Mathematical Analysis and its Contemporary Applications
Volume 6, Issue 3, 2024, 35–46
doi: 10.30495/maca.2024.2045703.1112
ISSN 2716-9898

$*$-K-frames in Hilbert C^*-modules

Sedigheh Hosseini

ABSTRACT. In this paper, we introduce the concept of $*$-K-frames in Hilbert C^*-modules. We find a necessary and sufficient condition on a $*$-Bessel sequence to be a $*$-K-frame. Using the concept of $*$-K-frames, we show that an adjointable operator is surjective. Moreover, we prove that the image of a $*$-K-frame under the action of an operator remains again a $*$-K-frame for Hilbert C^*-module $\mathcal{H}$. In addition, we show that under what conditions an adjointable operator can be invertible.

1. Introduction

Duffin and Schaeffer [7] introduced the concept of frames for the first time in 1952. They abstracted the fundamental notion of Gabor [10] to study signal processing and considered some problems in the non-harmonic Fourier series. However, it seems that the ideas were indicated in [7] did not attract much interest outside the realm of non-harmonic Fourier series. This gap was resolved by Daubechies, Grassman and Mayer [5] in 1986. In other words, they showed that quasiorthogonal expansions will be a useful tool in many areas of theoretical physics and applied mathematics. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields.

The theory of frames was generalized to different vectors in Hilbert spaces; for instance see [12], [16] and [18]. In other words, Frank and Larson [9] were the

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2020 *Mathematics Subject Classification.* 42C15, 46B20.
Key words and phrases. Frame, $*$-frame, $*$-K-frame, C^*-algebra, Hilbert C^*-module.

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first authors who introduced the notion of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces and then Jing [11] continued this topic. It is well-known that Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Moreover, the theory of Hilbert C^*-modules has more applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry and KK-theory. Note that there are some differences between Hilbert C^*-modules and Hilbert spaces. For instance, it is known that the Riesz representation theorem for continuous linear functionals on Hilbert spaces dose not extend to Hilbert C^*-modules [17] and there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [14]. On the other hand, every bounded operator on a Hilbert space has an adjoint while there are bounded operators on Hilbert C^*-modules do not have such property [15] and hence it is expected that some problems about frames and $*$-frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces.

The main purposes of the present paper are to introduce the $*$-K-frames, to consider the relation between $*$-frames and $*$-K-frames in a given Hilbert C^*-module $\mathcal{H}$. We obtain a necessary and sufficient condition on a $*$-Bessel sequence to be a $*$-K-frame. Moreover, we show that under which condition $*$-K-frame for $\mathcal{H}$ can be a $*$-M-frame, where K and M are adjointable operators. Using the concept of $*$-K-frames, we prove that an adjointable operator is surjective. Furthermore, we prove that the image of a $*$-K-frame under an adjointable operator is again a $*$-K-frame for the range. Finally, we show that under what conditions an adjointable operator can be invertible.

2. Basic definitions and Preliminaries

Let I be a countable index set. Throughout this paper, we assume that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. For information about frames in Hilbert spaces we refer to [3]. In what follows, all definitions about C^*-algebras are taken from [4] and [6]. An element a in $\mathcal{A}$ is called *positive* and denoted by $a \geq 0$ if $a = a^*$ and $\sigma(a) \subset R^+$, where $\sigma(a)$ is the spectrum of a. Moreover, $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$. The nonzero element a is called *strictly* nonzero if zero does not belong to $\sigma(a)$, and a is said to be *strictly positive* if it is strictly nonzero and positive. The absolute value of a is defined and denoted by $|a| := (a^*a)^{\frac{1}{2}}$. The relation $\leq$ given by $a \leq b$ if and only if $b - a$ is positive, defines a partial ordering on $\mathcal{A}$. Some elementary facts about $\leq$ are given in the following statements for each $a, b, c \in \mathcal{A}$.

(i) $a \leq \|a\|$;
(ii) $0 \leq a \leq b$ implies $\|a\| \leq \|b\|$, $ab \geq 0$, $a + b \geq 0$, and $at \leq bt$ for $t \in (0, 1)$;

(iii) If $a \leq b$, then $cac^* \leq cbc^*$. Moreover, if c commutes with a and b, then $ca \leq cb$ for $c \geq 0$;

(ii) If a and b are positive invertible elements and $a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$.

