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Cauchy additive functional equation with quality and certainty of the approximation in generalized Z-numbers

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Abstract. In this paper, we provide the generalized Ulam-Hyers stability of the Cauchy additive functional equation $F(X+Y)=F(X)+F(Y)$ with quality and certainty of the approximation in generalized Z-numbers in the sense of Rassias and Găvruţa.

1. Introduction

In many practical problems, the fuzzy probability approach can be an important component of decision-making. In the real world, we consider various aspects of uncertainty that are not always well represented in fuzzy sets of information uncertainty. To overcome this problem, Zadeh introduced the Z-number (Z-N) in 2011 [23]; for more on the subject, see Aliev et al. [4] and Allahviranloo et al. [5]. A Z-N is an ordered binary of the form (A, B), where the first component shows the fuzzy value and the second shows the uncertainty of the first. Based on the Z-N theory, we provide a model which considers both certainty and quality for the solution of a Cauchy additive functional equation.

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In 1940, Ulam [22] at the University of Wisconsin projected the following stability problem:

"When is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation? "

In 1941, Hyers [12] provided a favourable answer to the question of Ulam for Banach spaces. In 1950, Aoki [6] was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias [19] get a head in extending Hyers Theorem by weakening the condition for the Cauchy difference controlled by $(||x||^p + ||y||^p)$, $p \in [0, 1)$, to be unbounded. In 1982, J.M. Rassias [18] switched the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$, for $p, q \in R$. A generalization of all the above stability results were obtained by P. Găvruţa [11] in 1994 by swapping the unbounded Cauchy difference by a general control function $\varphi(x, y)$. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et al., [20] by considering the factor $||x||^{2p} + ||y||^{2p} + ||x||^p||y||^p$.

One of the most famous functional equation is the Cauchy additive functional equation

$$F(X + Y) = F(X) + F(Y) \tag{1}$$

having solution $F(X) = c\ X$. The functional equation (1) was first treated by Legendre (1791) and Gauss (1809). In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called a Cauchy additive functional equation in honour of A. L. Cauchy, see [15]. Stability analysis in the sense of Ulam-Hyers can be used to find an approximate solution for a wide selection of functional equations, we refer to [1, 4, 5, 7, 8, 10, 13, 14, 15, 16, 17, 21].

Now, we present the result due to Diaz and Margolis [9] for fixed point theory.

Theorem 1.1. [3] *Let (Ω, Δ) be a complete generalized metric space (GMS) and $\Xi : \Omega \longrightarrow \Omega$ such that for all $x, y \in \Omega$,*

$$d(\Xi x, \Xi y) \leq L d(x, y), L \in (0, 1).$$

Then, for each given $x \in \Omega$, either

$$d(\Xi^n x, \Xi^{n+1} x) = \infty,\ \forall\ n \geq 0,$$

or there exists a natural number n_0 such that
(FPC1) $d(\Xi^n x, \Xi^{n+1} x) < \infty$, for all $n \geq n_0$;
(FPC2) $\lim_{n\to\infty} \Xi^n x = y^*$;
(FPC3) Y^* is the unique fixed point of Ξ in the set $\Omega^* = \{y \in \Omega : d(\Xi^{n_0} x, Y) < \infty\}$;
(FPC4) $d(Y, y^*) \leq \frac{1}{1-L} d(Y, \Xi y)$, for all $Y \in \Omega^*$.

2. Basics of Generalized Z-number (GZ-N)

Now, we recall the basics of Generalized Z-number (GZ-N) as given in [3]. Let $\Theta_1 = [0,1]$, and let x_{Θ_1} be given as follows:

$$x_{\Theta_1} = \left\{ \text{diag } \Theta_1 = \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix} = \text{diag}[\theta_1, \cdots, \theta_n], \theta_1, \cdots, \theta_n \in \Theta_1 \right\}.$$

We write $\text{diag}[\theta_1, \cdots, \theta_n] \preceq \text{diag}[\kappa_1, \cdots, \kappa_n]$, when $\theta_i \leq \kappa_i$ for every $i = 1, \cdots, n$.

Definition 2.1. A mapping $\circledast : x_{\Theta_1} \times x_{\Theta_1} \to x_{\Theta_1}$ is called a generalized continuous t-norm (GCTN) if for all $\rho, \kappa, \varpi, y, \kappa_n, \varpi_n \in x_{\Theta_1}, \mathbf{1} = \text{diag}[1, \cdots, 1]$ the following conditions are satisfied:

(GCT1) $\varpi \circledast \mathbf{1} = \varpi$;
(GCT2) $\varpi \circledast \kappa = \kappa \circledast \varpi$;
(GCT3) $\varpi \circledast (\kappa \circledast \varpi) = (\varpi \circledast \kappa) \circledast \varpi$;
(GCT4) $\rho \preceq \kappa$ and $\varpi \preceq y$ imply that $\rho \circledast \varpi \preceq \kappa \circledast y$;
(GCT5) If $\lim_{n\to\infty} \kappa_n = \kappa$ and $\lim_{n\to\infty} \varpi_n = \varpi$, we have $\lim_{n\to\infty}(\kappa_n \circledast \varpi_n) = \kappa \circledast \varpi$.

