

Contact problem in thermo-elastic viscoplastic materials with damage

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ABSTRACT. In this paper, we consider a quasi-static contact problem with a simplified Tresca model of slip-dependent coefficient of friction in nonlinear thermo-elastic-viscoplastic materials with damage. An inclusion of the parabolic type describes the evolution of the damage. We derive a variational formulation of the problem and prove the existence of a unique weak solution. The proof is based on classical existence and uniqueness results for parabolic inequalities, differential equations, and fixed point arguments.

1. Introduction

Scientific research and recent papers in mechanics can be crystallized into two main parts: one devoted to the laws of behavior and the other to boundary conditions imposed on the body, our own contributions to this area are documented in the sources [10, 11, 12, 13, 14, 15, 16]. The boundary conditions reflect the interaction of the body with the outside world. Constitutive laws with internal variables are used in various publications to model the effect of internal variables on the behaviour of real bodies like metals, rocks, polymers, and so on, where the deformation rate depends on the internal variables. Some of the internal state variables considered

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by many authors include the spatial distribution of dislocations, the work-hardening of materials, the absolute temperature, and the damage field. See, for example, the references [2, 18, 19, 20] for the cases of hardening, temperature, and other internal state variables, and the references [7, 8] for the case of the damage field.

New in this paper is the study of the quasi-static evolution of damage in thermo-electroelastic materials, with frictional contact conditions in which the frictional coefficients depend on the slip according to Tresca's model. For this, we use thermo-elastic-viscoplastic materials. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \theta(s), \alpha(s)) ds,$$

in which $\mathbf{u}, \boldsymbol{\sigma}$ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ represents the absolute temperature, α is the damage field, \mathcal{A} and \mathcal{B} are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and \mathcal{G} is a nonlinear constitutive function which describes the viscoplastic behavior of the material. Also the differential inclusion used for the evolution of the temperature field is

$$\dot{\theta} - k_0 \Delta \theta = \Psi(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(\mathbf{u}), \theta, \alpha) + q.$$

In modelling the contact, we impose S is a given function, and consider the Tresca's model, i.e., the contact surface is given, taking into account that the friction coefficient varies during the slip process. Our model of slip-dependence is considered in geophysics and solid mechanics, corresponding to the dependence of the friction coefficients on the slip \mathbf{u}_τ , i.e., $\mu = \mu(|\mathbf{u}_\tau|)$. More precisely, the model of the contact reads

$$\sigma_\nu = -S, \quad \begin{cases} |\boldsymbol{\sigma}_\tau| \leq S\mu(|\mathbf{u}_\tau|), \\ |\boldsymbol{\sigma}_\tau| \leq S\mu(|\mathbf{u}_\tau|) \Rightarrow \dot{\mathbf{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| \leq S\mu(|\mathbf{u}_\tau|) \Rightarrow \exists \lambda \geq 0 \quad \text{such that} \quad -\lambda \boldsymbol{\sigma}_\tau = \dot{\mathbf{u}}_\tau, \end{cases}$$

such type of contact conditions was considered in various papers, see, for instance, [4, 9, 22].

In this work, we consider a version of a quasistatic contact problem with a slip-dependent coefficient of friction for elastic-viscoplastic materials with damage and thermal effects. The rest of the paper is organized as follows. In Section 2, we introduce some essential preliminaries. In Section 3, we present the mechanical problem, list the assumptions on the data, and give the variational formulation of the problem. In Section 4, we present the proof of the Theorem 4.1.

2. Preliminaries and Notations

In this section, we present the notations and some preliminary material which are necessary for the study of our problem, see reference [6]. We denote by \mathbb{S}^d the space of second order symmetric tensors on \mathbb{R}^d ($d = 2, 3$), while “.” and $\|\cdot\|$ will represent the inner product and the Euclidean norm on \mathbb{S}^d and \mathbb{R}^d respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ . Everywhere, the indices i, j run between 1 and d , the summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable.

We use the standard notations for Lebesgue and Sobolev spaces associated with Ω and Γ and introduce the spaces

$$\begin{aligned} H &= L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\}, & H_1 &= \{\mathbf{u} = (u_i) \mid \varepsilon(\mathbf{u}) \in \mathcal{H}\}, \\ \mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, & \mathcal{H}_1 &= \{\boldsymbol{\sigma} \in \mathcal{H} \mid \text{Div } \boldsymbol{\sigma} \in H\}. \end{aligned}$$

The operators of deformation ε and divergence are defined Div as follows

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H , H_1 , \mathcal{H} and \mathcal{H}_1 are real Hilbert spaces with the canonical inner products defined as follows

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_H &= \int u_i v_i dx \quad \forall \mathbf{u}, \mathbf{v} \in H, & (\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in H_1, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int \sigma_{ij} \tau_{ij} dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\text{Div } \boldsymbol{\sigma}, \text{Div } \boldsymbol{\tau})_H, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1. \end{aligned}$$

The associated norm in the space H , H_1 , \mathcal{H} and \mathcal{H}_1 , is denoted by $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. For every $\mathbf{v} \in H_1$, we also use the notation $\gamma \mathbf{v}$ for the trace of \mathbf{v} on Γ and we denote by v_ν and \mathbf{v}_τ the normal and the tangential components of \mathbf{v} on Γ given by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \quad (1)$$

The normal and tangential components of stress field $\boldsymbol{\sigma}$ on the boundary are defined by

$$\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \quad (2)$$

When $\boldsymbol{\sigma}$ is a regular function, the following Green is type formula holds,

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\text{Div } \boldsymbol{\sigma}, \mathbf{v})_H = \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \mathbf{v} da \quad \forall \mathbf{v} \in H_1. \quad (3)$$

Let $H_\Gamma = (H^{\frac{1}{2}}(\Gamma))^d$ and $\gamma : H_1 \rightarrow H_\Gamma$ be the trace map. We denote by V the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 : \gamma \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since $\text{meas}(\Gamma_1) > 0$, Korn's inequality holds, there exists a positive constant C_k depending only on Ω and Γ_1 such that