In this paper, the notation $a < b$ denotes $a \leq b$ with $a \neq b$.

Definition 2.1. [13] Let $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. Then, $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if it is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words

(i) $\langle x, y \rangle^* = \langle y, x \rangle$;

(ii) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(iii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, then it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle^{\frac{1}{2}}$. A C^*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b \rangle = ab^*$. The standard Hilbert $\mathcal{A}$-module $l_2(\mathcal{A})$ is defined by

$$l_2(A) := \{\{a_j\}_{j \in N} \subseteq A : \sum_{j \in N} a_j a_j^* \text{ converges in } \mathcal{A}\},$$

with $\mathcal{A}$-inner product $\langle \{a_j\}_{j \in N}, \{b_j\}_{j \in N} \rangle = \sum_{j \in N} a_j b_j^*$.

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$, $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. An $\mathcal{A}$-module map $T : \mathcal{H} \longrightarrow \mathcal{K}$ is said to be *adjointable* if there exists an operator $T^* : \mathcal{K} \longrightarrow \mathcal{H}$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

holds for all $x \in \mathcal{H}$, $y \in \mathcal{K}$. We denote by $\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$, the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and moreover $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ =$\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$.

In the sequel, we list some facts from [1], [8] and [19] which are useful tools to achieve our goals in this paper.

Theorem 2.2. [8, Theorem 1.3] *Let $\mathcal{F}, \mathcal{H}, \mathcal{K}$ be Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$. Let also $S \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{K}, \mathcal{H})$ and $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{F}, \mathcal{H})$ with $\overline{R(T^*)}$ orthogonally complemented. Then, the following statements are equivalent:*

(1) *$SS^* \leq \lambda TT^*$ for some $\lambda > 0$;*

(2) *there exists $\mu > 0$ such that $\|S^*z\| \leq \mu \|T^*z\|$ for all $z \in \mathcal{H}$;*

(3) *there exists $D \in L(\mathcal{K}, \mathcal{F})$ such that $S = TD$, i.e., $TX = S$ has a solution;*

(4) *$R(S) \subseteq R(T)$.*

Theorem 2.3. [19, Theorem 2.2] *Let $\mathcal{H}$,$\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $B \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$. Then, the Moore-Penrose inverse $B^\dagger$ of B exists if and only if B has closed range.*

Definition 2.4. Let $\mathcal{H}$ be a Hilbert C^*-module. A sequence $\{x_i\}_{i\in I}$ in H is called a *standard* frame for $\mathcal{H}$ if for each $x \in \mathcal{H}$, the series $\sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle$ is convergent in $\mathcal{A}$ and there exist two positive nonzero numbers C and D in such that

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle$$

for all $x \in \mathcal{H}$.

The elements C and D are called lower and upper frame bounds for $\{x_i\}_{i\in I}$, respectively. The frame is called *tight* if $C = D$ and called a *Parseval* if $C = D = 1$. If in the above we only need to have the upper bound, then $\{x_i\}_{i\in I}$ is called a *Bessel* sequence. The following definition is taken from from the above.

Definition 2.5. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. A sequence $\{x_i\}_{i\in I}$ in $\mathcal{H}$ is said to be a standard $*$-frame for $\mathcal{H}$ if for each $x \in \mathcal{H}$, the series $\sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle$ is convergent in $\mathcal{A}$ and there exist two strictly nonzero elements C and D in $\mathcal{A}$ such that

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle D^*$$

for all $x \in \mathcal{H}$.

The elements C and D are called $*$-frame bounds for $\{x_i\}_{i\in I}$. The $*$-frame is called tight if $C = D$ and called a Parseval if $C = D = 1$. If in the above we only need to have the upper bound, then $\{x_i\}_{i\in I}$ is called a $*$-Bessel sequence.