In this paper, we choose the minimum t-norm $\circledast_M = x_{\Theta_1} \times x_{\Theta_1} \to x_{\Theta_1}$ which is defined as follows:

$$\varpi \circledast_M \kappa = \text{diag}[\varpi_1, \cdots, \varpi_n] \circledast_M \text{diag}[\kappa_1, \cdots, \kappa_n] = \text{diag}[\min\{\varpi_1, \kappa_1\}, \cdots, \min\{\varpi_n, \kappa_n\}].$$

Definition 2.2. Let $\wp \in \mathbb{R}$ and $\wp \in (0,1]$, and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If X is a vector space over $\mathbb{K}$, A fuzzy set $\aleph_\wp : X \times (0,\infty) \to \Theta_1$ is a $\wp-$ fuzzy norm ($\wp-$FN) on X if and only if we have

($\wp-$FN1) $\aleph_\wp(x,\zeta) = 1$, if and only if $x = 0$ for $\zeta \in (0,\infty)$;
($\wp-$FN2) $\aleph_\wp(\gamma x,\zeta) = \aleph_\wp\left(x, \frac{\zeta}{\gamma}\right)$, for each $\gamma \neq 0 \in \mathbb{K}$, all $x \in X$ and for $\zeta \in (0,\infty)$;
($\wp-$FN3) $\aleph_\wp(x+y,\zeta+\delta) \geq \aleph_\wp(x,\zeta) \circledast \mu(Y,\delta)$, for all $x, Y \in X$ and for $\zeta, \delta \in (0,\infty)$;
($\wp-$FN4) $\lim_{\zeta\to+\infty} \aleph_\wp(x,\zeta) = 1$, for any $\zeta \in (0,\infty)$.

A $\wp-$Banach FN space is a complete $\wp-$FN space.

We now use the concept of probability distribution functions to measure the certainty of a vector [2], where we put

$$\epsilon_0(\zeta) = \begin{cases} 0, & \text{if } \zeta \leq 0, \\ 1, & \text{if } \zeta > 0. \end{cases}$$

Definition 2.3. Let $\wp \in (0,1)$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A $\wp-$random normed space ($\wp-$RNS) is a triple $(X, {}^\wp\mu, \circledast')$, where X is a vector space over $\mathbb{K}$, $\circledast'$ is a continuous $t-$norm and ${}^\wp\mu$ is a mapping from X into D^+ such that the following conditions hold:

($\wp-$RNS1) ${}^\wp\mu_x(\zeta) = \varepsilon_0(\zeta)$ for all $\zeta > 0$ if and only if $x = 0$;

($\wp$−RNS2) ${}^{\wp}\mu_{\alpha\ x}(\zeta) ={}^{\wp} \mu_x\Big(\dfrac{\zeta}{|\alpha|^{\wp}}\Big)$ for all $x \in X$ and $\alpha \neq 0$;

($\wp$−RNS3) ${}^{\wp}\mu_{x+y}(\zeta + \delta) \geq {}^{\wp}\mu_x(\zeta) \circledast' {}^{\wp}\mu_Y(\delta)$ for all $x, y \in X$ and $\zeta, \delta \geq 0$, where ${}^{\wp}\mu_x$ denotes the value of ${}^{\wp}\mu$ at a point $x \in X$.

Definition 2.4. Let $\wp \in \mathbb{R}$ and $\wp \in (0, 1)$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We define a matrix-valued function $\widetilde{Z} : X \times \mathbb{R}^+ \to x_{\Theta_1}$ with

$$\widetilde{Z}(x, \zeta) = \text{diag}[\aleph_{\wp}(x, \zeta), {}^{\wp}\mu_x(\zeta), \aleph_{\wp}(x, \zeta) \circledast {}^{\wp}\mu_x(\zeta)]$$

and call it a generalized Z-number (GZ-N), when for all $x, y \in X$ and $\zeta, \delta \geq 0$, $\alpha \neq 0$, the following conditions are satisfied:

(GZN1) $\widetilde{Z}(x, \zeta) = \text{diag}[1, 1, 1] = 1$ if and only if $x = 0$;

(GZN2) $\widetilde{Z}(\alpha\ x, \zeta) = \widetilde{Z}\Big(x, \frac{\zeta}{|\alpha|^{\wp}}\Big)$ for all $x \in X$ and $\alpha \neq 0$;

(GZN3) $\widetilde{Z}(x + y, \zeta + \delta) \succeq \widetilde{Z}(x, \zeta) \circledast_M \widetilde{Z}(Y, \delta)$.

