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Common fixed point results for minimal Ξ-commuting mappings in modular metric spaces

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Abstract. In this paper, we define the notions of Ξ-conditionally commuting, Ξ-conditionally compatible, Ξ-faintly compatible mappings and Ξ-reciprocal continuous mappings in setting of modular metric spaces and then prove common fixed point theorems for these mappings. In fact our results are generalization of the results of [**2**, **13**, **18**].

1. Introduction

The notion of modular metric space was introduced by Chistyakov with the time parameter λ (say) for the purpose to define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces in [**5**, **6**, **7**].This is a generalization of the classical modular spaces like Orlicz spaces (see [**11**]).

Throughout this paper $\mathbb{N}$ will denote the set of natural numbers. Let ψ be a non empty set. Throughout this paper, for a function $\Xi : (0, \infty) \times \Phi \times \Phi \to [0, \infty)$, we write $\Xi_\lambda(u, v) = \Xi(\lambda, u, v)$ for all $\lambda > 0$ and $u, v \in \Phi$.

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Definition 1.1. [3] Let Φ be a non empty set. A function $\Xi : (0, \infty) \times \Phi \times \Phi \to [0, \infty)$ is said to be a metric modular on Φ if it satisfies, for all $u, v, w \in \Phi$, the following condition:

(1) $\Xi_\lambda(u, v) = 0$ for all $\lambda > 0$ if and only if $u = v$,
(2) $\Xi_\lambda(u, v) = \Xi_\lambda(v, u)$ for all $\lambda > 0$,
(3) $\Xi_{\lambda+\mu}(u, v) \leq \Xi_\lambda(u, w) + \Xi_\mu(w, v)$ for all $\lambda, \mu > 0$.

If instead of (1) we have only the condition (1') $\Xi_\lambda(u, u) = 0$ for all $u \in \Phi, \lambda > 0$ then Ξ is said to be a pseudo modular (metric) on Φ.

An important property of the (metric) pseudo modular on set Φ is that the mapping $\lambda \mapsto \Xi_\lambda(u, v)$ is non increasing for all $u, v \in \Phi$.

Definition 1.2. [3] Let Ξ is a pseudo modular on Φ. Fixed $u_0 \in \Phi$. The set $\Phi_\Xi = \Phi_\Xi(u_0) = \{u \in \Phi : \Xi_\lambda(u, u_0) \to 0 \text{ as } \lambda \to \infty\}$ is said to be a modular metric space (around u_0).

Definition 1.3. [11] Let Φ_Ξ be a modular metric space.

(1) The sequence $\{u_\eta\}$ in Φ_Ξ is said to be Ξ-convergent to $u \in \Phi_\Xi$ if and only if there exists a number $\lambda > 0$, possibly depending on $\{u_\eta\}$ and u, such that $\lim_{n\to\infty} \Xi_\lambda(u_\eta, u) = 0$.
(2) The sequence $\{u_\eta\}$ in Φ_Ξ is said to be Ξ-Cauchy if there exists $\lambda > 0$, possibly depending on the sequence, such that $\Xi_\lambda(u_m, u_\eta) \to 0$ as $m, \eta \to \infty$
(3) A subset H of Φ_Ξ is said to be Ξ-complete if any Ξ-Cauchy sequence in H is a convergent sequence and its limit is in H.

Definition 1.4. [4] Let Ξ be a metric modular on Φ and Φ_Ξ be a modular metric space induced by Ξ. If Φ_Ξ is a Ξ-complete modular metric space and $\mathcal{T} : \Phi_\Xi \to \Phi_\Xi$ be an arbitrary mapping $\mathcal{T}$ is called a contraction if for each $u, v \in \Phi_\Xi$ and for all $\lambda > 0$ there exists $0 \leq \sigma < 1$ such that

$$\Xi_\lambda(\mathcal{T}u, \mathcal{T}v) \leq \sigma \Xi_\lambda(u, v).$$

2. Preliminaries

In 1976, Jungck [7] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. After that Sessa [18] introduced the notion of weakly commuting maps. Further, Jungck [8] introduced more generalized commutativity, the so-called compatibility, which is more general than that of weak commutativity. After then in 1998 , Jungck and Rhoades [9] introduced the notion of weakly compatible. In 2008, Al-Thagafi and Shahzad[1] weakened the notion of non-trivial weakly compatible maps by introducing a new notion of occasionally weakly compatible maps. In 2010 , Pant and Pant [12] redefined the notion of occasionally weakly compatible by conditional commutativity. In 2012, Pant and

Pant [17] defined new commutativity notion which is proper generalization of non-trivial compatibility as well as owc called conditional compatible. In 2013, Bisht, Shahzad[2] introduce the generalization of compatible and conditional compatible mappings named as faintly compatible mappings. The study of non-compatible maps is equally interesting and various fruitful results have been obtained using the aspect of non-compatibility (e.g. [13, 14, 15]). In 1999, Pant [16] introduced the concept of reciprocally continuous mappings.

Now we defined minimal Ξ-commuting mappings in setting of modular metric spaces as follows:

Definition 2.1. Let (Φ_λ, Ξ) be Modular metric spaces. Two self-mapping M and g defined on Φ_λ are called

(D1) Ξ-commuting mappings if $Mgu = gMu$ for all $u \in \Phi_\lambda$.

(D2) Ξ-weakly commuting if $\Xi_\lambda(Mgu, gMu) \leq \Xi_\lambda(Mu, gu)$ for each u in Φ_λ.

(D3) Ξ-compatible if $\lim_{n\to\infty} \Xi_\lambda (Mgu_n, gMu_n) = 0$, whenever $\{u_n\}$ is a sequence in Φ_λ such that $\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} gu_n = t$ for some $t \in \Phi_\lambda$ and $\lambda > 0$.

(D4) Ξ-weakly compatible if the pair commutes on the set of coincidence points i.e. if $Mu = gu$ implies $Mgu = gMu$.

(D5) Ξ-occasionally weakly compatible if there exists a coincidence point u in Φ_λ such that $Mu = gu$ implies $Mgu = gMu$;

(D6) Ξ-conditionally commuting if the pair commutes on a nonempty subset of the set of coincidence points whenever the set of coincidences is nonempty;

(D7) Ξ-conditionally compatible if and only if whenever the set of sequences $\{y_n\}$ satisfying $\lim_{n\to\infty} My_n = \lim_{n\to\infty} gy_n$ is nonempty, there exists a sequences $\{z_n\}$ such that $\lim_{n\to\infty} Mz_n = \lim_{n\to\infty} gz_n = t$ and $\lim_{n\to\infty} \Xi_\lambda (Mgz_n, gMz_n) = 0$;

(D8) Ξ-faintly compatible iff M and g are conditionally compatible and M and g commute on a non-empty subset of coincidence points whenever the set of coincidences is nonempty;

(D9) Ξ-non-compatible if (M, g) is not compatible, i.e., if there exists a sequence $\{x_n\}$ in Φ_λ such that $\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \Phi_\lambda$, and $\lim_{n\to\infty} \Xi_\lambda (Mgx_n, gMx_n) \neq 0$ or non-existent;

(D10) Ξ-reciprocally continuous if $\lim_{n\to\infty} \Xi_\lambda (Mgx_n, Mt) = 0$ and $\lim_{n\to\infty} \Xi_\lambda (gMx_n, gt) = 0$, whenever $\{x_n\}$ is a sequence in Φ_λ such that $\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \Phi_\lambda$.

Now we discuss the relations between various minimal commuting mappings in Modular metric spaces. Every Ξ-commuting mapping is Ξ-conditionally commuting mapping but converse need not be true in general.

Example 2.2. Let $\Phi_\lambda = [0, 4]$ be equipped with the modular metric space $\Xi_\lambda(u, v) = \frac{|u-v|}{\lambda}$. Define self -mappings M, g as $Mu = 2u - 1, gu = u^2$. Here M and g have coincidence point $u = 1$. Further, M and g are conditionally commuting since they commute at coincidence point $u = 1$ but M and g are not commuting . Ξ-weakly compatible mapping is Ξ-owc but converse is not true in general.

Example 2.3. Let $\Phi_\lambda = [0, \infty)$ with the modular metric space $\Xi_\lambda(u, v) = \frac{|u-v|}{\lambda}$. Define self -mappings M, g as $Mu = 2u, gu = u^2$. Here M and g have two coincidence point $u = 0, 2$. Further, M and g are owc since they commute at coincidence point $u = 0$ but M and g are not weakly compatible as they are not commute at $u = 2$. Every Ξ-owc mapping is Ξ-conditionally compatible but converse need not be true in general.

