbb{M} .

PROOF. Following the proof of Theorem 3.1, we can construct sequences $\{u_n\}$ and $\{v_n\}$ in \mathbb{M} , defined by equation (3), with $v_{2n} \neq v_{2n+1}$ for all $n \in \mathbb{N}$. Thus, according to condition (6), we have:

$$\begin{aligned} \rho(v_{2n}, v_{2n+1}) &= \rho(Fu_{2n}, Gu_{2n+1}) \\ &\preceq k_1\rho(Su_{2n+1}, Tu_{2n}) + k_2\rho(Fu_{2n}, Tu_{2n}) + k_3\rho(Gu_{2n+1}, Su_{2n+1}) \\ &\quad + k_4\rho(Tu_{2n}, Gu_{2n+1}) + k_5\rho(Su_{2n+1}, Fu_{2n}) \\ &= k_1\rho(v_{2n}, v_{2n-1}) + k_2\rho(v_{2n}, v_{2n-1}) + k_3\rho(v_{2n+1}, v_{2n}) \\ &\quad + k_4\rho(v_{2n-1}, v_{2n+1}) + k_5\rho(v_{2n}, v_{2n}) \\ &\preceq (k_1 + k_2)\rho(v_{2n-1}, v_{2n}) + k_3\rho(v_{2n}, v_{2n+1}) \\ &\quad + k_4r[\rho(v_{2n-1}, v_{2n}) + \rho(v_{2n}, v_{2n+1})] \\ &= (k_1 + k_2 + rk_4)\rho(v_{2n-1}, v_{2n}) + (k_3 + rk_4)\rho(v_{2n}, v_{2n+1}), \end{aligned}$$

which implies

$$\rho(v_{2n}, v_{2n+1}) \preceq \frac{k_1 + k_2 + rk_4}{1 - k_3 - rk_4} \rho(v_{2n-1}, v_{2n}) = \alpha \rho(v_{2n-1}, v_{2n}),$$

where $\alpha = \frac{k_1 + k_2 + rk_4}{1 - k_3 - rk_4} < \frac{1}{r}$. Therefore, for all $n \in \mathbb{N}$,

$$\rho(v_n, v_{n+1}) \preceq \alpha^n \rho(v_0, v_1).$$

Now, we assert that the sequence $\{v_n\}$ is a Cauchy sequence in (\mathbb{M}, ρ) . Let's consider $m, n \in \mathbb{N}$ such that $m > n$.

By utilizing assumption (iv) of Definition 2.1 and employing the condition $r\alpha < 1$ along with the same argument used in the proof of Theorem 3.1, we obtain:

$$|\rho(v_n, v_{n+m})| \leq \frac{r\alpha^n}{1 - r\alpha} |\rho(v_0, v_1)|.$$

Since $0 < \alpha < 1$, it follows that $\alpha^n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $|\rho(v_n, v_{n+m})| \rightarrow 0$ as $m, n \rightarrow \infty$. Consequently, according to Lemma 2.2, the sequence $\{v_n\}$, is Cauchy in (\mathbb{M}, ρ) . Furthermore, as in the proof of Theorem 3.1, if we consider $S\mathbb{M}$ as a complete subspace of \mathbb{M} , and $F\mathbb{M} \subseteq S\mathbb{M}$, there exists $v^* \in S\mathbb{M}$ such that $\rho(v_{2n}, v^*) \rightarrow 0$ as $n \rightarrow \infty$. Thus, we can find $u^* \in \mathbb{M}$ such that $v^* = Su^*$. We claim that $v^* = Gu^*$. The inequality

$$\begin{aligned} |\rho(Gu^*, v_{2n+1})| &\leq r[|\rho(Gu^*, v_{2n})| + |\rho(v_{2n}, v_{2n+1})|] \\ &\leq r[r[|\rho(Gu^*, v^*)| + \rho(v^*, v_{2n})]] + r|\rho(v_{2n}, v_{2n+1})| \end{aligned}$$