Theorem 2.6. [1, Theorem 1.1] *Let $\{x_i\}_{i\in I}$ in H be a standard $*$-frame for $\mathcal{H}$ with lower and upper $*$-frame bounds A and B, respectively. Then, the $*$-frame transform or pre-$*$-frame operator $T : \mathcal{H} \longrightarrow l_2(\mathcal{A})$ defined by $T(x) = \langle x, x_i\rangle_{i\in I}$ is an injective and closed range adjointable $\mathcal{A}$-module map and $\|T\| \leq \|B\|$. Moreover, the adjoint operator T^* is surjective and it is given by $T^*(e_i) = x_i$ for $i \in I$ where $\{e_i : i \in I\}$ is the standard basis for $l_2(\mathcal{A})$.*

Definition 2.7. Let $\{x_i\}_{i\in I}$ in $\mathcal{H}$ be a standard $*$-frame for $\mathcal{H}$ with pre-$*$-frame operator T and lower and upper $*$-frame bounds A and B, respectively. Consider the $*$-frame operator S as follows:

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \quad Sx := T^*Tx = \sum_{i\in I}\langle x, x_i\rangle x_i. \tag{1}$$

The $*$-frame operator has some similar properties with frame operator in ordinary frames, but the other properties are different. The main cause of differences is

$\mathcal{A}$-valued bounds. However, the reconstruction formula is given from the $*$-frame operator.

THEOREM 2.8. [1, Theorem 1.2] *Let $\{x_i\}_{i\in I}$ in $\mathcal{H}$ be a standard $*$-frame for $\mathcal{H}$ with $*$-frame operator S (defined in (1)) and lower and upper $*$-frame bounds A and B, respectively. Then, S is positive, invertible and adjointable. Also, the inequality $\|A^{-1}\|^2 \leq \|S\| \leq \|B\|^2$ holds, and the reconstruction formula $x = \sum_{i\in I}\langle x, S^{-1}x_i\rangle x_i$ is valid for $x \in \mathcal{H}$. Moreover, $\{x_i\}_{i\in I}$ is a set of module generators of $\mathcal{H}$.*

PROPOSITION 2.9. [1] *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in \mathrm{Hom}^*_{\mathcal{A}}(H, K)$. Then*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2$;*
(ii) *If T is surjective, then the adjointable map TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2$.*

3. Main results

Recall that $*$-frames are C^*-algebra version of frames. Actually, we need strictly positive elements of C^*-algebra $\mathcal{A}$ instead of positive real numbers.

Here, we remind the definition of K-frame for Hilbert C^*-modules. Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. A family $\{x_i\}_{i\in I}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ is a K-frame for $\mathcal{H}$, if there exist two positive constants A, B such that for each $x \in \mathcal{H}$,

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle,$$

for all $x \in \mathcal{H}$. The elements A and B are called lower and upper bounds of the K-frame, respectively. Motivated by the above, we have the next definition.

DEFINITION 3.1. Let $\mathcal{H}$ be a Hilbert C^*-module and $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. A sequence $\{x_i\}_{i\in I}$ in $\mathcal{H}$ is said to be a standard $*$-K-frame for $\mathcal{H}$ if for each $x \in \mathcal{H}$, the series $\sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle$ is convergent in $\mathcal{A}$ and there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle B^*,$$

for all $x \in \mathcal{H}$. The elements A and B are called $*$-K-frame bounds for $\{x_i\}_{i\in I}$. The $*$-K-frame is called *tight* if $A = B$ and called a *Parseval* if $A = B = 1$. If in the above we only need to have the upper bound, then $\{x_i\}_{i\in I}$ is called a $*$-K-*Bessel* sequence.

PROPOSITION 3.2. *Let $\{x_i\}_{i\in I}$ be a $*$-K-frame for $\mathcal{H}$ with lower and upper $*$-K-frame bounds A and B, respectively. Then, the $*$-K-frame operator S, defined in*

(1) *is invertible, positive,adjointable and bounded and also* $\|A^{-1}\|^{-2}\|K\|^2 \leq \|S\| \leq \|B\|^2$.