Example 2.4. Let $\Phi_\lambda = [0, \infty)$ with the modular metric space $\Xi_\lambda(u, v) = \frac{|u-v|}{\lambda}$. Define self -mappings M, g as $Mu = u^2$ for all $u \in \Phi_\lambda$, and

$$gu = \begin{cases} u + 2 \text{ if } & u \in [0, 4] \cup (9, \infty) \\ u + 12 & \text{if } \quad u \in (4, 9]. \end{cases}$$

Let $y_n = \left\{2 + \frac{1}{n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 4, \lim_{n\to\infty} gy_n = 4$ $\lim_{n\to\infty} Mgy_n = 16, \lim_{n\to\infty} gMy_n = 16, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = 0$. So, M and g are Ξ-conditionally compatible and $\{2\}$ is the only point of coincidence at which M and g do not commute so M and g are not Ξ-owc.

Every Ξ-compatible mapping is Ξ-weakly compatible but converse need not be true in general.

Example 2.5. Let $\Phi_\lambda = [2, 8]$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as space $\Xi_\lambda(u,\ v) = \frac{|u-v|}{\lambda}$. Define self-mappings M, g as

$$Mu = \begin{cases} 2 \text{ if } u = 2 \text{ or } u > 6 \\ \frac{4u}{3} \text{ if } 2 < u \leq 6, \end{cases} \qquad gu = \begin{cases} 2 & \text{if } u = 2 \\ 8 & \text{if } 2 < u \leq 6 \\ \frac{u}{3} & \text{if } \quad u > 6 \end{cases}$$

Let $y_n = \left\{6 + \frac{1}{n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 2, \lim_{n\to\infty} gy_n = 2$, $\lim_{n\to\infty} Mgy_n = \frac{8}{3}, \lim_{n\to\infty} gMy_n = 2, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = \frac{2}{3\lambda} \neq 0$. So, M and g are not Ξ-compatible. Now $M2 = 2 = g2$ implies $Mg2 = 2 = gM2$. Hence, M and g commute at coincidence point 2, so M and g are Ξ-weakly compatible. Ξ-Compatible mapping and Ξ-conditionally compatible mapping are independent each other.

Example 2.6. Let $\Phi_\lambda = [1, 6]$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as $\Xi_\lambda(u, v) = \frac{|u-v|}{\lambda}$. Define self -mappings M, g as

$$Mu = \begin{cases} 3 \text{ if } u \leq 3 \\ 6 \text{ if } u > 3, \end{cases} \quad gu = \begin{cases} 6 - u \text{ if } u \leq 3 \\ 1 \quad \text{ if } u > 3 \end{cases}$$

Let $y_n = \left\{3 - \frac{1}{n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 3, \lim_{n\to\infty} gy_n = 3$, $\lim_{n\to\infty} Mgy_n = 6, \lim_{n\to\infty} gMy_n = 3, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = \frac{3}{\lambda} \neq 0$. So, M and g are not Ξ-compatible.

Now $z_n = 3$ be a constant sequence then $\lim_{n\to\infty} Mz_n = 3, \lim_{n\to\infty} gz_n = 3$, $\lim_{n\to\infty} Mgz_n = 3, \lim_{n\to\infty} gMz_n = 3$, $\lim_{n\to\infty} \Xi_\lambda (Mgz_n, gMz_n) = 0$. So, M and g are Ξ-conditionally compatible .

Example 2.7. Let $\Phi_\lambda = [0, 1)$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as $\Xi_\lambda(u, v) = \frac{|\mathrm{u-v}|}{\lambda}$. Define self -mappings M, g as $Mu = u, \quad gux = \frac{1}{2} - u$. Let $y_n = \left\{\frac{1}{4} - \frac{1}{4^n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = \frac{1}{4}, \lim_{n\to\infty} gy_n = \frac{1}{4}$, $\lim_{n\to\infty} Mgy_n = \frac{1}{4}, \lim_{n\to\infty} gMy_n = \frac{1}{4}, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = 0$. So, M and g are Ξ-compatible but there does not exists any sequence $\{z_n\}$ in Φ_λ such that $\lim_{n\to\infty} Mz_n = \lim_{n\to\infty} Mz_n = t, \lim_{n\to\infty} \Xi_\lambda (Mgz_n, gMz_n) = 0$. So, M and g are not Ξ-conditionally compatible. Every Ξ-compatible mapping is Ξ-faintly compatible but converse need not be true in general.

Example 2.8. Let $\Phi_\lambda = [2, 5]$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as $\Xi_\lambda(u, v) = \frac{|\mathrm{u-v}|}{\lambda}$. Define self -mappings M, g as

$$Mu = \begin{cases} 2 \text{ if } u = 2 \text{ or } u > 6 \\ u + 1 \text{ if } 2 < u \leq 6, \end{cases} \qquad \mathrm{g}\ u = \begin{cases} 2 & \text{if } u = 2 \\ \frac{u+7}{3} & \text{if } 2 < u \leq 4 \\ \frac{u}{2} & \text{if } \quad u > 4. \end{cases}$$

Let $y_n = \left\{4 + \frac{1}{n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 2, \lim_{n\to\infty} gy_n = 2$, $\lim_{n\to\infty} Mgy_n = 3, \lim_{n\to\infty} gMy_n = 2, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = \frac{1}{\lambda} \neq 0$. So, M and g are not Ξ-compatible. Now $z_n = 2$ be a constant sequence then $\lim_{n\to\infty} Mz_n = 2, \lim_{n\to\infty} gz_n = 2, \lim_{n\to\infty} Mgz_n = 2, \lim_{n\to\infty} gMz_n = 2, \lim_{n\to\infty} \Xi_\lambda (Mgz_n, gMz_n) = 0$ and $\{2\}$ is the only point of coincidence in which M and g commute. So, M and g are Ξ-faintly compatible. Ξ-faintly compatible and Ξ-non-compatible are independent of each other.

Example 2.9. Let $\Phi_\lambda = [1, 5]$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as $\Xi_\lambda(u, v) = \frac{|\mathrm{u-v}|}{\lambda}$. Define self -mappings M, g as

$$Mu = \begin{cases} 4 & \text{if } 1 \leq u \leq 3 \\ 1 & \text{if } u > 3, \end{cases} \quad \mathrm{gu} = \begin{cases} 2 & \text{if } 1 \leq u \leq 3 \\ u - 2 & \text{if } u > 3. \end{cases}$$

Let $y_n = \left\{3 + \frac{1}{n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 1, \lim_{n\to\infty} gy_n = 1$, $\lim_{n\to\infty} Mgy_n = 4, \lim_{n\to\infty} gMy_n = 2, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = \frac{2}{\lambda} \neq 0$. So, M and g are Ξ-non-compatible and there is no point of coincidence of M and g. Therefore, M and g are not Ξ-faintly compatible.

Example 2.10. Let $\Phi_\lambda = [0, \infty)$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as $\Xi_\lambda(u, v) = \frac{|u-v|}{\lambda}$. Define self -mappings M, g as $Mu = u$ for all $u \in \Phi_\lambda$,

$$gu = \begin{cases} 0 & \text{if } u \in I^+ \cup \{0\} \\ 1 & \text{if } u \notin I^+ \end{cases}$$

Let $y_n = \left\{1 + \frac{1}{n+1}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 1$, $\lim_{n\to\infty} gy_n = 1$, $\lim_{n\to\infty} Mgy_n = 1, \lim_{n\to\infty} gMy_n = 1, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = 0$. So, M and g

are not Ξ-non-compatible. Now $z_n = 0$ be a constant sequence then $\lim_{n\to\infty} Mz_n = 0, \lim_{n\to\infty} gz_n = 0, \lim_{n\to\infty} Mgz_n = 0, \lim_{n\to\infty} gMz_n = 0, \lim_{n\to\infty} \Xi_\lambda (Mgz_n, gMz_n) = 0$ and $\{0\}$ is the only point of coincidence in which M and g commute. So, M and g are Ξ-faintly compatible. Every Ξ-occasionally weakly compatible is Ξ-faintly compatible but converse need not be true in general.