The above conditions are verified in [**3**].

Definition 2.5. Let $(X, \widetilde{Z})$ be a GZ-N. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\widetilde{Z}(x_n - x, \zeta) > 1 - \lambda$, for all $n \geq N$.

Definition 2.6. Let $(X, \widetilde{Z})$ be a GZ-N. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\widetilde{Z}(x_n - x_m, \zeta) > 1 - \lambda$, for all $n \geq m \geq N$.

Definition 2.7. Let $(X, \widetilde{Z})$ be a GZ-N. A GZ-N $(X, \widetilde{Z})$ is said to be complete if every Cauchy sequence in X is convergent to a point in X.

In order to investigate the generalized Ulam - Hyers stability of the functional equations (1) in GZ-N, for that, suppose that $F : \mathcal{F}_1 \to \mathcal{F}_2$ where $\mathcal{F}_1$ be a $\wp - N$ left $\mathcal{C}-$ module and $\mathcal{F}_2$ be a $\wp - N$ left Banach $\mathbb{C}-$ module.

3. Measure of the Quality and the Certainty of the Approximation of the Solution of Functional Equation (1) with GZ-N

Theorem 3.1. *Let $F : \mathcal{F}_1 \to \mathcal{F}_2$ is a function satisfying the inequality*

$$\widetilde{Z}\Big(F(X + Y) - F(X) - F(Y), \mathcal{C}\Big) \succeq A\Big(\Big(X, Y\Big), \mathcal{C}\Big) \tag{2}$$

where

$$A\Big((X, Y), \mathcal{C}\Big) = \text{diag}\Big[A_1((X, Y), \mathcal{C}), A_2((X, Y), \mathcal{C})), A_1((X, Y), \mathcal{C}) \circledast A_2((X, Y), \mathcal{C})\Big] \tag{3}$$

with $A_1 : \mathcal{F}_1^2 \times \mathbb{R}^+ \to \Theta_1, A_2 : \mathcal{F}_1^2 \to D^+$ *for all* $X, Y \in \mathcal{F}_1$ *and* $\mathcal{C} > 0$*. If there exists* $\mathcal{L} = \mathcal{L}(I)$ *be a function have the properties*

$$A\left(\left(Y, Y\right), \mathcal{C}\right) \succeq A\left(\left(\frac{Y}{2}, \frac{Y}{2}\right), \mathcal{C}\right), \ A\left(\left(P_I\ Y, P_I\ Y\right), P_I^{\wp}\ \mathcal{C}\right) \succeq A\left(\left(Y, Y\right), \frac{1}{\mathcal{L}}\ \mathcal{C}\right), \tag{4}$$

for all $Y \in \mathcal{F}_1$ *and* $\mathcal{C} > 0$*, then there exists a unique additive mapping* $\mathcal{G}(Y) : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ *satisfying the functional equation (1) and*

$$\widetilde{Z}\left(F(Y) - \mathcal{G}(Y), \mathcal{C}\right) \succeq A\left(\left(Y, Y\right), \frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}}\ \mathcal{C}\right) \tag{5}$$

where $A : \mathcal{F}_1^2 \to [0, \infty)$ *be function with the condition*

$$\lim_{D\to\infty} A\left(\left(P_I^D X, P_I^D Y\right), P_I^{D\wp}\mathcal{C}\right) = 1 \tag{6}$$

and

$$P_I = \begin{cases} 2 & I = 0; \\ \frac{1}{2} & I = 1. \end{cases} \tag{7}$$

for all $Y \in \mathcal{F}_1$ *and all* $\mathcal{C} > 0$*.*

Proof. Assume a set

$$\mathcal{X} = \{p/p : \mathcal{F}_1 \to \mathcal{F}_2,\ p(0) = 0\}$$

and introduce the generalized metric on the above set $\mathcal{X}$ as

$$d(p, q) = \inf\left\{K \in (0, \infty) : \widetilde{Z}(p(Y) - q(Y), \mathcal{C}) \succeq A\left((Y, Y), \frac{1}{K}\ \mathcal{C}\right)\right\}, \tag{8}$$

for all $Y \in \mathcal{F}_1$ and all $\mathcal{C} > 0$. It is easy to see that $(\mathcal{X}, d)$ is complete (See Theorem 3.1 of [3]). Define a function $T : \mathcal{X} \to \mathcal{X}$ by

$$Tp(Y) = \frac{1}{P_I} p(P_I Y), \quad \text{for all } Y \in \mathcal{F}_1. \tag{9}$$

Now $p, q \in \mathcal{X}$, $Y \in \mathcal{F}_1$ and all $\mathcal{C} > 0$, we have $d(p, q) \le K$. This implies that

$$\widetilde{Z}(p(Y) - q(Y), \mathcal{C}) \succeq A((Y, Y), \frac{1}{K}\ \mathcal{C}).$$