PROOF. By Theorem 2.8, S is invertible, positive and self-adjoint. It follows from the definition of a $*$-K-frame that

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle B^*$$

and

$$A\langle K^*x, K^*x\rangle A^* \leq \langle Sx, x\rangle \leq B\langle x, x\rangle B^*.$$

Hence

$$\|A^{-1}\|^{-2}\|\langle KK^*x, x\rangle \leq \|\langle Sx, x\rangle\| \leq \|B\|^2\langle x, x\rangle.$$

Taking the supremum on all $x \in \mathcal{H}$, where $\|x\| \leq 1$, we conclude that

$$\|A^{-1}\|^{-2}\|K\|^2 \leq \|S\| \leq \|B\|^2,$$

and so S is a bounded operator. □

In the next result we find a necessary and sufficient condition on a $*$-Bessel sequence to be a $*$-K-frame.

THEOREM 3.3. *Let* $\{x_i\}_{i\in I}$ *be a* $*$*-Bessel sequence for* $\mathcal{H}$*. Then, the* $\{x_i\}_{i\in I}$ *is a* $*$*-*K*-frame for* $\mathcal{H}$ *if and only if there exists a strictly nonzero element* A *in* C^**-algebra* $\mathcal{A}$ *such that* $S \geq AKK^*$*, where* S *is defined in* (1).

PROOF. It is known that $\{x_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$ with bounds A and B if and only if

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle B^*, \tag{2}$$

for all $x \in \mathcal{H}$. Relation (2) is equivalent to

$$A\langle KK^*x, x\rangle A^* \leq \langle\sum_{i\in I}\langle x, x_i\rangle x_i, x\rangle \leq B\langle x, x\rangle B^*, \tag{3}$$

for all $x \in \mathcal{H}$. Moreover, (3) holds if and only if

$$A\langle KK^*x, x\rangle A^* \leq \langle Sx, x\rangle \leq B\langle x, x\rangle B^*,$$

for all $x \in \mathcal{H}$. This completes the proof □

THEOREM 3.4. *Let* $\{x_i\}_{i\in I}$ *be a* $*$*-frame for* $\mathcal{H}$ *with upper and lower* $*$*-frame bounds* A *and* B*, respectively. Let also* $K \in End^*_{\mathcal{A}}(\mathcal{H})$ *is surjective. Then,* $\{x_i\}_{i\in I}$ *is a* $*$*-*K*-frame for* $\mathcal{H}$ *with upper* $*$*-*K*-frame bounds* $\frac{A}{\|K\|}$ *and* B*, respectively.*

PROOF. By the definition of $*$-frames, for each $x \in \mathcal{H}$, we have

$$A\langle x, x\rangle A^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle B^*.$$

By surjectivity of K and Proposition 2.9, we conclude that

$$A\langle K^*x, K^*x\rangle A^* = A\langle KK^*x, x\rangle A^* \leq A\|K\|^2\langle x, x\rangle A^*,$$

and hence

$$\frac{A}{\|K\|}\langle K^*x, K^*x\rangle\frac{A^*}{\|K\|} \leq A\langle x, x\rangle A^*.$$

The above relation implies that

$$\begin{aligned}\frac{A}{\|K\|}\langle K^*x, K^*x\rangle\frac{A^*}{\|K\|} &\leq A\langle x, x\rangle A^* \\ &\leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle B^*.\end{aligned}$$

This finishes the proof. □

Theorem 3.4 has an inverse version as follows.

THEOREM 3.5. *Let $\{x_i\}_{i\in I}$ be a $*$-K-frame for $\mathcal{H}$ with upper and lower $*$-K-frame bounds A and B, respectively. If the operator K is surjective, then $\{x_i\}_{i\in I}$ is a $*$-frame for $\mathcal{H}$.*

PROOF. By Proposition 2.1 from [2], $K \in \mathrm{End}^*_A(H)$ is surjective if and only if there is $M > 0$ such that

$$M\langle x, x\rangle \leq \langle K^*x, K^*x\rangle,$$

for all $x \in \mathcal{H}$. Since $\{x_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$, we obtain

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle B^*.$$

for all $x \in \mathcal{H}$ and hence

$$\begin{aligned}MA\langle x, x\rangle A^* &\leq A\langle K^*x, K^*x\rangle A^* \\ &\leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \\ &\leq B\langle x, x\rangle B^*.\end{aligned}$$

This means that $\{x_i\}_{i\in I}$ is a $*$-frame for $\mathcal{H}$. □

The following lemma is a fundamental result in obtaining some aims in this section.