Example 2.11. Let $\Phi_\lambda = [0, \infty)$ with modular metric $\Xi_\lambda : \Phi_\lambda \times \Phi_\lambda \to [1, \infty)$ be defined as $\Xi_\lambda(u, v) = \frac{|u-v|}{\lambda}$. Define self -mappings M, g as $Mu = u^2$, for all $u \in \Phi_\lambda$, and

$$gu = \begin{cases} u + 6 & \text{if } u \in [0, 9] \cup (11, \infty) \\ u + 72 & \text{if } u \in (9, 11]. \end{cases}$$

Let $y_n = \left\{3 + \frac{1}{n}\right\}$ be a sequence in Φ_λ then $\lim_{n\to\infty} My_n = 9, \lim_{n\to\infty} gy_n = 9$, $\lim_{n\to\infty} Mgy_n = 81, \lim_{n\to\infty} gMy_n = 81, \lim_{n\to\infty} \Xi_\lambda (Mgy_n, gMy_n) = 0$. Now $z_n = 3$ be a constant sequence then $\lim_{n\to\infty} Mz_n = 9, \lim_{n\to\infty} gz_n = 9, \lim_{n\to\infty} Mgz_n = 81, \lim_{n\to\infty} gMz_n = 81$, $\lim_{n\to\infty} \Xi_\lambda (Mgz_n, gMz_n) = 0$. So, M and g are Ξ-faintly compatible and $\{3\}$ is the only point of coincidence at which M and g do not commute so M and g are not Ξ-owc.

We have following relations between minimal Ξ-commuting mappings as given :
(D1) $\Rightarrow$ (D6), reverse implication is not true see Example 2.2.
(D3) $\not\Leftrightarrow$ (D7) see Examples 2.6 and 2.7.
(D3) $\Rightarrow$ (D4), reverse implication is not true see Example 2.5.
(D4) $\Rightarrow$ (D7), reverse implication is not true see Example 2.6.
(D3) $\not\Leftrightarrow$ (D5) see Examples 2.5 and 2.7.
(D5) $\not\Leftrightarrow$ (D9) see Examples 2.5 and 2.7.
(D5) $\Rightarrow$ $(D7)$, reverse implication is not true see Example 2.11.
(D5) $\Rightarrow$ (D6), reverse implication is not true see Example 2.2.
(D4) $\Rightarrow$ (D6), reverse implication is not true see Example 2.2.
(D3) $\Rightarrow$ (D8), reverse implication is not true see Example 2.8.
(D8) $\not\Leftrightarrow$ (D9) see Example 2.9 and 2.10.
(D5) $\Rightarrow$ (D8), reverse implication is not true see Example 2.11.
(D4) $\Rightarrow$ (D5), reverse implication is not true see Example 2.3.
(D4) $\Rightarrow$ (D6), reverse implication is not true see Example 2.2.

3. Main results

Now we prove common fixed point for minimal commuting maps as follows:

Theorem 3.1. *Let (Φ_λ, Ξ) be a complete modular metric space. Let M, B, S and T be self- mappings of Φ_λ into itself satisfying the following conditions:*
(C_1) $T(\Phi_\lambda) \subseteq M(\Phi_\lambda), S(\Phi_\lambda) \subseteq B(\Phi_\lambda)$
(C_2) *the pairs (M, S) and (B, T) are Ξ-conditionally commuting,*

(C_3)

$$\begin{aligned}&[1+p\Xi_1(Mu,Bv)]\,\Xi_1^2(Su,Tv)\\ \leq &p\max\left\{\begin{array}{c}\frac{1}{2}\left[\Xi_1^2(Mu,Su)\Xi_1(Bv,Tv)+\Xi_1(Mu,Su)\Xi_1^2(Bv,Tv)\right],\\ \Xi_1(Mu,Su)\Xi_2(Mu,Tv)\Xi_1(Bv,Su),\\ \Xi_2(Mu,Tv)\Xi_1(Bv,Su)\Xi_1(Bv,Tv)\end{array}\right\}\\ &+m(Mu,Bv)-\emptyset m(Mu,Bv)\end{aligned}$$

where,

$$m(Mu,Bv)=\max\left\{\begin{array}{c}\Xi_1^2(Mu,Bv),\Xi_1(Mu,Su)\Xi_1(Bv,Tv),\Xi_2(Mu,Tv)\Xi_1(Bv,Su),\\ \frac{1}{2}\left[\Xi_1(Mu,Su)\Xi_2(Mu,Tv)+\Xi_1(Bv,Su)\Xi_1(Bv,Tv)\right]\end{array}\right\}$$

$p\geq 0$ is a real number and $\emptyset:[0,\infty)\to[0,\infty)$ is a continuous function with $\emptyset(t)=0\Leftrightarrow t=0$ and $\emptyset(t)>0$ for each $t>0$. Then M,B,S and T have a unique common fixed point in Φ_λ.

PROOF. Let u_0 be an arbitrary point in Φ_λ. Choose a point $u_1\in\Phi_\lambda$ such that $v_0=Su_0=Bu_1$. For the point u_1, we can choose a point $u_2\in\Phi_\lambda$ such that $v_1=Tu_1=Mu_2$ as $T(\Phi_\lambda)\subseteq M(\Phi_\lambda)$. Continuing this process, we obtain a sequence $\{v_n\}$ in Φ_λ such that $v_{2n}=Su_{2n}=Bu_{2n+1}$ and $v_{2n+1}=Tu_{2n+1}=Mu_{2n+2}$. First, we show that $\{v_n\}$ is a Cauchy sequence in Φ_λ. There are two cases:

Case 1. If n is even, then from (C_3), putting $u=u_{2n},v=u_{2n+1}$ in inequality (C_3), we get

$$\begin{aligned}&[1+p\Xi_1(Mu_{2n},Bu_{2n+1})]\,\Xi_1^2(Su_{2n},Tu_{2n+1})\leq\\ &p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\Xi_1^2(Mu_{2n},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1})+\\ \Xi_1(Mu_{2n},Su_{2n})\,\Xi_1^2(Bu_{2n+1},Tu_{2n+1})\end{array}\right],\\ \Xi_1(Mu_{2n},Su_{2n})\,\Xi_2(Mu_{2n},Tu_{2n+1})\,\Xi_1(Bu_{2n+1},Su_{2n}),\\ \Xi_2(Mu_{2n},Tu_{2n+1})\,\Xi_1(Bu_{2n+1},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1})\end{array}\right\}\\ &+m(Mu_{2n},Bu_{2n+1})-\emptyset m(Mu_{2n},Bu_{2n+1})\end{aligned}$$

where,

$$\begin{aligned}&m(Mu_{2n},Bu_{2n+1})\\ &=\max\left\{\begin{array}{c}\Xi_1^2(Mu_{2n},Bu_{2n+1}),\Xi_1(Mu_{2n},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1}),\\ \Xi_2(Mu_{2n},Tu_{2n+1})\,\Xi_1(Bu_{2n+1},Su_{2n}),\\ \frac{1}{2}\left[\Xi_1(Mu_{2n},Su_{2n})\,\Xi_2(Mu_{2n},Tu_{2n+1})+\right.\\ \left.\Xi_1(Bu_{2n+1},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1})\right]\end{array}\right\}\end{aligned}$$

$$
\begin{aligned}
&[1+p\Xi_1(v_{2n-1},v_{2n})]\,\Xi_1^2(v_{2n},v_{2n+1}) \\
&\leq p\max\left\{\begin{array}{c}
\frac{1}{2}\left[\begin{array}{c}\Xi_1^2(v_{2n-1},v_{2n})\,\Xi_1(v_{2n},v_{2n+1})+\\ \Xi_1(v_{2n-1},v_{2n})\,\Xi_1^2(v_{2n},v_{2n+1})\end{array}\right],\\
\Xi_1(v_{2n-1},v_{2n})\,\Xi_2(v_{2n-1},v_{2n+1})\,\Xi_1(v_{2n},v_{2n}),\\
\Xi_2(v_{2n-1},v_{2n+1})\,\Xi_1(v_{2n},v_{2n})\,\Xi_1(v_{2n},v_{2n+1})
\end{array}\right\} \\
&\quad + m(v_{2n-1},v_{2n}) - \emptyset m(v_{2n-1},v_{2n})
\end{aligned}
$$

where,

$$
m(v_{2n-1},v_{2n}) = \max\left\{\begin{array}{c}
\Xi_1^2(v_{2n-1},v_{2n}),\,\Xi_1(v_{2n-1},v_{2n})\,\Xi_1(v_{2n},v_{2n+1}),\\
\Xi_2(v_{2n-1},v_{2n+1})\,\Xi_1(v_{2n},v_{2n})\\
\frac{1}{2}[\Xi_1(v_{2n-1},v_{2n})\,\Xi_2(v_{2n-1},v_{2n+1})\\
+\Xi_1(v_{2n},v_{2n})\,\Xi_1(v_{2n},v_{2n+1})]
\end{array}\right\}.
$$