Hence,

$$\begin{aligned} \widetilde{Z}\left(\frac{1}{P_I} p(P_I Y) - \frac{1}{P_I} q(P_I Y), \mathcal{C}\right) &\succeq A\left((P_I Y, P_I Y), P_I^{\wp} \frac{1}{K}\ \mathcal{C}\right) \\ &\succeq A\left((Y, Y), \frac{1}{\mathcal{L}K}\ \mathcal{C}\right). \end{aligned}$$

This implies that

$$\widetilde{Z}\left(Tp(Y) - Tq(Y)\mathcal{C}\right) \succeq A\left((Y, Y), \frac{1}{\mathcal{L}K}\ \mathcal{C}\right),$$

and so

$$d(Tp,Tq) \leq \mathcal{L}\ K,$$

i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant $\mathcal{L}$. Changing (X,Y) by (Y,Y) in (2), we arrive

$$\widetilde{Z}\left(F(2Y)-2F(Y),\mathcal{C}\right) \succeq A\left(\left(Y,Y\right),\mathcal{C}\right), \qquad \forall\ Y \in \mathcal{F}_1, \mathcal{C} > 0. \tag{10}$$

The above inequality can be rewritten by using $(GZN2)$ as

$$\widetilde{Z}\left(\frac{F(2Y)}{2}-F(Y),\frac{1}{2^{\wp}}\ \mathcal{C}\right) \succeq A\left(\left(Y,Y\right),\mathcal{C}\right).$$

Then

$$\widetilde{Z}\left(\frac{F(2Y)}{2}-F(Y),\mathcal{C}\right) \succeq A\left(\left(Y,Y\right),2^{\wp}\ \mathcal{C}\right), \tag{11}$$

for all $Y \in \mathcal{F}_1, \mathcal{C} > 0$.

With the help of (4), it follows from (11) for the case $I=0$ it reduces to

$$\widetilde{Z}\left(TF(Y)-F(Y),\mathcal{C}\right) \succeq A\left(\left(Y,Y\right),\frac{1}{\mathcal{L}}\ \mathcal{C}\right).$$

Then

$$d(TF,F) \leq \mathcal{L} = \mathcal{L}^{1-I} < \infty, \tag{12}$$

for all $Y \in \mathcal{F}_1, \mathcal{C} > 0$. Replacing $Y = \frac{Y}{2}$ in (10) and using $(GZN2)$, we have

$$\widetilde{Z}\left(F(Y)-2F\Big(\frac{Y}{2}\Big),\mathcal{C}\right) \succeq A\left(\Big(\frac{Y}{2},\frac{Y}{2}\Big),\mathcal{C}\right), \qquad \forall\ Y \in \mathcal{F}_1, \mathcal{C} > 0. \tag{13}$$

With the help of (4), it follows from (13) for the case $I=1$ it reduces to

$$\widetilde{Z}\left(F(Y)-TF(Y),\mathcal{C}\right) \succeq A\left(\left(Y,Y\right),\mathcal{C}\right).$$

Then

$$d(F,TF) \leq 1 = \mathcal{L}^{1-I} < \infty, \tag{14}$$

for all $Y \in \mathcal{F}_1, \mathcal{C} > 0$. Combining the above two cases, we arrive

$$d(F,TF) \leq \mathcal{L}^{1-I}. \tag{15}$$

Therefore (FPC1) of Theorem 1.1 holds. By (FPC2) of Theorem 1.1, it follows that there exists a unique fixed point $\mathcal{G}(Y)$ of T in $\mathcal{X}$ such that

$$\widetilde{Z}\left(\mathcal{G}(Y)-\lim_{D\to\infty}\frac{F(P_I^DY)}{P_I^D},\mathcal{C}\right) = 1, \qquad \forall\ Y \in \mathcal{F}_1, \mathcal{C} > 0. \tag{16}$$

Now, to show that $\mathcal{T}_2(Y)$ satisfies (1), changing (X,Y) by (P_I^DX, P_I^DY) and using (GZN2) in (2), we reach

$$\widetilde{Z}\Big(\frac{1}{P_I^D}\Big\{F\Big(P_I^D(X+Y)\Big)-F\Big(P_I^DX\Big)-F\Big(P_I^DY\Big)\Big\},\mathcal{C}\Big) \succeq A\Big(\Big(P_I^D\ X, P_I^D\ Y\Big),P_I^{D\wp}\mathcal{C}\Big) \tag{17}$$

for all $X, Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. Approaching $D \to \infty$ and using the definition of $\mathcal{G}(Y)$, (6), $(GZN1)$ in the above inequality, we see that $\mathcal{G}(Y)$ satisfies the functional equation (1) for all $X, Y \in \mathcal{F}_1$. Again by (FPC3) of Theorem 1.1, $\mathcal{G}(Y)$ is the unique fixed point of T in the set