LEMMA 3.6. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $\{x_i\}_{i\in I}$ be a $*$-K-Bessel sequence. Then, $\{Mx_i\}_{i\in I}$ is a $*$-Bessel sequence for all $M \in Hom^*_{\mathcal{A}}(\mathcal{H})$.*

Proof. By our assumption, there exists a strictly nonzero element B in $\mathcal{A}$ such that

$$\sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \leq B\langle x,x\rangle B^*,$$

for all $x \in \mathcal{H}$. Thus

$$\begin{aligned}\sum_{i\in I}\langle x,Mx_i\rangle\langle Mx_i,x\rangle &= \sum_{i\in I}\langle M^*x,x_i\rangle\langle x_i,M^*x\rangle \\ &\leq B\langle M^*x,M^*x\rangle B^* \\ &= B\langle MM^*x,x\rangle B^* \\ &\leq B\|M\|^2\langle x,x\rangle B^* \\ &= (B\|M\|)\langle x,x\rangle(B\|M\|)^*,\end{aligned}$$

for all $x \in \mathcal{H}$. □

The upcoming proposition shows that under which condition $*$-K-frame for $\mathcal{H}$ can be a $*$-M-frame, where $K, M \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$.

Proposition 3.7. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, $K \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ and $\{x_i\}_{i\in I}$ be a $*$-K-frame for $\mathcal{H}$. Suppose that $M \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ with $R(M) \subseteq R(K)$ and $\overline{R(K^*)}$ orthogonally complemented. Then, $\{x_i\}_{i\in I}$ is an $*$-M-frame for $\mathcal{H}$.*

Proof. Since $\{x_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$, there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle K^*x,K^*x\rangle A^* \leq \sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \leq B\langle x,x\rangle B^*, \tag{4}$$

for all $x \in \mathcal{H}$. Using Theorem 2.2 and the fact that $R(M) \subset R(K)$, we yield $MM^* \leq \lambda KK^*$ for some $\lambda > 0$ and so

$$\langle MM^*x,x\rangle \leq \lambda\langle KK^*x,x\rangle.$$

Hence

$$\frac{A}{\lambda}\langle MM^*x,x\rangle A^* \leq A\langle KK^*x,x\rangle A^*. \tag{5}$$

It follows from (4) and (5) that

$$\frac{A}{\lambda}\langle MM^*x,x\rangle A^* \leq \sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \leq B\langle x,x\rangle B^*,$$

and thus

$$\frac{A}{\sqrt{\lambda}}\langle MM^*x,x\rangle\frac{A^*}{\sqrt{\lambda}} \leq \sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \leq B\langle x,x\rangle B^*,$$

for all $x \in \mathcal{H}$. Therefore, $\{x_i\}_{i\in I}$ is an $*$-M-frame with bound $\frac{A}{\sqrt{\lambda}}$ and B for $\mathcal{H}$. □

Applying the concept of $*$-K-frames, one can prove that an adjointable operator is surjective. This is shown in the next result.

THEOREM 3.8. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $K \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ with the dense range. Suppose also $\{x_i\}_{i\in I}$ is a $*$-K-frame for H and $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ has closed range. If $\{Tx_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$, then T is surjective.*

PROOF. Assume that $K^*x = 0$ for $x \in \mathcal{H}$. For each $y \in \mathcal{H}$, we have $\langle Ky, x\rangle = \langle y, K^*x\rangle = 0$. Since $R(K)$ is dense in $\mathcal{H}$, we get $\langle z, x\rangle = 0$ for all $z \in \mathcal{H}$. Thus, $x = 0$ and K^* is injective. We shall show that T^* is injective as well. Note that $\{Tx_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$ with bounds A and B and hence

$$A\|K^*x\|^2A^* \leq \|\sum_{i\in I}\langle x, Tx_i\rangle\langle Tx_i, x\rangle\| \leq B\|x\|^2,$$

for $T^*x \in \mathcal{H}$. Consequently

$$A\|K^*x\|^2A^* \leq \|\sum_{i\in I}\langle T^*x, x_i\rangle\langle Tx_i, T^*x\rangle\| \leq B\|x\|^2.$$

If now $x \in N(T^*)$, then $T^*x = 0$ and so $\langle T^*x, x_i\rangle = 0$ for all i. Thus, $K^*x = 0$ by the last inequality. On the other hand, K^* is injective and hence $x = 0$ which yields T^* is injective. Therefore, $H = N(T^*) + \overline{R(T)} = \overline{R(T)} = R(T)$, and this completes the proof. $\square$

In the following theorem, we show that the image of a $*$-K-frame under an adjointable operator is again a $*$-K-frame for the range.