On using $\alpha_{2n}=\Xi_1(v_{2n},v_{2n+1})$ in (C$_3$), we have

$$
[1+p\alpha_{2n-1}]\,{\alpha_{2n}}^2 \leq p\max\left\{\begin{array}{c}
\frac{1}{2}\left[\begin{array}{c}{\alpha_{2n-1}}^2\alpha_{2n}+\\ \alpha_{2n-1}{\alpha_{2n}}^2\end{array}\right],\\
0,0
\end{array}\right\} + m(v_{2n-1},v_{2n}) - \emptyset m(v_{2n-1},v_{2n})
$$

where,

$$
m(v_{2n-1},v_{2n}) = \max\left\{\begin{array}{c}
\alpha_{2n-1}^2,\,\alpha_{2n-1}\alpha_{2n}\\
0,\\
\frac{1}{2}[\alpha_{2n-1}\Xi_2(v_{2n-1},v_{2n+1})+0]
\end{array}\right\}.
$$

Now, using triangular inequality and if $\alpha_{2n-1}<\alpha_{2n}$. Then after simplification, we get

$$
[1+p\alpha_{2n-1}]\,{\alpha_{2n}}^2 \leq p\max\left\{\begin{array}{c}
\frac{1}{2}\left[\begin{array}{c}{\alpha_{2n}}^2\alpha_{2n}+\\ \alpha_{2n}{\alpha_{2n}}^2\end{array}\right],\\
0,\\
0
\end{array}\right\} + m(v_{2n-1},v_{2n}) - \emptyset m(v_{2n-1},v_{2n})
$$

where $m(v_{2n-1},v_{2n}) = \max\left\{\begin{array}{c}{\alpha_{2n}}^2,\,\alpha_{2n}\alpha_{2n},\\ 0,\\ \frac{1}{2}[\alpha_{2n}(\alpha_{2n}+\alpha_{2n})]\end{array}\right\} = {\alpha_{2n}}^2$. Then

$$
[1+p\alpha_{2n}]\,{\alpha_{2n}}^2 \leq p{\alpha_{2n}}^3 + {\alpha_{2n}}^2 - \emptyset\left({\alpha_{2n}}^2\right) 0 \geq \emptyset\left({\alpha_{2n}}^2\right)
$$

which is a contradiction. Hence $\alpha_{2n}\leq\alpha_{2n-1}$.

Case 2 . If n is odd, in a similar way, we can obtain $\alpha_{2n+1}\leq\alpha_{2n}$. Therefore, sequence $\{\alpha_{2n}\}$ is monotone decreasing sequence which is bounded below by 0. So, there exists $r\geq 1$ such that $\alpha_{2n}\to r$ as $n\to\infty$. Suppose $r>0$, then from inequality (C_3), by putting $u=u_{2n}$ and $v=u_{2n+1}$ in (C_3), we have

$$\begin{aligned}
&[1+p\Xi_1(Mu_{2n},Bu_{2n+1})]\,\Xi_1^2(Su_{2n},Tu_{2n+1})\\
&\le p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\Xi_1^2(Mu_{2n},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1})+\\ \Xi_1(Mu_{2n},Su_{2n})\,\Xi_1^2(Bu_{2n+1},Tu_{2n+1})\end{array}\right]\\ \Xi_1(Mu_{2n},Su_{2n})\,\Xi_2(Mu_{2n},Tu_{2n+1})\,\Xi_1(Bu_{2n+1},Su_{2n})\\ \Xi_2(Mu_{2n},Tu_{2n+1})\,\Xi_1(Bu_{2n+1},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1})\end{array}\right\}\\
&+m(Mu_{2n},Bu_{2n+1})-\emptyset m(Mu_{2n},Bu_{2n+1}),
\end{aligned}$$

where

$$\begin{aligned}
&m(Mu_{2n},Bu_{2n+1})\\
&=\max\left\{\begin{array}{c}\Xi_1^2(Mu_{2n},Bu_{2n+1}),\Xi_1(Mu_{2n},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1}),\\ \Xi_2(Mu_{2n},Tu_{2n+1})\,\Xi_1(Bu_{2n+1},Su_{2n}),\\ \frac{1}{2}[\Xi_1(Mu_{2n},Su_{2n})\,\Xi_2(Mu_{2n},Tu_{2n+1})\\ +\Xi_1(Bu_{2n+1},Su_{2n})\,\Xi_1(Bu_{2n+1},Tu_{2n+1})]\end{array}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&[1+p\Xi_1(v_{2n-1},v_{2n})]\,{\Xi_1}^2(v_{2n},v_{2n+1})\\
&\le p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\Xi_1^2(v_{2n-1},v_{2n})\,\Xi_1(v_{2n},v_{2n+1})+\\ \Xi_1(v_{2n-1},v_{2n})\,\Xi_1^2(v_{2n},v_{2n+1})\end{array}\right],\\ \Xi_1(v_{2n-1},v_{2n})\,\Xi_2(v_{2n-1},v_{2n+1})\,\Xi_1(v_{2n},v_{2n}),\\ \Xi_2(v_{2n-1},v_{2n+1})\,\Xi_1(v_{2n},v_{2n})\,\Xi_1(v_{2n},v_{2n+1})\end{array}\right\}\\
&+m(v_{2n-1},v_{2n})-\emptyset m(v_{2n-1},v_{2n})
\end{aligned}$$

where

$$\begin{aligned}
&m(v_{2n-1},v_{2n})\\
&=\max\left\{\begin{array}{c}\Xi_1^2(v_{2n-1},v_{2n}),\Xi_1(v_{2n-1},v_{2n})\,\Xi_1(v_{2n},v_{2n+1}),\Xi_2(v_{2n-1},v_{2n+1})\,\Xi_1(v_{2n},v_{2n}\\ \frac{1}{2}[\Xi_1(v_{2n-1},v_{2n})\,\Xi_2(v_{2n-1},v_{2n+1})+\Xi_1(v_{2n},v_{2n})\,\Xi_1(v_{2n},v_{2n+1})]\end{array}\right\}
\end{aligned}$$

$$[1+p\alpha_{2n-1}]\,{\alpha_{2n}}^2\le p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\alpha_{2n-1}^2\alpha_{2n}+\\ \alpha_{2n-1}\alpha_{2n^2}\end{array}\right],\\ 0,\\ 0\end{array}\right\}+m(v_{2n-1},v_{2n})-\emptyset m(v_{2n-1},v_{2n})$$

where

$$m\left(v_{2n-1}, v_{2n}\right) = \max\left\{\begin{array}{c} {\alpha_{2n-1}}^2, \alpha_{2n-1}\alpha_{2n}, 0, \\ \frac{1}{2}\left[\alpha_{2n-1}\left(\alpha_{2n-1}+\alpha_{2n}\right)\right] \end{array}\right\}$$

$$[1+pr]r^2 \leq p\max\left\{\begin{array}{c} \frac{1}{2}\left[r^3+\right. \\ r^3 \\ 0, \\ 0 \end{array}\right\} + m\left(v_{2n}, v_{2n-1}\right) - \emptyset m\left(v_{2n}, v_{2n-1}\right)$$

where

$$m\left(v_{2n-1}, v_{2n}\right) = \max\left\{\begin{array}{c} r^2, r^2, 0, \\ r^2 \end{array}\right\} = r^2.[1+pr]r^2 \leq pr^3 + r^2 - \emptyset\left(r^2\right).$$

Then, $\emptyset\left(r^2\right) \leq 0$, since r is positive, then by property of $\emptyset$, we get $r = 0$, we conclude that $\lim_{n\to\infty} \alpha_{2n} = r = 0$. Now, we show $\{v_n\}$ to be a Cauchy sequence in Φ_λ. Suppose we assume that $\{v_n\}$ is not a Cauchy sequence. For $\epsilon > 0$, we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\Xi_8\left(v_{m(k)}, v_{n(k)}\right) \geq \epsilon, \quad \Xi_{\frac{1}{4}}\left(v_{m(k)}, v_{n(k-1)}\right) < \epsilon \tag{1}$$

and $n(k) > m(k) > k$. Now, we have

$$\begin{aligned} \epsilon \leq & \Xi_8\left(v_{m(k)}, v_{n(k)}\right) \\ \leq & \Xi_2\left(v_{m(k)}, v_{n(k)}\right) + \Xi_1\left(v_{m(k)}, v_{n(k)}\right) \\ \leq & \Xi_{\frac{1}{2}}\left(v_{m(k)}, v_{n(k-1)}\right) + \Xi_{\frac{1}{2}}\left(v_{n(k)-1}, v_{n(k)}\right) \\ \leq & \Xi_{\frac{1}{4}}\left(v_{m(k)}, v_{n(k-1)}\right) + \Xi_{\frac{1}{2}}\left(v_{n(k)-1}, v_{n(k)}\right) \\ \leq & \epsilon + \Xi_{\frac{1}{2}}\left(v_{n(k)-1}, v_{n(k)}\right). \end{aligned}$$