implies the sequences $\{\rho(Gu^*, v_{2n+1})\}$ and $\{\rho(Gu^*, v_{2n})\}$ are bounded. Therefore, there exists a strictly increasing mapping $\eta : \mathbb{N} \rightarrow \mathbb{N}$ such that $\{\rho(Gu^*, v_{2\eta(n)})\}$

and $\{\rho(Gu^*, v_{2\eta(n)+1})\}$ are convergent. Now, once again using assumption (iv) from Definition 2.1, in conjunction with (3) and (6), we derive:

$$\begin{aligned}
\rho(Su^*, Gu^*) &\leq r[\rho(Su^*, v_{2\eta(n)}) + \rho(v_{2\eta(n)}, Gu^*)] \\
&= r\rho(Su^*, v_{2\eta(n)}) + r\rho(Fu_{2\eta(n)}, Gu^*) \\
&\leq r\rho(Su^*, v_{2\eta(n)}) + r[k_1\rho(Su^*, Tu_{2\eta(n)}) \\
&\quad + k_2\rho(Fu_{2\eta(n)}, Tu_{2\eta(n)}) + k_3\rho(Gu^*, Su^*) \\
&\quad + k_4\rho(Tu_{2\eta(n)}, Gu^*) + k_5\rho(Su^*, Fu_{2\eta(n)})] \\
&= r\rho(v^*, v_{2\eta(n)}) + r[k_1\rho(v^*, v_{2\eta(n)-1}) \\
&\quad + k_2\rho(v_{2\eta(n)}, v_{2\eta(n)-1}) + k_3\rho(Gu^*, v^*) \\
&\quad + k_4\rho(v_{2\eta(n)-1}, Gu^*) + k_5\rho(v^*, v_{2\eta(n)})] \\
&\leq r\rho(v^*, v_{2\eta(n)}) + r[k_1\rho(v^*, v_{2\eta(n)-1}) \\
&\quad + k_2\rho(v_{2\eta(n)}, v_{2\eta(n)-1}) + k_3\rho(Gu^*, v^*) \\
&\quad + k_4r[\rho(v_{2\eta(n)-1}, v^*) + \rho(v^*, Gu^*)] + k_5\rho(v^*, v_{2\eta(n)})].
\end{aligned}$$

As $n \rightarrow \infty$, together with the assumption $r(k_3 + rk_4) < 1$, this implies that

$$\lim_{n \rightarrow \infty} |\rho(Su^*, Gu^*)| = 0.$$

Thus, $|\rho(Su^*, Gu^*)| = 0$, which means $v^* = Su^* = Gu^*$. Since $G\mathbb{M} \subseteq T\mathbb{M}$, there exists $w \in \mathbb{M}$ such that $v^* = Tw$. Furthermore, utilizing (6), we infer that:

$$\begin{aligned}
\rho(Fw, v^*) &= \rho(Fw, Gu^*) \\
&\leq k_1\rho(Su^*, Tw) + k_2\rho(Fw, Tw) + k_3\rho(Gu^*, Su^*) \\
&\quad + k_4\rho(Tw, Gu^*) + k_5\rho(Su^*, Fw) \\
&= (k_2 + k_5)\rho(Fw, v^*) \prec \rho(Fw, v^*).
\end{aligned}$$

A contradiction since $k_2 + k_5 < 1$. Hence, $|\rho(Fw, v^*)| = 0$, implying $v^* = Fw$. Consequently, this leads to: $v^* = Fw = Tw$; thus, $v^* = Su^* = Gu^* = Fw = Tw$. Since $\{G, S\}$ is weakly compatible, it follows that $GSu^* = SGu^*$, thereby implying $Gv^* = Sv^*$.