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}} \geq C_k \|\mathbf{v}\|_{H^1(\Omega)^d}, \quad \forall \mathbf{v} \in V.$$

We define inner product on V by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{v}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \|\mathbf{v}\|_V = \|\varepsilon(\mathbf{v})\|_{\mathcal{H}}, \quad \forall \mathbf{v}, \mathbf{v} \in V, \quad (4)$$

and let $\|\cdot\|_V$ be the associated norm. It follows that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, (\cdot, \cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant \tilde{C}_0 , depending only on Ω , Γ_1 and Γ_3 , such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \|\mathbf{v}\|_V, \quad \forall \mathbf{v} \in V. \quad (5)$$

Let $T > 0$, for every real Banach space V we use the notations $C(0, T; V)$ and $C^1(0, T; V)$ for the space of continuous and continuously differentiable functions from $[0, T]$ to V , respectively; $C(0, T; V)$ and $C^1(0, T; V)$ is a real Banach space with the norm

$$\|\mathbf{f}\|_{C(0, T; V)} = \max_{t \in [0, T]} \|\mathbf{f}(t)\|_V.$$

$$\|\mathbf{f}\|_{C^1(0, T; V)} = \max_{t \in [0, T]} \|\mathbf{f}(t)\|_V + \max_{t \in [0, T]} \|\dot{\mathbf{f}}(t)\|_V.$$

Now let's assume the following:

The operator $A : V \rightarrow V'$ satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists a constant } m_A > 0 \text{ such that} \\ \quad (A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_X \geq m \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V, \\ \text{(b) There exists a constant } L > 0 \text{ such that} \\ \quad \|A\mathbf{u}_1 - A\mathbf{u}_2\|_{V'} \leq L \|\mathbf{u}_1 - \mathbf{u}_2\|_V, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \end{array} \right. \quad (6)$$

The functional $j : V \times V \rightarrow \mathbb{R}$ satisfies:

$$\left\{ \begin{array}{l} \text{(a) For all } j(\mathbf{u}, \cdot) \text{ is convex and s.c.i. on } V, \\ \text{(b) There exists a constant } m_j > 0 \text{ such that} \\ \quad j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ \quad \leq m_j \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V. \end{array} \right. \quad (7)$$

The operator $B : V \rightarrow V'$ satisfies there exists a constant $L_B > 0$ such that

$$\|B\mathbf{u}_1 - B\mathbf{u}_2\|_{V'} \leq L_B \|\mathbf{u}_1 - \mathbf{u}_2\|_V, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \quad (8)$$

The \mathbf{f} function satisfies

$$\mathbf{f} \in C(0, T; V). \quad (9)$$

The initial values

$$\mathbf{u}_0 \in V. \quad (10)$$

In the section 3 of the paper, we will use an abstract result on quasi-variational evolutionary inequalities. This result concerns problems of the following type.

Problem P_1 . Find \mathbf{u} such that

$$\begin{aligned} (A\dot{\mathbf{u}}(t), v - \dot{\mathbf{u}}(t))_V + (B\mathbf{u}(t), v - \mathbf{u}(t))_V + j(\mathbf{u}(t), v) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geq (\mathbf{f}(t), v - \dot{\mathbf{u}}(t))_V, \end{aligned} \quad (11)$$

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (12)$$

Theorem 2.1. *Let (6)-(10) satisfied. Then*

(1) *There exists a unique solution $\mathbf{u} \in C^1([0, T]; V)$ to the problem p_1 .*

(2) *If \mathbf{u}_1 and \mathbf{u}_2 are two solutions of problem p_1 corresponding to the data $\mathbf{f}_1, \mathbf{f}_2 \in C([0, T]; V)$, then there exists $c > 0$ such that*

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq c(\|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_V + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V), \quad \forall t \in [0, T]. \quad (13)$$

The proof of the above theorem, we use an abstract existence and unique result which may be found in [17].

Problem P_2 . Find $\alpha(t) \in K$ such that

$$\langle \dot{\alpha}(t), \zeta - \alpha(t) \rangle_{V' \times V} + a(\dot{\alpha}(t), \zeta - \alpha(t)) \geq (\phi(t), \zeta - \alpha(t))_{L^2(\Omega)}, \quad (14)$$

$$\alpha(0) = \alpha_0. \quad (15)$$

Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V . We write $V \subset H \subset V'$ and we say that the inclusions above define a Gelfand triple.

The following is a standard result for parabolic variational inequalities (see, e.g., [21]).

Theorem 2.2. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V . Assume that $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants λ and γ ,*

$$a(\alpha, \alpha) + \gamma\|\alpha\|_H^2 \geq \lambda\|\alpha\|_V^2, \quad \forall \alpha \in V.$$

Then, for every $\alpha_0 \in K$ and $\phi \in L^2(0, T; H)$, there exists a unique function $\alpha \in H^1(0, T; H) \cap L^2(0, T; V)$, such that $\alpha(0) = \alpha_0$, $\alpha(t) \in K$ for all $t \in [0, T]$, where α is the unique solution of Problem P_2 .

3. Mechanical and Variational Formulations

We give the physical setting of the contact problem. A deformable body occupies the reference configuration $\Omega \subset \mathbb{R}^d (d = 2, 3)$, which is a bounded domain with a smooth boundary Γ . The body is described by the elastic-viscoplastic constitutive law with damage. and the process is quasistatic in the time interval of interest $[0, T]$.