Letting $k \to \infty$, we get $\lim_{k\to\infty} \Xi_2\left(v_{m(k)}, v_{n(k)}\right) = \lim_{k\to\infty} \Xi_1\left(v_{m(k)}, v_{n(k)}\right) = \epsilon$. Again using triangular inequality , we have

$$\begin{aligned} \epsilon \leq & \Xi_8\left(v_{m(k)}, v_{n(k)}\right) \\ \leq & \Xi_4\left(v_{m(k)}, v_{n(k)}\right) \\ \leq & \Xi_2\left(v_{n(k)}, v_{n(k)+1}\right) + \Xi_2\left(v_{m(k)}, v_{n(k)+1}\right). \end{aligned} \tag{2}$$

We get

$$\begin{aligned} \epsilon - \Xi_2\left(v_{n(k)}, v_{n(k)+1}\right) \leq & \Xi_2\left(v_{m(k)}, v_{n(k)+1}\right) \\ \leq & \Xi_1\left(v_{m(k)}, v_{n(k)+1}\right) \\ \leq & \Xi_{\frac{1}{4}}\left(v_{m(k)}, v_{n(k)+1}\right) \\ \leq & \Xi_{\frac{1}{8}}\left(v_{m(k)}, v_{n(k)}\right) + \Xi_{\frac{1}{8}}\left(v_{n(k)}, v_{n(k)+1}\right). \end{aligned}$$

Taking limits as $k \to \infty$, we have

$$\lim_{k\to\infty} \Xi_1\left(v_{m(k)}, v_{n(k)+1}\right) = \lim_{k\to\infty} \Xi_2\left(v_{m(k)}, v_{n(k)+1}\right) = \epsilon. \tag{3}$$

Now from the triangular inequality, we have

$$\epsilon \leq \Xi_2\left(v_{m(k)}, v_{n(k)}\right) \leq \Xi_1\left(v_{m(k)}, v_{m(k)+1}\right) + \Xi_1\left(v_{m(k)+1}, v_{n(k)}\right).$$

We get

$$\begin{aligned}\epsilon - \Xi_1\left(v_{m(k)}, v_{m(k)+1}\right) \leq & \Xi_1\left(v_{m(k)+1}, v_{n(k)}\right) \\ \leq & \Xi_{\frac{1}{2}}\left(v_{n(k)}, v_{m(k)-1}\right) + \Xi_{\frac{1}{2}}\left(v_{m(k)+1}, v_{m(k)-1}\right) \\ \leq & \Xi_{\frac{1}{2}}\left(v_{n(k)}, v_{m(k)-1}\right) + \Xi_{\frac{1}{4}}\left(v_{m(k)-1}, v_{m(k)}\right) + \Xi_{\frac{1}{4}}\left(v_{m(k)}, v_{m(k)+1}\right).\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \Xi_1\left(v_{m(k)+1}, v_{n(k)}\right) = \epsilon. \tag{4}$$

Again, from the triangular inequality, we have

$$\Xi_8\left(v_{m(k)}, v_{n(k)}\right) \leq \Xi_4\left(v_{n(k)}, v_{n(k)+1}\right) + \Xi_4\left(v_{n(k)+1}, v_{m(k)}\right).$$

We get

$$\Xi_8\left(v_{m(k)}, v_{n(k)}\right) \leq \Xi_4\left(v_{n(k)}, v_{n(k)+1}\right) + \Xi_2\left(v_{m(k)+1}, v_{m(k)}\right) + \Xi_2\left(v_{m(k)+1}, v_{n(k)+1}\right).$$

Then

$$\begin{aligned}&\Xi_8\left(v_{m(k)}, v_{n(k)}\right) - \Xi_4\left(v_{n(k)}, v_{n(k)+1}\right) - \Xi_2\left(v_{m(k)+1}, v_{m(k)}\right) \leq \Xi_2\left(v_{m(k)+1}, v_{n(k)+1}\right) \\ \leq & \Xi_1\left(v_{m(k)+1}, v_{m(k)}\right) + \Xi_1\left(v_{n(k)+1}, v_{m(k)}\right).\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \Xi_2\left(v_{m(k)+1}, v_{n(k)+1}\right) = \epsilon. \tag{5}$$

Since

$$\begin{aligned}\Xi_2\left(v_{m(k)+1}, v_{n(k)+1}\right) \leq & \Xi_1\left(v_{m(k)+1}, v_{n(k)+1}\right) \\ \leq & \Xi_{\frac{1}{2}}\left(v_{m(k)+1}, v_{m(k)}\right) + \Xi_{\frac{1}{2}}\left(v_{m(k)}, v_{n(k)+1}\right) \\ \leq & \Xi_{\frac{1}{2}}\left(v_{m(k)}, v_{n(k)+1}\right) \\ \leq & \Xi_{\frac{1}{4}}\left(v_{m(k)}, v_{m(k)-1}\right) + \Xi_{\frac{1}{4}}\left(v_{m(k)-1}, v_{n(k)-1}\right) \\ \leq & \Xi_{\frac{1}{8}}\left(v_{m(k)-1}, v_{n(k)}\right) + \Xi_{\frac{1}{8}}\left(v_{n(k)-1}, v_{n(k)}\right).\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \Xi_1\left(v_{m(k)+1}, v_{n(k)+1}\right) = \epsilon. \tag{6}$$

On putting $u = u_{m(k)}$ and $v = u_{n(k)}$ in (C_3), we get

$$\left[1 + p\Xi_1\left(Mu_{m(k)}, Bu_{n(k)}\right)\right]\Xi_1^2\left(Su_{m(k)}, Tu_{n(k)}\right) \leq$$
$$p\max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c}\Xi_1^2\left(Mu_{m(k)}, Su_{m(k)}\right)\Xi_1\left(Bu_{n(k)}, Tu_{n(k)}\right) + \\ \Xi_1\left(Mu_{m(k)}, Su_{m(k)}\right)\Xi_1^2\left(Bu_{n(k)}, Tu_{n(k)}\right)\end{array}\right] \\ \Xi_1\left(Mu_{m(k)}, Su_{m(k)}\right)\Xi_2\left(Mu_{m(k)}, Tu_{n(k)}\right)\Xi_1\left(Bu_{n(k)}, Su_{m(k)}\right), \\ \Xi_2\left(Mu_{m(k)}, Tu_{n(k)}\right)\Xi_1\left(Bu_{n(k)}, Su_{m(k)}\right)\Xi_1\left(Bu_{n(k)}, Tu_{n(k)}\right)\end{array}\right\}$$
$$+ m\left(Mu_{m(k)}, Bu_{n(k)}\right) - \emptyset m\left(Mu_{m(k)}, Bu_{n(k)}\right)$$

where

$$m\left(Mu_{m(k)}, Bu_{n(k)}\right)$$
$$= \max\left\{\begin{array}{l} \Xi_1^2\left(Mu_{m(k)}, Bu_{n(k)}\right), \Xi_1\left(Mu_{m(k)}, Su_{m(k)}\right)\Xi_1\left(Bu_{n(k)}, Tu_{n(k)}\right), \\ \Xi_2\left(Mu_{m(k)}, Tu_{n(k)}\right)\Xi_1\left(Bu_{n(k)}, Su_{m(k)}\right), \\ \frac{1}{2}\left[\Xi_1\left(Mu_{m(k)}, Su_{m(k)}\right)\Xi_2\left(Mu_{m(k)}, Tu_{n(k)}\right)\right. \\ \left.+\Xi_1\left(Bu_{n(k)}, Su_{m(k)}\right)\Xi_1\left(Bu_{n(k)}, Tu_{n(k)}\right)\right]\end{array}\right\}$$

Then

$$\left[1 + p\Xi_1\left(v_{m(k)-1}, v_{n(k)-1}\right)\right]\Xi_1^2\left(v_{m(k)}, v_{n(k)}\right)$$
$$\leq p\max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c}\Xi_1^2\left(v_{m(k)-1}, v_{m(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{n(k)}\right) + \\ \Xi_1\left(v_{m(k)-1}, v_{m(k)}\right)\Xi_1^2\left(v_{n(k)-1}, v_{n(k)}\right)\end{array}\right] \\ \Xi_1\left(v_{m(k)-1}, v_{m(k)}\right)\Xi_2\left(v_{m(k)-1}, v_{n(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{m(k)}\right), \\ \Xi_2\left(v_{m(k)-1}, v_{n(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{m(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{n(k)}\right)\end{array}\right\}$$
$$+ m\left(v_{m(k)-1}, v_{n(k)-1}\right) - \emptyset m\left(v_{m(k)-1}, v_{n(k)-1}\right)$$

where,

$$m\left(v_{m(k)-1}, v_{n(k)-1}\right)$$
$$= \max\left\{\begin{array}{l} \Xi_1^2\left(v_{m(k)-1}, v_{n(k)-1}\right), \Xi_1\left(v_{m(k)-1}, v_{m(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{n(k)}\right), \\ \Xi_2\left(v_{m(k)-1}, v_{n(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{m(k)}\right), \\ \frac{1}{2}\left[\Xi_1\left(v_{m(k)-1}, v_{m(k)}\right)\Xi_2\left(v_{m(k)-1}, v_{n(k)}\right)\right. \\ \left.+\Xi_1\left(v_{n(k)-1}, v_{m(k)}\right)\Xi_1\left(v_{n(k)-1}, v_{n(k)}\right)\right]\end{array}\right\}.$$

Letting $k \to \infty$ and using (1)-(6), we get $[1 + p\epsilon]\epsilon^2 \leq \epsilon^2 - \emptyset\left(\epsilon^2\right)$ a contradiction. Thus, $\{v_n\}$ is a Cauchy sequence in Φ_λ.