Now, let us prove that v^* is indeed a fixed-point of G . If $Gv^* \neq v^*$, then according to (6):

$$\begin{aligned}
\rho(v^*, Gv^*) &= \rho(Fw, Gv^*) \\
&\leq k_1\rho(Sv^*, Tw) + k_2\rho(Fw, Tw) + k_3\rho(Gv^*, Sv^*) \\
&\quad + k_4\rho(Tw, Gv^*) + k_5\rho(Sv^*, Fw) \\
&= (k_1 + k_4 + k_5)\rho(v^*, Gv^*) \prec \rho(v^*, Gv^*),
\end{aligned}$$

a contradiction since $k_1 + k_4 + k_5 < 1$. This implies that $|\rho(v^*, Gv^*)| = 0$. Therefore, $v^* = Gv^*$. Hence, $v^* = Gv^* = Sv^*$. Similarly, due to the weak compatibility of

$\{F, T\}$, we observe that $v^* = Fv^* = Tv^*$. Indeed, since

$$\begin{aligned} \rho(Fv^*, v^*) &= \rho(Fv^*, Gv^*) \\ &\leq k_1\rho(Sv^*, Tv^*) + k_2\rho(Fv^*, Tv^*) + k_3\rho(Gv^*, Sv^*) \\ &\quad + k_4\rho(Tv^*, Gv^*) + k_5\rho(Sv^*, Fv^*) \\ &= (k_1 + k_4 + k_5)\rho(Fv^*, v^*), \end{aligned}$$

this implies that $|\rho(Fv^*, v^*)| < |\rho(Fv^*, v^*)|$, a contradiction since $(k_1 + k_4 + k_5) < 1$. Thus, $v^* = Fv^*$, and v^* is a common fixed-point of F, G, S , and T . In a similar manner, it is evident that v^* is the common fixed-point of F, G, S and T wherever \mathbb{TM} is complete. Finally, to prove the uniqueness of v^* , let us suppose v_1^* and v_2^* , where $v_1^* \neq v_2^*$, are common fixed-points of F, G, S and T . Using (6), we obtain:

$$\begin{aligned} \rho(v_1^*, v_2^*) &= \rho(Fv_1^*, Gv_2^*) \\ &\leq k_1\rho(Sv_2^*, Tv_1^*) + k_2\rho(Fv_1^*, Tv_1^*) + k_3\rho(Gv_2^*, Sv_2^*) \\ &\quad + k_4\rho(Tv_1^*, Gv_2^*) + k_5\rho(Sv_2^*, Fv_1^*) \\ &= (k_1 + k_4 + k_5)\rho(v_1^*, v_2^*) < \rho(v_1^*, v_2^*), \end{aligned}$$

a contradiction since $k_1 + k_4 + k_5 < 1$. Thus, $|\rho(v_1^*, v_2^*)| = 0$. Therefore, $v_1^* = v_2^*$. This concludes the proof of our theorem. \square

Remark 4.1. When $F = G, S = T = I$, in Theorem 4.1, we obtain the following Hardy-Rogers fixed-point for single-valued mappings in a complete complex-valued b-metric space.

Corollary 4.2. *Let (\mathbb{M}, ρ) be a complete complex-valued b-metric space with a coefficient $r \geq 1$, and let $F : \mathbb{M} \rightarrow \mathbb{M}$ be a self-map satisfying the following condition:*

$$\rho(Fu, Fv) \leq k_1\rho(u, v) + k_2\rho(u, Fu) + k_3\rho(v, Fv) + k_4\rho(u, Fv) + k_5\rho(v, Fu),$$

for all $u, v \in \mathbb{M}$, where $k_i \geq 0$ satisfies the condition (7). Then F has a unique fixed-point in \mathbb{M} .

Remark 4.2. This result has been established in [4, Theorem 2], under the restriction

$$k_1 + rk_2 + k_3 + 2rk_4 + rk_5 < 1.$$

Remark 4.3. If $S = T = I, F = G$ and $k_2 = k_3 = k_4 = k_5 = 0$ in Theorem 4.1, we obtain the significant outcome of the Banach fixed-point theorem for single-valued mappings in a complete complex-valued b-metric space [14, Theorem 2.1], as indicated in Corollary 3.4 above.