The boundary $\partial\Omega$ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with $meas(\Gamma_1) > 0$. In addition, we clampe the body on Γ_1 and so the displacement field vanishes there. Surface tractions of density f_0 act on $\Gamma_2 \times (0, T)$ and a volume force of density f_2 is applied in $\Omega \times (0, T)$. The body is in contact on Γ_3 with a foundation, the contact is supposed to be friction-associated to Tresca's law, in which the normal stress is prescribed on Γ_3 , and the coefficient of friction depends on the slip. We admit a possible external heat source applied in $\Omega \times (0, T)$, given by the function q . The classical formulation of the mechanical problem may be stated as follows

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{S}^d$, the damage field $\alpha : \Omega \times [0, T] \rightarrow \mathbb{R}$ and the temperature $\theta : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) \\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \theta(s), \alpha(s)) ds, \end{aligned} \quad \text{in } \Omega \times (0, T), \quad (16)$$

$$0 = \text{Div } \boldsymbol{\sigma} + f_0, \quad \text{in } \Omega \times (0, T), \quad (17)$$

$$\dot{\theta} - k_0 \Delta \theta = \Psi(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}})), \varepsilon(\mathbf{u}), \theta, \alpha + q, \quad \text{in } \Omega \times (0, T), \quad (18)$$

$$\dot{\alpha} - k_1 \Delta \alpha + \partial\varphi_K(\alpha) \ni \Theta(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}})), \varepsilon(\mathbf{u}), \theta, \alpha, \quad \text{in } \Omega \times (0, T), \quad (19)$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_1 \times (0, T), \quad (20)$$

$$\sigma\nu = f_2, \quad \text{on } \Gamma_2 \times (0, T), \quad (21)$$

$$\sigma_\nu(t) = -S, \quad \text{on } \Gamma_3 \times (0, T), \quad (22)$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \leq S\mu(|\mathbf{u}_\tau(t)|), \\ |\boldsymbol{\sigma}_\tau(t)| \leq S\mu(|\mathbf{u}_\tau(t)|) \Rightarrow \dot{\mathbf{u}}_\tau(t) = 0, \\ |\boldsymbol{\sigma}_\tau(t)| \leq S\mu(|\mathbf{u}_\tau(t)|), \\ \Rightarrow \exists \lambda \geq 0 \text{ such that } -\lambda\boldsymbol{\sigma}_\tau(t) = \dot{\mathbf{u}}_\tau(t), \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \quad (23)$$

$$k_0 \frac{\partial \theta}{\partial \nu} + \mathbf{B}\theta = 0, \quad \text{on } \Gamma \times (0, T), \quad (24)$$

$$\frac{\partial \alpha}{\partial \nu} = 0, \quad \text{on } \Gamma \times (0, T), \quad (25)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0, \quad \text{in } \Omega. \quad (26)$$

The equation (16) is the thermo-elastic-viscoplastic constitutive law where \mathcal{A} and \mathcal{B} are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and \mathcal{G} is a nonlinear constitutive function which describes the viscoplastic behavior of the material. Equation (17) represents the equilibrium equation posed on the domain Ω . Equation (18) represents the energy conservation where Ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. The evolution of

the damage field is modeled by the inclusion of the parabolic type given by relation (19) where Θ is the source function of the damage, (20)-(21) are displacement and tensile boundary conditions, respectively.

Conditions (22)-(23) characterize the contact boundary conditions. Since S is a given function, condition (22) states that the normal stress σ_ν is prescribed on Γ_3 , Condition (23) states that if there is a contact, the tangential shear σ_τ cannot exceed the normal stress S multiplied by the value of the friction coefficient $\mu(|\mathbf{u}_\tau|)$. (24)-(25) represent, respectively a Fourier boundary condition for the temperature and a homogeneous Neumann boundary condition for the damage field on Γ . In (26) we consider the initial conditions where \mathbf{u}_0 is the displacement- initially, and α_0 the initial damage.

For the study of the mechanical problem (16) -(26) we consider the following hypotheses. The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}_d \longrightarrow \mathbb{S}_d$ satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|, \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \quad \text{a.e } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2, \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \boldsymbol{\varepsilon} \in \mathbb{S}_d. \\ (d) \text{ The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{array} \right. \quad (27)$$

The elastic function $\mathcal{B} : \Omega \times \mathbb{S}_d \times \mathbb{R} \longrightarrow \mathbb{S}_d$ satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|, \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \forall \text{ a.e } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \boldsymbol{\varepsilon} \in \mathbb{S}_d. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, 0) \in \mathcal{H}. \end{array} \right. \quad (28)$$

The plasticity operator $\mathcal{G} : \Omega \times \mathbb{S}_d \times \mathbb{S}_d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{S}_d$ satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \theta_1, \alpha_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \theta_2, \alpha_2)\| \\ \quad \leq L_{\mathcal{G}} (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| + \|\alpha_1 - \alpha_2\|), \\ \quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \quad \forall \theta_1, \theta_2 \in \mathbb{R}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \theta, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}_d, \forall \theta, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0, 0) \in \mathcal{H}. \end{array} \right. \quad (29)$$

The function $\Psi : \Omega \times \mathbb{S}_d \times \mathbb{S}_d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists a constant } L_\Psi > 0 \text{ such that} \\ \|\Psi(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \theta_1, \alpha_1) - \Psi(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \theta_2, \alpha_2)\| \\ \leq L_\Psi (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \quad \forall \theta_1, \theta_2 \in \mathbb{R}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \Psi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \theta, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}_d, \forall \theta, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \Psi(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0, 0) \in L^2(\Omega). \end{array} \right. \quad (30)$$

The function of the damage source $\Theta : \Omega \times \mathbb{S}_d \times \mathbb{S}_d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists a constant } L_\Theta > 0 \text{ such that} \\ \|\Theta(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \theta_1, \alpha_1) - \Theta(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \theta_2, \alpha_2)\| \\ \leq L_\Theta (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \quad \forall \theta_1, \theta_2 \in \mathbb{R}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \Theta(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \theta, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}_d, \forall \theta, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \Theta(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0, 0) \in L^2(\Omega). \end{array} \right. \quad (31)$$