Suppose that $M\left(\Phi_\lambda\right)$ is complete there exists $u \in M\left(\Phi_\lambda\right)$ such that $v_{2n+1} = Mu_{2n+2} = \mathrm{T}u_{2n+1} \to u$, as $n \to \infty$. Consequently, we can find $v \in \Phi_\lambda$ such that

$$Mv = u. \tag{7}$$

We claim $Sv = u$. Putting $u = v, v = u_{2n+1}$ in hypothesis (C_3), we get

$$\begin{aligned} &[1 + p\Xi_1(Mv, Bu_{2n+1})]\,\Xi_1^2(Sv, Tu_{2n+1}) \\ &\leq p\max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \Xi_1^2(Mv,Sv)\Xi_1(Bu_{2n+1},Tu_{2n+1}) + \\ \Xi_1(Mv,Sv)\Xi_1^2(Bu_{2n+1},Tu_{2n+1}) \end{array}\right], \\ \Xi_1(Mv,Sv)\Xi_2(Mv,Tu_{2n+1})\Xi_1(Bu_{2n+1},Sv), \\ \Xi_2(Mv,Tu_{2n+1})\Xi_1(Bu_{2n+1},Sv)\Xi_1(Bu_{2n+1},Tu_{2n+1}) \end{array}\right\} \\ &+ m(Mv, Bu_{2n+1}) - \emptyset m(Mv, Bu_{2n+1}) \end{aligned}$$

where,

$$\begin{aligned} &m(Mv, Bu_{2n+1}) \\ &= \max\left\{\begin{array}{c} \Xi_1^2(Mv,Bu_{2n+1}), \Xi_1(Mv,Sv)\Xi_1(Bu_{2n+1},Tu_{2n+1}), \\ \Xi_2(Mv,Tu_{2n+1})\Xi_1(Bu_{2n+1},Sv), \\ \frac{1}{2}\left[\Xi_1(Mv,Sv)\Xi_2(Mv,Tu_{2n+1}) + \Xi_1(Bu_{2n+1},Sv)\Xi_1(Bu_{2n+1},Tu_{2n+1})\right] \end{array}\right\}. \end{aligned}$$

Then

$$\begin{aligned} &[1 + p\Xi_1(Mv, v_{2n})]\,\Xi_1^2(Sv, v_{2n+1}) \\ &\leq p\max\left\{\begin{array}{c} \frac{1}{2}\left[\Xi_1^2(Mv,Sv)\Xi_1(v_{2n},v_{2n+1}) + \Xi_1(Mv,Sv)\Xi_1^2(v_{2n},v_{2n+1})\right], \\ \Xi_1(Mv,Sv)\Xi_2(Mv,v_{2n+1})\Xi_1(v_{2n},Sv), \\ \Xi_2(Mv,v_{2n+1})\Xi_1(v_{2n},Sv)\Xi_1(v_{2n},v_{2n+1}) \end{array}\right\} \\ &+ m(Mv, v_{2n}) - \emptyset m(Mv, v_{2n}) \end{aligned}$$

where,

$$m(Mv, v_{2n}) = \max\left\{\begin{array}{c} \Xi_1^2(Mv,v_{2n}), \Xi_1(Mv,Sv)\Xi_1(v_{2n},v_{2n+1}), \\ \Xi_2(Mv,v_{2n+1})\Xi_1(v_{2n},Sv), \\ \frac{1}{2}\left[\Xi_1(Mv,Sv)\Xi_2(Mv,v_{2n+1}) + \Xi_1(v_{2n},Sv)\Xi_1(v_{2n},v_{2n+1})\right] \end{array}\right\}$$

Then,

$$\begin{aligned} [1 + p\Xi_1(u,u)]\,\Xi_1^2(Sv,u) \leq p\max\left\{\begin{array}{c} \frac{1}{2}\left[\Xi_1^2(u,Sv)\Xi_1(u,u) + \Xi_1(u,Sv)\Xi_1^2(u,u)\right] \\ \Xi_1(u,Sv)\Xi_2(u,u)\Xi_1(u,Sv) \\ \Xi_2(u,u)\Xi_1(u,Sv)\Xi_1(u,u) \end{array}\right\} \\ + m(u,u) - \emptyset m(u,u), \end{aligned}$$

where,

$$m(u,u) = \max\left\{\begin{array}{c} \Xi_1^2(u,u), \Xi_1(u,Sv)\Xi_1(u,u), \Xi_2(u,u)\Xi_1(u,Sv), \\ \frac{1}{2}\left[\Xi_1(u,Sv)\Xi_2(u,u) + \Xi_1(u,Sv)\Xi_1(u,u)\right] \end{array}\right\} = 0,$$

i. e., $\Xi_1^2(Sv,u) = 0$, this implies

$$u = Sv. \tag{8}$$

Since $u = Sv \in S(\Phi_\lambda) \subseteq B(\Phi_\lambda)$, there exists $w \in \Phi_\lambda$ such that

$$u = Bw. \tag{9}$$

We claim $Tw = u$. Now using (C_3), putting $u = v, v = w$, we have

$$\begin{aligned}
&[1+p\Xi_1(Mv,Bw)]\,\Xi_1^2(Sv,Tw)\\
\leq & p\max\left\{\begin{array}{c} \frac{1}{2}\left[\Xi_1^2(Mv,Sv)\Xi_1(Bw,Tw)+\Xi_1(Mv,Sv)\Xi_1^2(Bw,Tw)\right], \\ \Xi_1(Mv,Sv)\Xi_2(Mv,Tw)\Xi_1(Bw,Sv), \\ \Xi_2(Mv,Tw)\Xi_1(Bw,Sv)\Xi_1(Bw,Tw) \end{array}\right\}\\
&+m(Mv,Bw)-\emptyset m(Mv,Bw),
\end{aligned}$$

where,

$$\begin{aligned}
&m(Mv,Bw)\\
=&\max\left\{\begin{array}{c} \Xi_1^2(Mv,Bw),\Xi_1(Mv,Sv)\Xi_1(Bw,Tw),\Xi_2(Mv,Tw)\Xi_1(Bw,Sv), \\ \frac{1}{2}\left[\Xi_1(Mv,Sv)\Xi_2(Mv,Tw)+\Xi_1(Bw,Sv)\Xi_1(Bw,Tw)\right] \end{array}\right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
&[1+p\Xi_1(u,u)]\,\Xi_1^2(u,Tw)\\
\leq & p\max\left\{\begin{array}{c} \frac{1}{2}\left[\Xi_1^2(u,u)\Xi_1(u,Tw)+\Xi_1(u,u)\Xi_1^2(u,Tw)\right], \\ \Xi_1(u,u)\Xi_2(u,Tw)\Xi_1(u,u), \\ \Xi_2(u,Tw)\Xi_1(u,u)\Xi_1(u,Tw) \end{array}\right\}+m(u,u)-\emptyset m(u,u)
\end{aligned}$$

where,

$$m(u,u)=\max\left\{\begin{array}{c} \Xi_1^2(u,u),\Xi_1(u,u)\Xi_1(u,Tw),\Xi_2(Mv,Tw)\Xi_1(u,u), \\ \frac{1}{2}\left[\Xi_1(u,u)\Xi_2(u,Tw)+\Xi_1(u,u)\Xi_1(u,Tw)\right] \end{array}\right\}$$

This implies that $\Xi_1^2(u,Tw)=0$, hence,

$$u = Tw. \tag{10}$$

Using (8) and (10), we get

$$u = Mv = Sv. \tag{11}$$

that is, u is point of coincidence of M and S. Using (9) and (10), we get

$$u = Bw = Tw \tag{12}$$

that is, w is coincidence point of B and T. Now taking (M,S) as conditionally commuting:

Case 1. M and S commute at v, then

$$u = Sv = Mv \text{ implies that } Mu = MSv = SMv = Su = w_1.$$

Case 2. B and T commute at w, then

$$u = Bw = Tw \text{ implies that } Tu = TBw = BTw = Bu = w_2.$$

We claim that $w_1 = w_2$. Now, using (C_3) on putting $u = w_1, v = w_2$,

$$\begin{aligned}&[1 + p\Xi_1(w_1, w_2)]\,\Xi_1^2(w_1, w_2)\\ \leq& p\max\left\{\begin{array}{c}\frac{1}{2}\left[\Xi_1^2(w_1, w_1)\,\Xi_1(w_2, w_2) + \Xi_1(w_1, w_1)\,\Xi_1^2(w_2, w_2)\right],\\ \Xi_1(w_1, w_1)\,\Xi_2(w_1, w_2)\,\Xi_1(w_2, w_1),\\ \Xi_2(w_1, w_2)\,\Xi_1(w_2, w_1)\,\Xi_1(w_2, w_2)\end{array}\right\}\\ &+ m(w_1, w_2) - \emptyset m(w_1, w_2)\end{aligned}$$

where,

$$m(w_1, w_2) = \max\left\{\begin{array}{c}\Xi_1^2(w_1, w_2), \Xi_1(w_1, w_1)\,\Xi_1(w_2, w_2), \Xi_2(w_1, w_2)\,\Xi_1(w_2, w_1),\\ \frac{1}{2}\left[\Xi_1(w_1, w_1)\,\Xi_2(w_1, w_2) + \Xi_1(w_2, w_1)\,\Xi_1(w_2, w_2)\right]\end{array}\right\}.$$

This implies that $\Xi_1^2(w_1, w_2) = 0$, hence, $w_1 = w_2$. Therefore, $Su = Mu = Tu = Bu$, i.e., u is common coincident point of S, M, B and T. Now, we claim that $Tu = u$. Now using (C_3), putting $u = v, v = u$,

$$\begin{aligned}&[1 + p\Xi_1(Mv, Bu)]\,\Xi_1^2(Sv, Tu)\\ \leq& p\max\left\{\begin{array}{c}\frac{1}{2}\left[\Xi_1^2(Mv, Sv)\Xi_1(Bu, Tu) + \Xi_1(Mv, Sv)\Xi_1^2(Bu, Tu)\right],\\ \Xi_1(Mv, Sv)\Xi_2(Mv, Tu)\Xi_1(Bu, Sv),\\ \Xi_2(Mv, Tu)\Xi_1(Bu, Sv)\Xi_1(Bu, Tu)\end{array}\right\}\\ &+ m(Mv, Bu) - \emptyset m(Mv, Bu)\end{aligned}$$

where,

$$\begin{aligned}m(Mv, Bu) =& \max\left\{\begin{array}{c}\Xi_1^2(Mv, Bu), \Xi_1(Mv, Sv)\Xi_1(Bu, Tu),\\ \Xi_2(Mv, Tu)\Xi_1(Bu, Sv),\\ \frac{1}{2}\left[\Xi_1(Mv, Sv)\Xi_2(Mv, Tu) + \Xi_1(Bu, Sv)\Xi_1(Bu, Tu)\right]\end{array}\right\}\\ =&\Xi_1^2(u, Tu).\end{aligned}$$

This implies that $u = Tu$. Hence, $u = Su = Mu = Tu = Bu$, i.e., u is common fixed point of M, B, S and T. Case 3. M and S are not commute at v, then by virtue of conditionally commute of M and S there exists a point w in Φ_λ such that

$$y = Sw = Mw \text{ implies that } My = MSw = SMw = Sy. \tag{13}$$

Case 4. B and T are not commute at w, then by virtue of conditionally commute of B and T there exists a point z in Φ_λ such that $p = Bz = Tz$. This implies that

$$Tp = TBz = BTz = Bp. \tag{14}$$

Combining (13) and (14), we get again M, B, S and T have common fixed point. Uniqueness can be easily found from (C_3). The proof for cases in which $B(\Phi_\lambda), T(\Phi_\lambda), S(\Phi_\lambda)$ are complete are similar and are therefore omitted. □

Theorem 3.2. *Let M, B, S and T be self-mappings of a modular metric space (Φ_λ, Ξ) satisfying (C_3) and the following:*

(C_4) *the pairs (M, S) and (B, T) be non-compatible, reciprocally continuous, conditionally compatible. Then M, B, S and T have unique common fixed point.*

Proof. From (C_4), non-compatibility of (M, S) and (B, T) implies there exists a sequences $\{u_n\}$ and $\{v_n\}$ in Φ_λ such that $\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Su_n = t_1$, for some $t_1 \in \Phi_\lambda$ and $\lim_{n\to\infty} \Xi_\lambda(\mathrm{SM}\, u_n, \mathrm{MS}\, u_n) \neq 0$. Also, $\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Bu_n = t_2$ for some $t_2 \in \Phi_\lambda$ and $\lim_{n\to\infty} \Xi_\lambda(\mathrm{TB}v_n, \mathrm{BT}v_n) \neq 0$.

Since pairs (M, S) and (B, T) are conditionally compatible therefore, conditional compatible of (M, S) and (B, T) implies that there exists sequences $\{z_n\}$ and $\{z'_n\}$ in Φ_λ satisfying $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Mz_n = u$ for some $u \in \Phi_\lambda$ and $\lim_{n\to\infty} \Xi_\lambda(\mathrm{S}Mz_n, MSz_n) = 0$.

Also, $\lim_{n\to\infty} Bz'{}_n = \lim_{n\to\infty} Tz'_n = v$ for some $v \in \Phi_\lambda$ and $\lim_{n\to\infty} \Xi_\lambda(\mathrm{BT}z', \mathrm{TB}'n) = 0$. Morover, pairs (M, S) and (B, T) are reciprocally continuous so, $\lim_{n\to\infty} SMz_n = Su$, $\lim_{n\to\infty} MSz_n = Mu$ and so $Su = Mu$. Also, $\lim_{n\to\infty} TBz'_n = \lim_{n\to\infty} BTz'_n = Bv$ and so $Bv = Tv$. We claim that $u = v$. Putting $u = z_n, v = z'_n$ in (C_3), we have

$$
\begin{aligned}
&[1 + p\Xi_1(Mz_n, Bz'_n)]\, \Xi_1^2(Sz_n, Tz'_n) \\
\leq & p \max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \Xi_1^2(Mz_n, Sz_n)\, \Xi_1(Bz'_n, Tz'_n) + \\ \Xi_1(Mz_n, Sz_n)\, \Xi_1^2(Bz'_n, Tz'_n) \end{array} \right] \\ \Xi_1(Mz_n, Sz_n)\, \Xi_2(Mz_n, Tz'_n)\, \Xi_1(Bz'{}_n, Sz_n), \\ \Xi_2(Mz_n, Tz'_n)\, \Xi_1(Bz'{}_n, Sz_n)\, \Xi_1(Bz'_n, Tz'_n) \end{array} \right\} \\
&+ m(Mz_n, Bz'_n) - \emptyset m(Mz_n, Bz'_n)
\end{aligned}
$$

where,

$$
\begin{aligned}
&m(Mz_n, Bz'_n) \\
=& \max \left\{ \begin{array}{c} \Xi_1^2(Mz_n, Bz'_n), \Xi_1(Mz_n, Sz_n)\, \Xi_1(Bz'_n, Tz'_n), \\ \Xi_2(Mz_n, Tz'_n)\, \Xi_1(Bz'_n, Sz_n), \\ \frac{1}{2}[\Xi_1(Mz_n, Sz_n)\, \Xi_2(Mz_n, Tz'_n) + \Xi_1(Bz'_n, Sz_n)\, \Xi_1(Bz'_n, Tz'_n)] \end{array} \right\}.
\end{aligned}
$$