THEOREM 3.9. *Let $K \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ and $\{x_i\}_{i\in I}$ be a $*$-K-frame for $\mathcal{H}$. If $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ with closed range such that $\overline{R(TK)}$ is orthogonal complemented and $KT = TK$, then $\{Tx_i\}_{i\in I}$ is a $*$-K-frame for $R(T)$.*

PROOF. Note that Theorem 2.3 shows that if T has closed range, then T has Moore-Penrose inverse operator $T^\dagger$ such that $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$. Hence, $TT^\dagger|_{R(T)} = I_{R(T)}$ and $(TT^\dagger)^* = I^* = I = TT^\dagger$. For every $x \in R(T)$ we have

$$\begin{aligned}\langle K^*x, K^*x\rangle &= \langle (TT^\dagger)^*K^*x, (TT^\dagger)^*K^*x\rangle \\ &= \langle (T^\dagger)^*T^*K^*x, (T^\dagger)^*T^*K^*x\rangle \\ &\leq \|(T^\dagger)^*\|^2\langle T^*K^*x, T^*K^*x\rangle,\end{aligned}$$

for all $x \in \mathcal{H}$ and so

$$\|(T^\dagger)^*\|^{-2}\langle K^*x, K^*x\rangle \leq \langle T^*K^*x, T^*K^*x\rangle,$$

for all $x \in \mathcal{H}$. it is known that $\{x_i\}_{i\in I}$ is a $*$-K-frame and $R(T^*K^*) \subset R(K^*T^*)$. If A is a lower bound, then by Theorem 2.2 there exists some $\lambda > 0$ such that

$$\begin{aligned}\sum_i \langle x, Tx_i\rangle\langle Tx_i, x\rangle &= \sum_i \langle T^*x, x_i\rangle\langle x_i, T^*x\rangle \\ &\geq A\langle K^*T^*x, K^*T^*x\rangle A^* \\ &\geq \lambda A\langle T^*K^*x, T^*K^*x\rangle A^* \\ &\geq \lambda A\|(T^\dagger)^*\|^{-2}\langle K^*x, K^*x\rangle A^*.\end{aligned}$$

This is the lower inequality for $\{Tx_i\}_{i\in I}$. On the other hand, by Lemma 3.6, $\{Tx_i\}_{i\in I}$ is a Bessel sequence and therefore $\{Tx_i\}_{i\in I}$ is a $*$-K-frame for Hilbert module $R(T)$. □

Using an adjointable operator, we can construct a new $*$-K-frame for $\mathcal{H}$ as follows.

THEOREM 3.10. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, $K \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ and $\{x_i\}_{i\in I}$ be a $*$-K-frame for $\mathcal{H}$. If $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ is a co-isometry such that $R(T^*K^*) \subseteq R(K^*T^*)$ with $\overline{R(TK)}$ orthogonal complemented, then $\{Tx_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$.*

PROOF. Applying Lemma 3.6, we observe that $\{Tx_i\}_{i\in I}$ is a Bessel sequence. By Theorem 2.2, there exist $\lambda > 0$ such that $\|T^*K^*x\|^2 \leq \lambda\|K^*T^*x\|^2$ for all $x \in \mathcal{H}$. Assume A is a lower bound for the $*$-K-frame $\{x_i\}_{i\in I}$. Since T is a co-isometry, we get

$$\begin{aligned}\frac{A}{\lambda}\|K^*x\|^2A^* &= \frac{A}{\lambda}\|T^*K^*x\|^2A^* \leq A\|K^*T^*x\|^2A^* \\ &\leq \sum_i \langle T^*x, x_i\rangle\langle x_i, T^*x\rangle \\ &= \sum_i \langle x, Tx_i\rangle\langle Tx_i, x\rangle,\end{aligned}$$

which implies that $\{Tx_i\}_{i\in I}$ is a $*$-K-frame for $\mathcal{H}$. □

We close this paper by the incoming result, shows that under what condition an adjointable operator is invertible.