The coefficient of friction satisfies

$$\left\{ \begin{array}{l} (a) \mu : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \\ (b) \text{ There exists } L_\mu > 0 \text{ such that} \\ \|\mu(\mathbf{x}, r_1) - \mu(\mathbf{x}, r_2)\| \leq L_\mu \|r_1 - r_2\|, \quad \forall r_1, r_2 \in \mathbb{R}_+, \quad \text{a.e. } \mathbf{x} \in \Gamma_3, \\ (c) \text{ The mapping } \mathbf{x} \longmapsto \mu(\mathbf{x}, r) \text{ is Lebesgue measurable on } \Gamma_3, \quad \forall r \in \mathbb{R}_+, \\ (d) \text{ The mapping } \mathbf{x} \longmapsto \mu(\mathbf{x}, 0) \in L^2(\Gamma_3). \end{array} \right. \quad (32)$$

We assume that the body forces, surface tractions, volume heat source and the functions \mathbf{a} and \mathbf{m} , satisfy

$$f_0 \in C(0, T, H), \quad f_2 \in C(0, T, L^2(\Gamma_2)^d), \quad q \in L^2(0, T, L^2(\Omega)). \quad (33)$$

$$\mathbf{u}_0 \in V, \quad \theta_0 \in V, \alpha_0 \in K. \quad (34)$$

$$\mathbf{B} > 0, \quad k_i > 0; \quad i = 0, 1. \quad (35)$$

$$S \in L^\infty(\Gamma_3) \text{ and } S \geq 0 \text{ a.e. on } \Gamma_3. \quad (36)$$

We define the mapping $f : [0, T] \rightarrow V$ by

$$(f(t), \mathbf{v})_V = \int_\Omega f_0(t) \cdot \mathbf{v} dx + \int_{\Gamma_2} f_2(t) \cdot \mathbf{v} da, \quad \forall \mathbf{v} \in V, t \in (0, T). \quad (37)$$

Note that condition (33) implies that

$$f \in C(0, T; V). \quad (38)$$

We define the bilinear form $a_i : V \times V \rightarrow \mathbb{R}$, ($i = 0, 1$).

$$a_0(\zeta, \xi) = k_0 \int_{\Omega} \nabla \zeta \cdot \nabla \xi dx + \mathbf{B} \int_{\Gamma} \zeta \xi d\gamma. \quad (39)$$

$$a_1(\zeta, \xi) = k_1 \int_{\Omega} \nabla \zeta \cdot \nabla \xi dx. \quad (40)$$

Now let $j : V \times V \rightarrow \mathbb{R}$, the mapping defined by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} S\mu(|\mathbf{u}_\tau|) |\mathbf{v}_\tau| da. \quad (41)$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (16)-(26).

Problem PV.

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) \\ &\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)), \theta(s), \alpha(s)) ds. \end{aligned} \quad (42)$$

$$\begin{aligned} (\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geq (f, (\mathbf{v} - \dot{\mathbf{u}}(t)))_V, \quad \forall \mathbf{v} \in V, t \in [0, T]. \end{aligned} \quad (43)$$

$$\begin{aligned} (\dot{\theta}(t), \mathbf{v})_{L^2(\Omega)} + a_0(\theta(t), \mathbf{v}) = (q(t), w)_{L^2(\Omega)} + \\ (\Psi(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}})(t)), \varepsilon(\mathbf{u}(t)), \theta(t), \alpha(t)), \mathbf{v})_{L^2(\Omega)}. \end{aligned} \quad (44)$$

$$\begin{aligned} \alpha(t) \in K, \quad (\dot{\alpha}(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t)) \\ \geq (\Theta(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}})(t)), \varepsilon(\mathbf{u}(t)), \theta(t), \alpha(t)), \zeta - \alpha(t))_{L^2(\Omega)}, \end{aligned} \quad (45)$$

$$\forall \zeta \in K, t \in [0, T],$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0. \quad (46)$$

Our main existence and uniqueness result for Problem PV is the following.

4. Existence and Uniqueness

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that (27)-(41) hold, Then there exists a unique solution $(\mathbf{u}, \boldsymbol{\sigma}, \theta, \alpha)$ to problem PV. Moreover, the solution has the regularity*

$$C^1(0, T; V). \quad (47)$$

$$\boldsymbol{\sigma} \in C(0, T; \mathcal{H}). \quad (48)$$

$$\theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega)). \quad (49)$$

$$\alpha \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \quad (50)$$

Quartet $(\mathbf{u}, \boldsymbol{\sigma}, \theta, \alpha)$ which satisfy (42)-(46) are called a weak solution of the contact problem P . We conclude that, under the assumptions (27)-(41), the mechanical problem (16)-(26) has a unique weak solution satisfying (47)-(50).

The proof of Theorem 4.1, is carried out is several steps and is based on the following abstract result for evolutionary quasi-variational inequality.

We denote by C a constant whose value may change from line to line when no confusing can arise.

Let $\eta \in C(0, T; \mathcal{H})$ be given and consider the following variational problems.