Corollary 4.3. *Let (\mathbb{M}, ρ) be a complex-valued b-metric space with a coefficient $r \geq 1$, and let $F : \mathbb{M} \rightarrow \mathbb{M}$ be a self-map satisfying the following condition:*

$$\rho(Fu, Fv) \leq k_1\rho(u, v) + k_2\rho(u, Fu) + k_3\rho(v, Fv),$$

for all $u, v \in \mathbb{M}$, where $k_i \geq 0$ and $r(k_1 + k_2) + k_3 < 1$. Then F has a unique fixed-point in \mathbb{M} .

Remark 4.4. This Corollary, established by [4, Corollary 8], under the restriction $k_1 + rk_2 + k_3 < 1$, introduces the Reich fixed-point theorem for single-valued mappings in the context of complete complex-valued b-metric space. For the corresponding result in complete b-metrics, refer to [12].

Remark 4.5. If we set $F = G, S = T = I$, $k_1 = k_2 = k_3 = 0$, and $k_4 = k_5 = k$ in Theorem 4.1, we obtain the following Chatterjea fixed-point version of the single-valued mappings in a complete complex-valued b-metric space.

Corollary 4.4. [4, Corollary 7] *Let (\mathbb{M}, ρ) be a complete complex-valued b-metric space with a coefficient $r \geq 1$, and let $F : \mathbb{M} \rightarrow \mathbb{M}$ be a self-map satisfying the following condition:*

$$\rho(Fu, Fv) \preceq k[\rho(u, Fv) + \rho(v, Fu)],$$

for all $u, v \in \mathbb{M}$, where $k \geq 0$ and $2rk < 1$. Then F has a unique fixed-point in \mathbb{M} .

Remark 4.6. If we set $F = G, S = T = I$, $k_1 = k_4 = k_5 = 0$, and $k_3 = k_4 = k$ in Theorem 4.1, we obtain the following Kannan fixed-point version of single-valued mapping in a complex-valued b-metric space.

Corollary 4.5. [4, Corollary 6] *Let (\mathbb{M}, ρ) be a complete complex-valued b-metric space with a coefficient $r \geq 1$, and let $F : \mathbb{M} \rightarrow \mathbb{M}$ be a self-map satisfying the following condition:*

$$\rho(Fu, Fv) \preceq k[\rho(u, Fu) + \rho(v, Fv)],$$

for all $u, v \in \mathbb{M}$, where $k \geq 0$ and $2rk < 1$. Then F has a unique fixed-point.

To illustrate Theorems 3.1 and 4.1, we present the following example:

Example 4.7. Consider the set $\mathbb{M} = [0, \infty)$ endowed with the complete complex-valued b-metric function $\rho(u, v) = i|u - v|^2$ for all $u, v \in \mathbb{M}$ with $r = 2$. Define the mappings $F, G, S, T : \mathbb{M} \rightarrow \mathbb{M}$ as follows:

$$Fu = \frac{u}{9}, \quad Gu = \frac{u}{18}, \quad Su = \frac{u}{6}, \quad Tu = \frac{u}{3}.$$

We observe that $F\mathbb{M} \subseteq S\mathbb{M}$ and $G\mathbb{M} \subseteq T\mathbb{M}$. The sets $F\mathbb{M}, G\mathbb{M}, S\mathbb{M}$, and $T\mathbb{M}$ are complete subspaces of \mathbb{M} . For each $u \in \mathbb{M}$, we have $FTu = \frac{u}{27} = TFu$ and $GSu = \frac{u}{108} = SGu$. Therefore, the pairs $\{F, T\}$ and $\{G, S\}$ are commuting mapping and hence are weakly compatible.