THEOREM 3.11. *Let $K \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ with dense range and $\{x_i\}_{i\in I}$ be a $*$-K-frame for $\mathcal{H}$. Suppose that $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ with closed range. If $\{Tx_i\}_{i\in I}$ and $\{T^*x_i\}_{i\in I}$ are $*$-K-frame for $\mathcal{H}$, then $T$is invertible.*

PROOF. By Lemma 3.6, T is surjective. It follows from the definition of $*$-K-frame for $\mathcal{H}$ that there exist elements A and B in $\mathcal{A}$ such that for each $x \in \mathcal{H}$, we have

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_i \langle x, T^*x_i\rangle\langle T^*x_i, x\rangle \leq B\langle x, x\rangle B^*.$$

Furthermore, for $x \in N(T)$, we have

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_i \langle x, T^*x_i\rangle\langle T^*x_i, x\rangle = 0.$$

Hence, $\langle K^*x, K^*x\rangle = 0$ and thus $x \in N(K^*)$. On the other hand, $K \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H})$ has dense range and so K^* is injective. This implies that T is injective. □

References

[1] A. Alijani and M. Dehghan, *$*$-frames in Hilbert C^*-modules*, U. P. B. Sci. Bull. Series A., **73** (4) (2011), 89–106.

[2] Lj. Arambasic, *On frames for countably generated Hilbert C^*-modules*, Proc. Amer. Math. Soc., **135**(2) (2007), 469–478.

[3] O. Christensen, *An Introduction to Frames and Riesz Bases*, Brikhouser, 2016.

[4] J. B. Conway, *A Course in Operator Theory*, AMS, 2000.

[5] I. Daubechies, A. Grassman, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986), 1271–1283.

[6] F. R. Davidson, *C^*-Algebra by Example*, Fields Ins. Monog, 1996.

[7] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Proc. Amer. Math. Soc., **72** (1952), 341–366.

[8] X. Fang, J. Yu, and H. Yao, *Solutions to operator equations on Hilbert C^*-modules*, Linear Algebra Appl., **431**(11) (2009), 2142–2153.

[9] M. Frank and D. R. Larson, *A module frame concept for Hilbert C^*-modules*, Functional and Harmonic Analysis of Wavelets (San Antonio, TX, Jan. 1999), Contemp. Math., **247** (2000), 207–233.

[10] D. Gabor, *Theory of communications*, J. Electr. Eng. (London), **49**(3) (1946), 429–457.

[11] W. Jing, *Frames in Hilbert C^*-modules*, Ph.D. Thesis, University of Central Florida, 2006.

[12] A. Khosravi and B. Khosravi, *Frames and bases in tensor products of Hilbert spaces and Hilbert C^*-modules*, Proc. Indian Acad. Sci., **117**(1) (2007), 1–12.

[13] E. C. Lance, *Hilbert C^*-modules, A Toolkit for Operator Algebraists*, University of Leeds, Cambridge University Press, 1995.

[14] B. Magajna, *Hilbert C^*-modules in which all closed submodules are complemented*, Proc. Amer. Math. Soc., **125**(3) (1997), 849–852.

[15] V. M. Manuilov, *Adjointability of operators on Hilbert C^*-modules*, Acta Math. Univ. Comenianae, LXV (2) (1996), 161-169.

[16] A. Nejati and A. Rahimi, *Generalized frames in Hilbert spaces*, Bull. Iran. Math. Soc., **35**(1) (2009), 97–109.

[17] N. E. Wegge Olsen, *K-Theory and C^*-Algebras, A Friendly Approach*, Oxford University Press, Oxford, England, 1993.

[18] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl., **322** (2006), 437–452.

[19] Q. Xu and L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C^*-modules*. Linear Algebra Appl., **428**(4) (2008), 992–1000.

Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran

Email address: sedigheh.hosseini496@gmail.com,

Received : *June* 2024
Accepted : *July* 2024