$$\mathcal{Y} = \{\mathcal{G}(Y) \in \mathcal{X} : d(F, \mathcal{G}(Y)) < \infty\},$$

such that

$$d(F(Y) - \mathcal{G}(Y)) \leq K\ A(Y, Y).$$

Finally by (FPC4) of Theorem 1.1, we obtain

$$d(F, \mathcal{G}) \leq \frac{1}{1 - \mathcal{L}}\ d(F, \mathcal{G}).$$

This implies that

$$d(F, \mathcal{G}) \leq \frac{\mathcal{L}^{1-I}}{1 - \mathcal{L}}$$

which yields

$$\widetilde{Z}\Big(F(Y) - \mathcal{G}(Y), \mathcal{C}\Big) \succeq A\left(\Big(Y, Y\Big), \frac{(1 - \mathcal{L})}{\mathcal{L}^{1-I}}\ \mathcal{C}\right).$$

this completes the proof of the theorem. □

From the above theorem, we have the following corollaries concerning some stabilities of (1).

Corollary 3.2. *Let E be positive number and $F : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a function fulfilling the inequality*

$$\widetilde{Z}\Big(F(X + Y) - F(X) - F(Y), \mathcal{C}\Big) \succeq \operatorname{diag}\left(\frac{\mathcal{C}}{E}, e^{-\frac{1}{\frac{\mathcal{C}}{E}}}, \frac{\mathcal{C}}{E} \circledast e^{\frac{1}{\frac{\mathcal{C}}{E}}}\right), \tag{18}$$

for all $X, Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. Then, there exists a unique additive mapping $\mathcal{G}(Y) : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ satisfying the functional equation (1) and

$$\widetilde{Z}\Big(\mathcal{G}(Y) - F(Y), \mathcal{C}\Big) \succeq \operatorname{diag}\left(\frac{|2^{\wp} - 1|\ \mathcal{C}}{E}, e^{-\frac{1}{\frac{|2^{\wp}-1|\ \mathcal{C}}{E}}}, \frac{|2^{\wp} - 1|\ \mathcal{C}}{E} \circledast e^{\frac{1}{\frac{|2^{\wp}-1|\ \mathcal{C}}{E}}}\right), \tag{19}$$

for all $Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$.

Proof. From (4), we see

$$\widetilde{Z}'\left(A\Big(Y, Y\Big), \mathcal{C}\right) = \widetilde{Z}'\left(A\left(\frac{Y}{2}, \frac{Y}{2}\right), \mathcal{C}\right) = \operatorname{diag}\left(\frac{\mathcal{C}}{E}, e^{-\frac{1}{\frac{\mathcal{C}}{E}}}, \frac{\mathcal{C}}{E} \circledast e^{\frac{1}{\frac{\mathcal{C}}{E}}}\right) \tag{20}$$

and

$$\begin{aligned} &\widetilde{Z}'\left(A\left(P_I\ Y, P_I\ Y\right), P_I^{\wp}\ \mathcal{C}\right) \\ &= \operatorname{diag}\left(\frac{P_I^{\wp}\ \mathcal{C}}{E}, e^{-\frac{1}{P_I^{\wp}\ \frac{\mathcal{C}}{E}}}, \frac{P_I^{\wp}\ \mathcal{C}}{E} \circledast e^{\frac{1}{\frac{P_I^{\wp}\ \mathcal{C}}{E}}}\right) = \widetilde{Z}'\left(A\Big(Y, Y\Big), \frac{1}{\mathcal{L}}\ \mathcal{C}\right), \end{aligned} \tag{21}$$

for all $Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. It follows from (5), (20) and (21):

For $I=0$:

$$\mathcal{L}={P_I}^{\wp}=\frac{1}{2^{\wp}}=2^{-\wp}.$$

Then

$$\frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}}=\frac{(1-2^{-\wp})}{(2^{-\wp})^{1-0}}=(2^{\wp}-1)$$

Hence,

$$\widetilde{Z}\Big(F(Y)-\mathcal{G}(Y),\mathcal{C}\Big)\succeq \operatorname{diag}\left(\frac{(2^{\wp}-1)\ \mathcal{C}}{E},e^{-\frac{1}{\frac{(2^{\wp}-1)\ \mathcal{C}}{E}}},\frac{(2^{\wp}-1)\ \mathcal{C}}{E}\circledast e^{\frac{1}{\frac{(2^{\wp}-1)\ \mathcal{C}}{E}}}\right).$$

For $I=1$:

$$\mathcal{L}=\frac{1}{P_I^{\wp}}=\frac{1}{\frac{1}{2^{\wp}}}=2^{\wp}.$$

We have,

$$\frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}}=\frac{(1-2^{\wp})}{(2^{\wp})^{1-1}}=(1-2^{\wp})$$