Furthermore, conditions (1) and (6) are satisfied. Indeed, for all $u, v \in \mathbb{M}$, we have:

$$\rho(Fu, Gv) = i|Fu - Gv|^2 = \frac{i}{324}|v - 2u|^2,$$

$$\rho(Sv, Tu) = i|Sv - Tu|^2 = \frac{i}{36}|v - 2u|^2,$$

$$\rho(Tu, Fu) = i|Tu - Fu|^2 = \frac{4i}{81}|u|^2,$$

$$\rho(Sv, Gv) = i|Sv - Gv|^2 = \frac{i}{81}|v|^2,$$

$$\rho(Tu, Gv) = i|Tu - Gv|^2 = \frac{i}{324}|v - 6u|^2,$$

$$\rho(Sv, Fu) = i|Sv - Fu|^2 = \frac{i}{36}|v - \frac{2}{3}u|^2.$$

We also have:

$$\max \left\{ \rho(Sv, Tu), \frac{\rho(Tu, Fu)\rho(Sv, Gv)}{1 + \rho(Fu, Gv)} \right\} = \rho(Sv, Tu) = \frac{i}{36}|v - 2u|^2,$$

since

$$\frac{\rho(Tu, Fu)\rho(Sv, Gv)}{1 + \rho(Fu, Gv)} = \frac{\frac{-4}{6561}|u|^2|v|^2}{1 + \frac{i}{324}|v - 2u|^2}.$$

Additionally,

$$\min \{ \rho(Tu, Gv), \rho(Sv, Fu) \} \preceq \rho(Tu, Sv),$$

since

$$\rho(Tu, Gv) = \frac{i}{324}|v - 6u|^2 \preceq \frac{i}{4}|3v - 6u|^2 = \frac{i}{36}|v - 2u|^2,$$

and

$$\rho(Sv, Fu) = \frac{i}{36}|v - \frac{2}{3}u|^2 \preceq \frac{i}{36}|v - 2u|^2.$$

This implies that condition (1) holds. Furthermore, since

$$\begin{aligned} \rho(Fu, Tu) + \rho(Gv, Sv) &= \frac{4i}{81}|u|^2 + \frac{i}{81}|v|^2 \\ &= \frac{i}{81}|2u|^2 + \frac{i}{81}|v|^2 \\ &= \frac{i}{81}| - 2u|^2 + \frac{i}{81}| - v|^2 \\ &\preceq \frac{i}{81}|v - 2u|^2 + \frac{i}{81}|v - 2u|^2, \end{aligned}$$

it follows that condition (6) is also satisfied. Therefore, all conditions of Theorems 3.1 and 4.1 are satisfied, and F, G, S and T have a unique common fixed-point at 0 in \mathbb{M} .

5. Application to Linear Nonhomogeneous Systems with Positivity Constrains.

Linear nonhomogeneous systems of equations are crucial for modeling a wide range of real-world phenomena in fields such as engineering, physics, biology, economics, and control systems.

In this section, we apply the results from Corollary 3.3 to investigate the conditions necessary for a unique common solution to two systems of linear nonhomogeneous equations encountered in various practical contexts.

Consider the following systems of equations:

$$Au = b \quad \text{and} \quad Bu = b, \quad (8)$$

where $A = [a_{jk}]$ and $B = [b_{jk}]$ are $n \times n$ matrices, $a_{jk}, b_{jk} \in \mathbb{R}$, $1 \leq j, k \leq n$, and $u = (u_1, \dots, u_n)^T$ and $b = (b_1, \dots, b_n)^T$ are n -dimensional vectors in \mathbb{C}^n .

The solutions to these systems correspond to two vector subspaces of \mathbb{C}^n , denoted as U and V . Any vector $u \in U \cap V$ is a common solution to both systems given in equation (8).

Theorem 5.1. *Let $\mathbb{M} = \mathbb{C}^n$, and define the complex metric $\rho(u, v)$ as*

$$\rho(u, v) = \sqrt{1+i} \left(\sum_{j=1}^n |u_j - v_j|^2 \right)^{\frac{1}{2}}.$$