Problem \mathcal{P}_η . Find a displacement field $\mathbf{u}_\eta : [0, T] \rightarrow V$, such that

$$\begin{aligned} (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_\eta(t)), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{\mathcal{H}} \\ + j(\mathbf{u}_\eta(t), \mathbf{v}) - j(\mathbf{u}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \geq (f, \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V \\ \text{a.e. } t \in (0, T), \quad \text{for all } \mathbf{v} \in V, \end{aligned} \quad (51)$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \quad (52)$$

We have the following result for \mathcal{P}_η

Lemma 4.2. *There is a unique solution $\mathbf{u}_\eta \in C^1([0, T]; V)$ to the problem (51)-(52). Moreover, if \mathbf{u}_{η_1} , and \mathbf{u}_{η_2} Are two solutions of the problem (51)-(52) corresponding to the data $\eta_1, \eta_2 \in C([0, T]; \mathcal{H})$, then there exists $c > 0$ such that*

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq c(\|\eta_1(t) - \eta_2(t)\|_{\mathcal{H}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V), \quad \forall t \in [0, T] \quad (53)$$

PROOF. We will apply Theorem 2.1 in the Hilbert space V endowed with the scalar product $(\cdot, \cdot)_V$ and the associated norm $\|\cdot\|_V$ defined by (4). We use the Riesz-Fréchet representation theorem to define the operators $A : V \rightarrow V$, $B : V \rightarrow V$ and the function $\mathbf{f} : [0, T] \rightarrow V$ by equalities

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad (54)$$

$$(B\mathbf{u}, \mathbf{v})_V = (\mathcal{B}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad (55)$$

$$(\mathbf{f}(t), \mathbf{v})_V = (f(t), \mathbf{v})_V - (\boldsymbol{\eta}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}}. \quad (56)$$

For all $\mathbf{u}, \mathbf{v} \in V$ and $t \in [0, T]$. It follows from the assumptions (27)-(28), we have

$$\begin{aligned} (A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V &= (\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))_{\mathcal{H}} \\ &\geq m_{\mathcal{A}} \|\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\|_{\mathcal{H}}^2 \geq C \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2. \end{aligned}$$

And according to Korn's inequality, it comes

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq C \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2.$$

Also for all $\mathbf{u}_1, \mathbf{u}_2 \in V$ and $\mathbf{v} \in V$ we have

$$\begin{aligned} \|(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{v})_V\| &= \|(\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{v}))\|_{\mathcal{H}}, \\ &\leq \|\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2))\|_{\mathcal{H}} \|\varepsilon(\mathbf{v})\|_{\mathcal{H}}, \\ &\leq L_{\mathcal{A}} \|\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\|_{\mathcal{H}} \|\varepsilon(\mathbf{v})\|_{\mathcal{H}}, \\ &\leq L_{\mathcal{A}} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}\|_V. \end{aligned}$$

So

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_V \leq C \|\mathbf{u}_1 - \mathbf{u}_2\|_V.$$

Similarly, we find

$$\|B\mathbf{u}_1 - B\mathbf{u}_2\|_V \leq C \|\mathbf{u}_1 - \mathbf{u}_2\|_V.$$

Then the operators A and B satisfy the conditions (6), (8), respectively. We use (5) to see that the functional j defined in (41) satisfies the condition (7) (a). By means of (32) and (5) once again we obtain

$$\begin{aligned} j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ \leq c \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V. \end{aligned} \quad (57)$$

For all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1$ and $\mathbf{v}_2 \in V$, which shows that the functional j satisfies the condition (7) (b) on V . Moreover, using (33) it is easy to see that f defined by (37) satisfies $f \in C(0, T; V)$ and, taking into consideration that $\eta \in C([0, T]; \mathcal{H})$, we conclude from (56) that $\mathbf{f} \in C(0, T; V)$. Finally, note that (41) shows that the condition (10) is also satisfied. Now using (54)-(56) we notice that Lemma 4.2.

Is a direct consequence of Theorem 2.1 .(1), (2), which completes the proof. \square

For $\phi \in C(0, T; L^2(\Omega))$, we consider the following variational problem.

Problem \mathcal{P}_ϕ . Find the temperature $\theta_\phi : [0, T] \rightarrow L^2(\Omega)$ which is solution of the variational problem

$$\left(\dot{\theta}_\phi(t), \mathbf{v} \right)_{L^2(\Omega)} + a_0(\theta_\phi(t), \mathbf{v}) = (\phi(t) + q(t), \mathbf{v})_{L^2(\Omega)} \quad \forall \mathbf{v} \in L^2(\Omega), \text{ a.e. } t \in (0, T), \quad (58)$$

$$\theta_\phi(0) = \theta_0, \quad \text{in } \Omega. \quad (59)$$

Lemma 4.3. For all $\phi \in C(0, T; L^2(\Omega))$, there exists a unique solution θ_ϕ to the auxiliary problem \mathcal{P}_ϕ satisfying (49).

PROOF. See [11]. □

For $\lambda \in L^2(0, T; L^2(\Omega))$, we consider the following variational problem.

Problem \mathcal{P}_λ . Find the damage field $\alpha_\lambda : [0, T] \rightarrow H^1(\Omega)$,

$$\begin{aligned} \alpha_\lambda(t) \in K, \quad (\dot{\alpha}_\lambda(t), \xi - \alpha_\lambda(t))_{L^2(\Omega)} + a(\alpha_\lambda(t), \xi - \alpha_\lambda(t)) \\ \geq (\lambda(t), \xi - \alpha_\lambda(t))_{L^2(\Omega)}, \forall \xi \in K, \text{ a.e. } t \in (0, T), \end{aligned} \quad (60)$$

$$\alpha_\lambda(0) = \alpha_0. \quad (61)$$

Lemma 4.4. *There exists a unique solution α_λ to the auxiliary problem \mathcal{P}_λ satisfying (50).*

PROOF. The inclusion mapping of $(H^1(\Omega), \|\cdot\|_{H^1(\Omega)})$ into $(L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(\cdot, \cdot)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $(H^1(\Omega))$. We have

$$(\alpha, \varsigma)_{(H^1(\Omega))' \times H^1(\Omega)} = (\alpha, \varsigma)_{L^2(\Omega)}, \quad \forall \alpha \in L^2(\Omega), \varsigma \in H^1(\Omega),$$

and we note that K is a closed convex set in $(H^1(\Omega))$, using the definition (40) of the bilinear form a , for all $\vartheta, \varsigma \in H^1(\Omega)$, we have $a(\vartheta, \varsigma) = a(\varsigma, \vartheta)$ and

$$|a(\vartheta, \varsigma)| \leq k_0 \|\nabla \vartheta\|_H \|\nabla \varsigma\|_H \leq c \|\vartheta\|_{H^1(\Omega)} \|\varsigma\|_{H^1(\Omega)}.$$