Therefore,

$$\widetilde{Z}\Big(F(Y)-\mathcal{G}(Y),\mathcal{C}\Big)\succeq \operatorname{diag}\left(\frac{(1-2^{\wp})\ \mathcal{C}}{E},e^{-\frac{1}{\frac{(1-2^{\wp})\ \mathcal{C}}{E}}},\frac{(1-2^{\wp})\ \mathcal{C}}{E}\circledast e^{\frac{1}{\frac{(1-2^{\wp})\ \mathcal{C}}{E}}}\right).$$

□

Corollary 3.3. *Assume that E is a positive number and J is a real number with $J\neq 1$. Let $F:\mathcal{F}_1\longrightarrow\mathcal{F}_2$ be a function fulfilling the inequality*

$$\widetilde{Z}\Big(F(X+Y)-F(X)-F(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\{||X||_{\wp}^{J}+||Y||_{\wp}^{J}\}},e^{-\frac{\{||X||_{\wp}^{J}+||Y||_{\wp}^{J}\}}{\frac{\mathcal{C}}{E}}},\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\{||X||_{\wp}^{J}+||Y||_{\wp}^{J}\}}\circledast e^{-\frac{\{||X||_{\wp}^{J}+||Y||_{\wp}^{J}\}}{\frac{\mathcal{C}}{E}}}\right), \tag{22}$$

for all $X,Y\in\mathcal{F}_1$ and $\mathcal{C}>0$. Then, there exists a unique additive mapping $\mathcal{G}(Y):\mathcal{F}_1\longrightarrow\mathcal{F}_2$ satisfying the functional equation (1) and

$$\widetilde{Z}\Big(\mathcal{G}(Y)-F(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{|2^{\wp}-2^{J\wp}|\ \mathcal{C}}{2^{J\wp}E}}{\frac{|2^{\wp}-2^{J\wp}|\ \mathcal{C}}{2^{J\wp}E}+\frac{2}{2^J}||Y||_{\wp}^{J}},e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^{J}}{\frac{|2^{\wp}-2^{J\wp}|\ \mathcal{C}}{2^{J\wp}E}}},\frac{\frac{|2^{\wp}-2^{J\wp}|\ \mathcal{C}}{2^{J\wp}E}}{\frac{|2^{\wp}-2^{J\wp}|\ \mathcal{C}}{2^{J\wp}E}+\frac{2}{2^J}||Y||_{\wp}^{J}}\circledast e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^{J}}{\frac{|2^{\wp}-2^{J\wp}|\ \mathcal{C}}{2^{J\wp}E}}}\right), \tag{23}$$

for all $Y\in\mathcal{F}_1$ and $\mathcal{C}>0$.

PROOF. From (4), we see

$$\widetilde{Z}'\left(A\Big(Y,Y\Big),\mathcal{C}\right) = \widetilde{Z}'\left(A\left(\frac{Y}{2},\frac{Y}{2}\right),\mathcal{C}\right)$$
$$= \operatorname{diag}\left(\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\frac{2}{2^J}||Y||_{\wp}^J}, e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^J}{\frac{\mathcal{C}}{E}}}, \frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\frac{2}{2^J}||Y||_{\wp}^J}\circledast e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^J}{\frac{\mathcal{C}}{E}}}\right) \tag{24}$$

and

$$\widetilde{Z}'\left(A\left(P_I\, Y, P_I\, Y\right), P_I^{\wp}\, \mathcal{C}\right)$$
$$= \operatorname{diag}\left(\frac{\frac{(P_I^{1-J})^{\wp}\ \mathcal{C}}{E}}{\frac{(P_I^{1-J})^{\wp}\ \mathcal{C}}{E}+2||Y||_{\wp}^J}, e^{-\frac{2||Y||_{\wp}^J}{\frac{(P_I^{1-J})^{\wp}\ \mathcal{C}}{E}}}, \frac{\frac{(P_I^{1-J})^{\wp}\ \mathcal{C}}{E}}{\frac{(P_I^{1-J})^{\wp}\ \mathcal{C}}{E}+2||Y||_{\wp}^J}\circledast e^{-\frac{2||Y||_{\wp}^J}{\frac{(P_I^{1-J})^{\wp}\ \mathcal{C}}{E}}}\right)$$
$$= \widetilde{Z}'\left(A\Big(Y,Y\Big),\frac{1}{\mathcal{L}}\,\mathcal{C}\right), \tag{25}$$

for all $Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. It follows from (5), (24) and (25):

For $I = 0$,

$$\mathcal{L} = (P_I^{1-J})^{\wp} = \frac{1}{(2^{1-J})^{\wp}} = 2^{J\wp-\wp}.$$

Thus,

$$\frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}} = \frac{(1-2^{J\wp-\wp})}{(2^{J\wp-\wp})^{1-0}} = \frac{(2^{\wp}-2^{J\wp})}{2^{J\wp}}.$$