Here $u = (u_1, \dots, u_n)^T$ and $v = (v_1, \dots, v_n)^T$ are n -dimensional vectors in \mathbb{M} , with $u_j \succ 0$ and $v_j \succ 0$, for all $1 \leq j \leq n$. Assume that $a_{jk}, b_{jk} \in \mathbb{R}$ such that $b_{jk} > a_{jk} > 0$, for all $1 \leq j, k \leq n$. Suppose that α_{jk} and β_{jk} are defined as

$$\alpha_{jk} = \begin{cases} a_{jk} & \text{if } j \neq k \\ a_{jk} + 1 & \text{if } j = k, \end{cases} \quad \beta_{jk} = \begin{cases} b_{jk} & \text{if } j \neq k \\ b_{jk} + 1 & \text{if } j = k, \end{cases}$$

and set $\alpha = \max_{1 \leq j, k \leq n} \alpha_{jk}$. Let $b = (b_1, \dots, b_n)^T$ be an n -dimensional vector in \mathbb{M} such that $b_j \succ 0$, for all $1 \leq j \leq n$. If

$$n\alpha < \left(\frac{1}{2}\right)^{\frac{3}{2}},$$

then the two systems given in (8) have a unique common solution.

PROOF. On \mathbb{M} , we define the mappings $F : \mathbb{M} \rightarrow \mathbb{M}$ and $G : \mathbb{M} \rightarrow \mathbb{M}$ by $Fu = (A + I)u - b$ and $Gu = (B + I)u - b$.

To establish that the two linear systems $Au = b$ and $Bu = b$ have a unique common solution, it is sufficient to show that F and G satisfy the contraction condition

stated in Corollary 3.3. We have

$$\begin{aligned}\rho(Fu, Gv) &= \sqrt{1+i} \left(\sum_{j=1}^n |(Fu)_j - (Gv)_j|^2 \right)^{\frac{1}{2}} \\ &= \sqrt{1+i} \left(\sum_{j=1}^n \left| \sum_{k=1}^n \alpha_{jk} u_k - \beta_{jk} v_k \right|^2 \right)^{\frac{1}{2}},\end{aligned}$$

$$\alpha_{jk} = \begin{cases} a_{jk} & \text{if } j \neq k \\ a_{jk} + 1 & \text{if } j = k, \end{cases} \quad \text{and} \quad \beta_{jk} = \begin{cases} b_{jk} & \text{if } j \neq k \\ b_{jk} + 1 & \text{if } j = k. \end{cases}$$

Since $b_{jk} > a_{jk} > 0$, for all $1 \leq j, k \leq n$, it follows that

$$\begin{aligned}\rho(Fu, Gv) &\leq \sqrt{1+i} \left(\sum_{j=1}^n \left| \sum_{k=1}^n \alpha_{jk} (u_k - v_k) \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^n \max_{1 \leq k \leq n} \alpha_{jk}^2 \right)^{\frac{1}{2}} \sqrt{1+i} \left(\sum_{j=1}^n \sum_{k=1}^n |u_k - v_k|^2 \right)^{\frac{1}{2}} \\ &= (n\alpha^2)^{\frac{1}{2}} \sqrt{1+i} \left(n \sum_{k=1}^n |u_k - v_k|^2 \right)^{\frac{1}{2}} \\ &= n\alpha\rho(u, v),\end{aligned}$$

where $\alpha = \max_{j,k} \alpha_{jk}$. Thus, we have

$$\rho(Fu, Gv) \leq n\alpha\rho(u, v).$$

Therefore, F and G satisfy the contraction condition stated in Corollary 3.3 with $r = \sqrt{2}$. By applying Corollary 3.3, we conclude that the linear systems in (8) have a unique common solution, thereby completing the proof. \square

Conclusion

This paper introduces fixed-point theorems for quadruple mappings that satisfy various contraction conditions using weakly compatible mappings in complete complex valued b-metric spaces. An example illustrates the practical application of these theorems. The findings advance and refine previous research, offering a comprehensive framework for understanding quadruple weakly compatible mappings. Additionally, we present an application to determine the existence and uniqueness of a common solution for two systems of linear nonhomogeneous equations under positivity constraints.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, ASSIUT UNIVERSITY, ASSIUT, EGYPT

Email address: saddeek@aun.edu.eg

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, ASSIUT UNIVERSITY, ASSIUT, EGYPT

Email address: gihen.17290498@science.aun.edu.eg,

Received : June 2024
Accepted : September 2024