Therefore, a is continuous and symmetric. Thus, for all $\vartheta \in H^1(\Omega)$, we have

$$a(\vartheta, \vartheta) = k_0 \|\nabla \vartheta\|_H^2,$$

so

$$a(\vartheta, \vartheta) + (k_0 + 1) \|\vartheta\|_{L^2(\Omega)}^2 \geq k_0 \left(\|\nabla \vartheta\|_H^2 + \|\vartheta\|_{L^2(\Omega)}^2 \right),$$

which implies

$$a(\vartheta, \vartheta) + c_0 \|\vartheta\|_{L^2(\Omega)}^2 \geq c_1 \|\vartheta\|_{H^1(\Omega)}^2 \text{ with } c_0 = k_0 + 1 \text{ and } c_1 = k_0.$$

Finally, we use that condition $\alpha_0 \in K$, and we use a standard result for parabolic variational inequalities (see [3], p. 124), we find that there exists a unique function $\alpha \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, such that $\alpha(0) = \alpha_0$, $\alpha(t) \in K$ for all $t \in [0, T]$ and for almost all $t \in (0, T)$

$$(\dot{\alpha}_\lambda(t), \varsigma - \alpha_\lambda)_{(H^1(\Omega))' \times H^1(\Omega)} + a(\alpha_\lambda(t), \varsigma - \alpha_\lambda(t)) \geq (\lambda(t), \varsigma - \alpha_\lambda(t))_{L^2(\Omega)},$$

for all $\varsigma \in K$. □

Problem $\mathcal{P}_{\eta,\phi,\lambda}$. Find the stress field $\sigma_{\eta,\phi,\lambda} : [0; T] \rightarrow \mathcal{H}$ which is a solution of the problem

$$\begin{aligned} \sigma_{\eta,\phi,\lambda}(t) &= \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t))) \\ &+ \int_0^t \mathcal{G}(\sigma_{\eta,\phi,\lambda}(s), \varepsilon(\mathbf{u}_\eta(s)), \theta_\phi(s), \alpha_\lambda(s)) ds, \forall t \in [0, T]. \end{aligned} \quad (62)$$

Lemma 4.5. *There exists a unique solution of Problem $\mathcal{P}_{\eta,\phi,\lambda}$, and it satisfies (48). Moreover, if $\mathbf{u}_{\eta_i}, \theta_{\phi_i}, \alpha_{\lambda_i}$ and $\sigma_{\eta_i, \phi_i, \lambda_i}$ represent the solutions of problems $\mathcal{P}_{\eta_i}, \mathcal{P}_{\phi_i}, \mathcal{P}_{\lambda_i}$ and $\mathcal{P}_{\eta_i, \phi_i, \lambda_i}$, respectively, for $i = 1, 2$, then there exists $C > 0$ such that*

$$\begin{aligned} \|\sigma_{\eta_1, \phi_1, \lambda_1}(t) - \sigma_{\eta_2, \phi_2, \lambda_2}(t)\|_{\mathcal{H}}^2 &\leq C (\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V^2 \\ &+ \int_0^t (\|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_V^2 + \|\theta_{\phi_1}(s) - \theta_{\phi_2}(s)\|_{L^2(\Omega)}^2 + \|\alpha_{\lambda_1}(s) - \alpha_{\lambda_2}(s)\|_V^2) ds). \end{aligned} \quad (63)$$

PROOF. Let the mapping $\sum_{\eta,\phi,\lambda} : C(0, T; \mathcal{H}) \rightarrow C(0, T; \mathcal{H})$, given by

$$\begin{aligned} \sum_{\eta,\phi,\lambda} \sigma_{\eta,\phi,\lambda}(t) &= \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t))) \\ &+ \int_0^t \mathcal{G}(\sigma_{\eta,\phi,\lambda}(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(s))), \varepsilon(\mathbf{u}_\eta(s)), \theta_\phi(s), \alpha_\lambda(s)) ds. \end{aligned} \quad (64)$$

Let $\sigma_{\eta_i, \phi_i, \lambda_i} = \sigma_i \in W^{1,p}(0, T; \mathcal{H})$, $i = 1, 2$ and $t_1 \in (0, T)$, we use the assumption (29) and the Hölder inequality, we find

$$\left\| \sum_{\eta,\phi,\lambda} \sigma_1(t_1) - \sum_{\eta,\phi,\lambda} \sigma_2(t_1) \right\|_{\mathcal{H}}^2 \leq L_G^2 T \int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds. \quad (65)$$

We have more

$$\begin{aligned} \left\| \sum_{\eta,\phi,\lambda} \left(\sum_{\eta,\phi,\lambda} \sigma_1(t_1) \right) - \sum_{\eta,\phi,\lambda} \left(\sum_{\eta,\phi,\lambda} \sigma_2(t_1) \right) \right\|_{\mathcal{H}}^2 \\ \leq L_G^2 T \int_0^{t_1} \left\| \sum_{\eta,\phi,\lambda} \sigma_1(t_1) - \sum_{\eta,\phi,\lambda} \sigma_2(t_1) \right\|_{\mathcal{H}}^2 dt_1 \\ \leq L_G^4 T^2 \int_0^{t_1} \int_0^{t_2} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds dt_2. \end{aligned}$$

We generalize the inequality above by recurrence, we obtain for all $t_1, t_2, \dots, t_n \in [0, T]$.

$$\left\| \sum_{\eta,\phi,\lambda}^{(n)} \sigma_1(t_n) - \sum_{\eta,\phi,\lambda}^{(n)} \sigma_2(t_n) \right\|_{\mathcal{H}}^2 \leq L_G^{2n} T^n \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_n} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds dt_n \dots dt_2.$$