Hence,

$$\widetilde{Z}\Big(F(Y)-\mathcal{G}(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{(2^{\wp}-2^{J\wp})\ \mathcal{C}}{2^{J\wp}E}}{\frac{(2^{\wp}-2^{J\wp})\ \mathcal{C}}{2^{J\wp}E}+\frac{2}{2^J}||Y||_{\wp}^J}, e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^J}{\frac{(2^{\wp}-2^{J\wp})\ \mathcal{C}}{2^{J\wp}E}}}, \frac{\frac{(2^{\wp}-2^{J\wp})\ \mathcal{C}}{2^{J\wp}E}}{\frac{(2^{\wp}-2^{J\wp})\ \mathcal{C}}{2^{J\wp}E}+\frac{2}{2^J}||Y||_{\wp}^J}\circledast e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^J}{\frac{(2^{\wp}-2^{J\wp})\ \mathcal{C}}{2^{J\wp}E}}}\right).$$

For $I = 1$:

$$\mathcal{L} = \frac{1}{(P_I^{1-J})^{\wp}} = \frac{1}{\frac{1}{(2^{1-J})^{\wp}}} = 2^{\wp-J\wp}.$$

Then

$$\frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}} = \frac{(1-2^{\wp-J\wp})}{(2^{\wp-J\wp})^{1-1}} = \frac{(2^{J\wp}-2^{\wp})}{2^{J\wp}}.$$

Thus,

$$\widetilde{Z}\Big(F(Y)-\mathcal{G}(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{(2^{J\wp}-2^{\wp})\ \mathcal{C}}{2^{J\wp}E}}{\frac{(2^{J\wp}-2^{\wp})\ \mathcal{C}}{2^{J\wp}E}+\frac{2}{2^J}||Y||_{\wp}^{J}}, e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^{J}}{\frac{(2^{J\wp}-2^{\wp})\ \mathcal{C}}{2^{J\wp}E}}}, \frac{\frac{(2^{J\wp}-2^{\wp})\ \mathcal{C}}{2^{J\wp}E}}{\frac{(2^{J\wp}-2^{\wp})\ \mathcal{C}}{2^{J\wp}E}+\frac{2}{2^J}||Y||_{\wp}^{J}} \circledast e^{-\frac{\frac{2}{2^J}||Y||_{\wp}^{J}}{\frac{(2^{J\wp}-2^{\wp})\ \mathcal{C}}{2^{J\wp}E}}}\right).$$

□

Corollary 3.4. *Assume E is a positive number and J is a real number with $J \neq \frac{1}{2}$. Let $F : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a function fulfilling the inequality*

$$\widetilde{Z}\Big(F(X+Y)-F(X)-F(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}}, e^{-\frac{\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}}{\frac{\mathcal{C}}{E}}}, \frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}} \circledast e^{-\frac{\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}}{\frac{\mathcal{C}}{E}}}\right), \tag{26}$$

for all $X, Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. Then, there exists a unique additive mapping $\mathcal{G}(Y) : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ satisfying the functional equation (1) and

$$\widetilde{Z}\Big(\mathcal{G}(Y)-F(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}, e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}}, \frac{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}} \circledast e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}}\right), \tag{27}$$

for all $Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$.

Proof. From (4), we see

$$\widetilde{Z}'\left(A\Big(Y,Y\Big),\mathcal{C}\right) = \widetilde{Z}'\left(A\left(\frac{Y}{2},\frac{Y}{2}\right),\mathcal{C}\right)$$
$$= \left\{ \operatorname{diag}\left(\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}, e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{\mathcal{C}}{E}}}, \frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}} \circledast e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{\mathcal{C}}{E}}}\right)\right. \tag{28}$$

and

$$\begin{aligned}
&\widetilde{Z}'\left(A\left(P_I\, Y, P_I\, Y\right), P_I^{\wp}\, \mathcal{C}\right)\\
&\succeq \operatorname{diag}\left(\frac{\frac{(P_I^{1-2J})^{\wp}\ \mathcal{C}}{E}}{\frac{(P_I^{1-2J})^{\wp}\ \mathcal{C}}{E}+||Y||_{\wp}^{2J}}, e^{-\frac{||Y||_{\wp}^{2J}}{\frac{(P_I^{1-2J})^{\wp}\ \mathcal{C}}{E}}}, \frac{\frac{(P_I^{1-2J})^{\wp}\ \mathcal{C}}{E}}{\frac{(P_I^{1-2J})^{\wp}\ \mathcal{C}}{E}+||Y||_{\wp}^{2J}} \circledast e^{-\frac{||Y||_{\wp}^{2J}}{\frac{(P_I^{1-2J})^{\wp}\ \mathcal{C}}{E}}}\right)\\
&= \widetilde{Z}'\left(\ A\Big(Y,Y\Big), \frac{1}{\mathcal{L}}\ \mathcal{C}\right), \qquad (29)
\end{aligned}$$

for all $Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. It follows from (5), (28) and (29):

For $I = 0$

$$\mathcal{L} = (P_I^{1-2J})^{\wp} = \frac{1}{(2^{1-2J})^{\wp}} = 2^{2J\wp-\wp}.$$

This implies that

$$\frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}} = \frac{(1-2^{2J\wp-\wp})}{(2^{2J\wp-\wp})^{1-0}} = \frac{(2^{\wp}-2^{2J\wp})}{2^{2J\wp}}.$$

Hence,

$$\begin{aligned}
&\widetilde{Z}\Big(F(Y)-\mathcal{G}(Y),\mathcal{C}\Big)\\
&\succeq \operatorname{diag}\left(\frac{\frac{(2^{\wp}-2^{2J\wp})\ \mathcal{C}}{2^{2J\wp}E}}{\frac{(2^{\wp}-2^{2J\wp})\ \mathcal{C}}{2^{2J\wp}E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}, e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{(2^{\wp}-2^{2J\wp})\ \mathcal{C}}{2^{2J\wp}E}}}, \frac{\frac{(2^{\wp}-2^{2J\wp})\ \mathcal{C}}{2^{2J\wp}E}}{\frac{(2^{\wp}-2^{2J\wp})\ \mathcal{C}}{2^{2J\wp}E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}} \circledast e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{(2^{\wp}-2^{2J\wp})\ \mathcal{C}}{2^{2J\wp}E}}}\right).
\end{aligned}$$

For $I = 1$,

$$\mathcal{L} = \frac{1}{(P_I^{1-2J})^{\wp}} = \frac{1}{\frac{1}{(2^{1-2J})^{\wp}}} = (2^{1-2J})^{\wp}.$$

This implies thst

$$\frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}} = \frac{(1-(2^{1-2J})^{\wp})}{((2^{1-2J})^{\wp})^{1-1}} = \frac{(2^{2J\wp}-2^{\wp})}{2^{2J\wp}}.$$

Hence,

$$\begin{aligned}
&\widetilde{Z}\Big(F(Y)-\mathcal{G}(Y),\mathcal{C}\Big)\\
&\succeq \operatorname{diag}\left(\frac{\frac{(2^{2J\wp}-2^{\wp})\ \mathcal{C}}{2^{2J\wp}E}}{\frac{(2^{2J\wp}-2^{\wp})\ \mathcal{C}}{2^{2J\wp}E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}, e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{(2^{2J\wp}-2^{\wp})\ \mathcal{C}}{2^{2J\wp}E}}}, \frac{\frac{(2^{2J\wp}-2^{\wp})\ \mathcal{C}}{2^{2J\wp}E}}{\frac{(2^{2J\wp}-2^{\wp})\ \mathcal{C}}{2^{2J\wp}E}+\frac{1}{2^{2J}}||Y||_{\wp}^{2J}} \circledast e^{-\frac{\frac{1}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{(2^{2J\wp}-2^{\wp})\ \mathcal{C}}{2^{2J\wp}E}}}\right).
\end{aligned}$$

□

Corollary 3.5. *Let E be a positive number and J be a real number with $J \neq \frac{1}{2}$. Let also $F : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a function fulfilling the inequality*

$$\widetilde{Z}\Big(F(X+Y)-F(X)-F(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\{||X||_{\wp}^{2J}+||Y||_{\wp}^{2J}\}+\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}}, e^{-\frac{\{||X||_{\wp}^{2J}+||Y||_{\wp}^{2J}\}+\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}}{\frac{\mathcal{C}}{E}}},\right.$$
$$\left.\frac{\frac{\mathcal{C}}{E}}{\frac{\mathcal{C}}{E}+\{||X||_{\wp}^{2J}+||Y||_{\wp}^{2J}\}+\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}} \circledast e^{-\frac{\{||X||_{\wp}^{2J}+||Y||_{\wp}^{2J}\}+\{||X||_{\wp}^{J}||Y||_{\wp}^{J}\}}{\frac{\mathcal{C}}{E}}}\right), \tag{30}$$

for all $X, Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$. Then, there exists a unique additive mapping $\mathcal{G}(Y) : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ satisfying the functional equation (1) and

$$\widetilde{Z}\Big(\mathcal{G}(Y)-F(Y),\mathcal{C}\Big)$$
$$\succeq \operatorname{diag}\left(\frac{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}+\frac{3}{2^{2J}}||Y||_{\wp}^{2J}}, e^{-\frac{\frac{3}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}}, \frac{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}+\frac{3}{2^{2J}}||Y||_{\wp}^{2J}} \circledast e^{-\frac{\frac{3}{2^{2J}}||Y||_{\wp}^{2J}}{\frac{|2^{\wp}-2^{2J\wp}|\ \mathcal{C}}{2^{2J\wp}E}}}\right), \tag{31}$$

for all $Y \in \mathcal{F}_1$ and $\mathcal{C} > 0$.

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