Thus, we can deduce by integrating with respect to $[0, T]$ the following inequality

$$\left\| \sum_{\eta, \phi, \lambda}^{(n)} \sigma_1 - \sum_{\eta, \phi, \lambda}^{(n)} \sigma_2 \right\|_{\mathcal{H}}^2 \leq \frac{L_{\mathcal{G}}^{2n} T^{2n}}{n!} \|\sigma_1 - \sigma_2\|_{\mathcal{H}}^2. \quad (66)$$

Then from (66), for n sufficiently large, the operator $\sum_{\eta, \phi, \lambda}^{(n)}$, is a contraction on space $C(0, T; \mathcal{H})$ and according to the Banach fixed point theorem, there is a single element $\sigma_{\eta, \phi, \lambda} \in C(0, T; \mathcal{H})$ such that $\sum_{\eta, \phi, \lambda}^{(n)} \sigma_{\eta, \phi, \lambda} = \sigma_{\eta, \phi, \lambda}$, which represents the unique solution of problem $\mathcal{P}_{\eta, \phi, \lambda}$. Moreover, if \mathbf{u}_{η_i} , θ_{ϕ_i} , α_{λ_i} and $\sigma_{\eta_i, \phi_i, \lambda_i}$, represents the solutions of problem \mathcal{P}_{η_i} , \mathcal{P}_{ϕ_i} , \mathcal{P}_{λ_i} and $\mathcal{P}_{\eta_i, \phi_i, \lambda_i}$ respectively.

For $i = 1, 2$. designate $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\sigma_{\eta_i, \phi_i, \lambda_i} = \sigma_i$, $\theta_{\phi_i} = \theta_i$, $\alpha_{\lambda_i} = \alpha_i$. We have

$$\sigma_i(t) = \mathcal{B}(\varepsilon(\mathbf{u}_i(t))) + \int_0^t \mathcal{G}(\sigma_i(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_i(s))), \varepsilon(\mathbf{u}_i(s)), \theta_i(s), \alpha_i(s)) ds, \text{ a.e. } t \in (0, T),$$

we use the assumption (28)-(29), we find

$$\begin{aligned} \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}}^2 &\leq C \left(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds \right. \\ &\quad + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds \\ &\quad \left. + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2 ds \right). \end{aligned}$$

We use the Gronwall argument in the resulting inequality to deduce (63). \square

We now consider the operator

$$\Lambda : C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega)) \rightarrow C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega)),$$

defined by

$$\Lambda(\eta, \phi, \lambda)(t) = (\Lambda^1(\eta, \phi, \lambda)(t), \Lambda^2(\eta, \phi, \lambda)(t), \Lambda^3(\eta, \phi, \lambda)(t)), \quad (67)$$

and

$$\Lambda^1(\eta, \phi, \lambda)(t) = \int_0^t \mathcal{G}(\sigma_{\eta, \phi, \lambda}, \varepsilon(\mathbf{u}_{\eta}(s)), \theta_{\phi}(s), \alpha_{\lambda}(s)) ds. \quad (68)$$

$$\Lambda^2(\eta, \phi, \lambda)(t) = \Psi(\sigma_{\eta, \phi, \lambda}, \varepsilon(\mathbf{u}_{\eta}(s)), \theta_{\phi}(s), \alpha_{\lambda}(s)). \quad (69)$$

$$\Lambda^3(\eta, \phi, \lambda)(t) = \Theta(\sigma_{\eta, \phi, \lambda}, \varepsilon(\mathbf{u}_{\eta}(s)), \theta_{\phi}(s), \alpha_{\lambda}(s)). \quad (70)$$

We have the following result.

Lemma 4.6. *The mapping Λ has a fixed point $(\eta^*, \phi^*, \lambda^*) \in C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$,*

such that $\Lambda(\eta^, \phi^*, \lambda^*) = (\eta^*, \phi^*, \lambda^*)$.*

PROOF. Let $(\eta_1, \phi_1, \lambda_1), (\eta_2, \phi_2, \lambda_2) \in C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i, \theta_{\phi_i} = \theta_i$ and $\alpha_{\lambda_i} = \alpha_i$, for $i = 1, 2$. using (29)-(31), we have

$$\begin{aligned}
& \|\Lambda(\eta_1, \chi_1, \phi_1)(t) - \Lambda(\eta_1, \chi_1, \phi_1)(t)\|_{\mathcal{H} \times L^2(\Omega) \times L^2(\Omega)} \\
& \leq L_{\mathcal{G}} \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \\
& \quad + \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}) ds \\
& \quad + L_{\Psi} (\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \\
& \quad + \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}) \\
& \quad + L_{\Theta} (\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \\
& \quad + \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}) ,
\end{aligned} \tag{71}$$

so we obtain

$$\begin{aligned}
& \|\Lambda(\eta_1, \phi_1, \lambda_1)(t) - \Lambda(\eta_1, \phi_1, \lambda_1)(t)\|_{\mathcal{H} \times L^2(\Omega) \times L^2(\Omega)} \\
& \leq C (\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\| + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \\
& \quad + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)} + \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}) \\
& \quad + \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \\
& \quad + \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}) ds .
\end{aligned} \tag{72}$$

This implies that

$$\begin{aligned}
& \|\Lambda(\eta_1, \phi_1, \lambda_1)(t) - \Lambda(\eta_1, \phi_1, \lambda_1)(t)\|_{\mathcal{H} \times L^2(\Omega) \times L^2(\Omega)} \\
& \leq C \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \\
& \quad + \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}) ds .
\end{aligned} \tag{73}$$

We use estimate (63) to obtain

$$\begin{aligned}
& \|\Lambda(\eta_1, \phi_1, \lambda_1)(t) - \Lambda(\eta_1, \phi_1, \lambda_1)(t)\|_{\mathcal{H} \times L^2(\Omega) \times L^2(\Omega)} \leq C \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \\
& \quad + \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} + \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}) ds .
\end{aligned} \tag{74}$$

On the other hand, since $\mathbf{u}_i(t) = \int_0^t \dot{\mathbf{u}}_i(s) ds + \mathbf{u}_0$, we know that for a.e. $t \in (0, T)$,

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 ds, \tag{75}$$

and using this inequality in (53) yields

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq C \left(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{\mathcal{H}} + \int_0^t |\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)|_V ds \right). \quad (76)$$

Next, we apply Gronwall's inequality to deduce

$$\int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}} ds, \quad \text{for all } t \in [0; T], \quad (77)$$

which also implies, using a variant of (75), that

$$\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 \leq C \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}}^2 ds, \quad \text{a.e. } t \in [0, T]. \quad (78)$$

For the temperature, if we take the substitution $\phi = \phi_1, \phi = \phi_2$ in (58) and subtracting the two obtained equations, we deduce by choosing $v = \theta_1 - \theta_2$ as test function and $t \in [0, T]$, such that

$$\begin{aligned} \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 + C_1 \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds \\ \leq \int_0^t \|\phi_1(s) - \phi_2(s)\|_{L^2(\Omega)} \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} ds. \end{aligned}$$

We find

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds \leq C \int_0^t \|\phi_1(s) - \phi_2(s)\|_{L^2(\Omega)}^2 ds.$$

From this inequality, combined with Gronwall's inequality, we deduce that

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leq C \int_0^t \|\phi_1(s) - \phi_2(s)\|_{L^2(\Omega)}^2 ds, \quad (79)$$

Form (60), deduced that

$$\begin{aligned} (\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a_1 (\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \\ \leq (\lambda_1 - \lambda_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \quad \text{a.e. } t \in (0, T). \end{aligned}$$

integrate inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, and inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$, we find

$$\frac{1}{2} \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \leq C \int_0^t (\lambda_1(s) - \lambda_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} ds,$$

which implies

$$\begin{aligned} \|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \\ \leq C \left(\int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2 ds \right). \end{aligned}$$

This inequality combined with the Gronwall inequality leads to

$$\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \leq C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{L^2(\Omega)}^2 ds, \forall t \in [0, T]. \quad (80)$$

Using (74), (78), (79) and (80), we find

$$\begin{aligned} & \|\Lambda(\eta_1, \phi_1, \lambda_1) - \Lambda(\eta_2, \phi_2, \lambda_2)\|_{\mathcal{H} \times L^2(\Omega) \times L^2(\Omega)} \\ & \leq C \|(\eta_1 - \eta_2, \phi_1 - \phi_2, \lambda_1 - \lambda_2)\|_{\mathcal{H} \times L^2(\Omega) \times L^2(\Omega)}. \end{aligned} \quad (81)$$

Reiterant cette inegalite m lead time a

$$\begin{aligned} & \|\Lambda^m(\eta_1, \phi_1, \lambda_1) - \Lambda^m(\eta_2, \phi_2, \lambda_2)\|_{C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))}^2 \\ & \leq \frac{C^m T^m}{m!} \|(\eta_1 - \eta_2, \phi_1 - \phi_2, \lambda_1 - \lambda_2)\|_{C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))}^2. \end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on $C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$. Hence, Banach's fixed point theorem shows that Λ admits a unique fixed point $(\eta^*, \phi^*, \lambda^*) \in C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$. \square

Now we have every thing that is required to prove theorem 4.1.

Existence. Let $(\eta^*, \phi^*, \lambda^*) \in C(0, T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$ be the fixed point of Λ defined by (68)-(70) and denote

$$\mathbf{u} = \mathbf{u}_{\eta^*}, \quad \boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \boldsymbol{\sigma}_{\eta^*, \phi^*, \lambda^*}. \quad (82)$$

$$\theta = \theta_{\phi^*}, \quad \alpha = \alpha_{\lambda^*}. \quad (83)$$

We prove that $(\mathbf{u}, \boldsymbol{\sigma}, \theta, \alpha)$ satisfies (42)-(46) and (47)-(50). Indeed, we write (60) for $\eta^* = \eta$, $\phi^* = \phi$, $\lambda^* = \lambda$ and use (82)-(83) to obtain that (42) is satisfied. Now we consider (51) for $\eta^* = \eta$ and use the first equality in (82) to find

$$\begin{aligned} & (\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}(t)), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + (\boldsymbol{\eta}^*(t), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \\ & + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}, \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V}, \\ & \text{a.e. } t \in (0, T), \quad \text{for all } \mathbf{v} \in V. \end{aligned} \quad (84)$$

The equalities $\Lambda^1(\eta^*, \phi^*, \lambda^*) = \eta^*$, $\Lambda^2(\eta^*, \phi^*, \lambda^*) = \phi^*$ and $\Lambda^3(\eta^*, \phi^*, \lambda^*) = \lambda^*$ combined with (68)-(70) and (82)-(83) show that for all $\mathbf{v} \in V$,

$$(\boldsymbol{\eta}^*(t), \mathbf{v})_{\mathcal{H} \times V} = \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\mathbf{u}(s)), \theta(s), \alpha(s)) ds, \varepsilon(\mathbf{v}) \right)_{\mathcal{H}}. \quad (85)$$

$$\phi^*(t) = \Psi(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \theta(t), \alpha(t)). \quad (86)$$

$$\lambda^*(t) = \Theta(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \theta(t), \alpha(t)). \quad (87)$$

We now substitute (85) in (84) and use (42) to see that (43) is satisfied. We write (58) for $\phi = \phi^*$ and use (83) and (86) to find that (44) is satisfied.

Also, we write (60) for $\lambda = \lambda^*$ and using (83) and (87) to find that (45) is satisfied. Next (46), and regularities (47), (48), (49) and (50) follow Lemmas 4.2, 4.3 and 4.4. The regularity $\sigma \in C(0, T; \mathcal{H})$ follows from Lemma 4.5.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ .

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