Taking limit $n \to \infty$, we have $u = v$. So, $Mu = Bu = Su = Tu$. Next, we claim that $Su = u$. Putting $u = u, v = z'_n$ in (C_3), we have

$$
\begin{aligned}
&[1 + p\Xi_1(Mu, Bz'_n)]\, \Xi_1^2(Su, Tz'_n) \\
\leq \quad & p \max \left\{ \begin{array}{c} \frac{1}{2}[\Xi_1^2(Mu, Su)\Xi_1(Bz'_n, Tz'_n) + \Xi_1(Mu, Su)\Xi_1^2(Bz'_n, Tz'_n)] \\ \Xi_1(Mu, Su)\Xi_2(Mu, Tz'_n)\, \Xi_1(Bz'{}_n, Su) \\ \Xi_2(Mu, Tz'_n)\, \Xi_1(Bz'_n, Su)\, \Xi_1(Bz'_n, Tz'{}_n) \end{array} \right\} \\
&+ m(Mu, Bz'_n) - \emptyset m(Mu, Bz'_n)
\end{aligned}
$$

where,

$$m\left(Mu, Bz_n'\right) = \max\left\{\begin{array}{c} \Xi_1^2\left(Mu, Bz_n'\right), \Xi_1(Mu, Su)\Xi_1\left(Bz_n', Tz_n'\right), \Xi_2\left(Mu, Tz_n'\right)\Xi_1\left(Bz_n', Su\right), \\ \frac{1}{2}\left[\Xi_1(Mu, Su)\Xi_2\left(Mu, Tz_n'\right) + \Xi_1\left(Bz_n', Su\right)\Xi_1\left(Bz_n', Tz_n'\right)\right] \end{array}\right\}$$

Taking limit $n \to \infty$, we have $Su = u$. Therefore, $u = Mu = Bu = Su = Tu$. Hence, u is common fixed point of M, S, B and T. Uniqueness, can be easily found from (C_3). □

Theorem 3.3. *Let M, B, S and T be self-mappings of a modular metric space (Φ_λ, Ξ) satisfying (C_3) and*

(C_5) the pairs (M, S) and (B, T) be non-compatible, reciprocally continuous, faintly compatible.

Then M, B, S and T have unique common fixed point.

Proof. From (C_5), non-compatibility of (M, S) and (B, T) implies there exists a sequences $\{u_n\}$ and $\{v_n\}$ in Φ_λ such that $\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Su_n = t_1$, for some $t_1 \in \Phi_\lambda$ and $\lim_{n\to\infty} \Xi_\lambda\left(\mathrm{SM}u_n, \mathrm{MS}u_n\right) \neq 0$. Also, $\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Bu_n = t_2$ for some $t_2 \in \Phi_\lambda$ and $\lim_{n\to\infty} \Xi_\lambda\left(\mathrm{TB}v_n, \mathrm{BT}v_n\right) \neq 0$.

Since pairs (M, S) and (B, T) are faintly compatible therefore, conditional compatible of (M, S) and (B, T) implies that there exists sequences $\{z_n\}$ and $\{z_n'\}$ in Φ_λ satisfying

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Mz_n = u,$$

for some $u \in \Phi_\lambda$ and

$$\lim_{n\to\infty} \Xi_\lambda\left(\mathrm{SM}z_n, \mathrm{MS}_n\right) = 0.$$

Also,

$$\lim_{n\to\infty} Bz'{}_n = \lim_{n\to\infty} Tz_n' = v,$$

for some $v \in \Phi_\lambda$ and

$$\lim_{n\to\infty} \Xi_\lambda\left(\mathrm{BT}z_n', \mathrm{TB}z_n'\right) = 0.$$

Also, pairs (M, S) and (B, T) are reciprocally continuous so, $\lim_{n\to\infty} SMz_n = Su, \lim_{n\to\infty} MSz_n = Mu$ and so $Su = Mu$. Also,

$$\lim_{n\to\infty} TBz_n' = Tv, \lim_{n\to\infty} BTz_n' = Bv$$

and so $Bv = Tv$. Since pairs (M, S) and (B, T) are faintly compatible, $SMu = MSu$ and so $SSu = SMu = MMu$ and also, $TBv = BTv$ and so $TTv = TBv = BTv = BBv$.

First we claim that $Su = Tv$. Putting $u = u, v = v$ in (C_3), we get

$$\begin{aligned}&[1+p\Xi_1(Mu,Bv)]\,\Xi_1^2(Su,Tv)\\ \leq& p\max\left\{\begin{array}{c}\frac{1}{2}\left[\Xi_1^2(Mu,Su)\Xi_1(Bv,Tv)+\Xi_1(Mu,Su)\Xi_1^2(Bv,Tv)\right],\\ \Xi_1(Mu,Su)\Xi_2(Mu,Tv)\Xi_1(Bv,Su)\\ \Xi_2(Mu,Tv)\Xi_1(Bv,Su)\Xi_1(Bv,Tv)\end{array}\right\}\\ &+m(Mu,Bv)-\emptyset m(Mu,Bv),\end{aligned}$$

where,

$$\begin{aligned}&m(Mu,Bv)\\ =&\max\left\{\begin{array}{c}\Xi_1^2(Mu,Bv),\Xi_1(Mu,Su)\Xi_1(Bv,Tv),\Xi_2(Mu,Tv)\Xi_1(Bv,Su),\\ \frac{1}{2}\left[\Xi_1(Mu,Su)\Xi_2(Mu,Tv)+\Xi_1(Bv,Su)\Xi_1(Bv,Tv)\right]\end{array}\right\}.\end{aligned}$$

Then

$$\begin{aligned}&[1+p\Xi_1(Su,Tv)]\,\Xi_1^2(Su,Tv)\\ \leq& p\max\left\{\begin{array}{c}\frac{1}{2}\left[\Xi_1^2(Su,Su)\Xi_1(Tv,Tv)+\Xi_1(Su,Su)\Xi_1^2(Tv,Tv)\right],\\ \Xi_1(Su,Su)\Xi_2(Su,Tv)\Xi_1(Tv,Su),\\ \Xi_2(Su,Tv)\Xi_1(Tv,Su)\Xi_1(Tv,Tv)\end{array}\right\}\\ &+m(Su,Tv)-\emptyset m(Su,Tv)\end{aligned}$$

where,

$$\begin{aligned}&m(Su,Tv)\\ =&\max\left\{\begin{array}{c}\Xi_1^2(Su,Tv),\Xi_1(Su,Su)\Xi_1(Tv,Tv),\Xi_2(Su,Tv)\Xi_1(Tv,Su)\\ \frac{1}{2}\left[\Xi_1(Su,Su)\Xi_2(Su,Tv)+\Xi_1(Tv,Su)\Xi_1(Tv,Tv)\right]\end{array}\right\}\\ =&\Xi_1^2(Su,Tv).\end{aligned}$$

This implies that $Su = Tv$. Next, we claim that $SSu = Su$. Putting $u = Su, v = v$ in (C_3), we have

$$\begin{aligned}&[1+p\Xi_1(MSu,Bv)]\,\Xi_1^2(SSu,Tv)\\ \leq& p\max\left\{\begin{array}{c}\frac{1}{2}\left[\Xi_1^2(MSu,SSu)\Xi_1(Bv,Tv)+\Xi_1(MSu,SSu)\Xi_1^2(Bv,Tv)\right],\\ \Xi_1(MSu,SSu)\Xi_2(MSu,Tv)\Xi_1(Bv,SSu),\\ \Xi_2(MSu,Tv)\Xi_1(Bv,SSu)\Xi_1(Bv,Tv)\end{array}\right\}\\ &+m(M\mathrm{Su},Bv)-\emptyset m(M\mathrm{Su},Bv)\end{aligned}$$

where,

$$\begin{aligned}&m(MSu,Bv)\\ =&\max\left\{\begin{array}{c}\Xi_1^2(MSu,Bv),\Xi_1(MSu,SSu)\Xi_1(Bv,Tv),\Xi_2(MSu,Tv)\Xi_1(Bv,SSu),\\ \frac{1}{2}\left[\Xi_1(MSu,SSu)\Xi_2(MSu,Tv)+\Xi_1(Bv,SSu)\Xi_1(Bv,Tv)\right]\end{array}\right\}.\end{aligned}$$

Then,

$$
\begin{aligned}
&[1 + p\Xi_1(SSu, Tv)]\, {\Xi_1}^2(SSu, Tv)\\
\leq &p \max \left\{ \begin{array}{c} \frac{1}{2}\left[\Xi_1^2(SSu, SSu)\Xi_1(Tv, Tv) + \Xi_1(SSu, SSu)\Xi_1^2(Tv, Tv)\right], \\ \Xi_1(SSu, SSu)\Xi_2(SSu, Tv)\Xi_1(Tv, SSu), \\ \Xi_2(SSu, Tv)\Xi_1(Su, SSu)\Xi_1(Tv, Tv) \end{array} \right\}\\
&+ m(SSu, Tv) - \emptyset m(SSu, Tv)
\end{aligned}
$$

where,

$$
\begin{aligned}
&m(SSu, Tv)\\
= &\max \left\{ \begin{array}{c} \Xi_1^2(SSu, Su), \Xi_1(SSu, SSu)\Xi_1(Tv, Tv), \Xi_2(SSu, Su)\Xi_1(Su, SSu), \\ \frac{1}{2}\left[\Xi_1(SSu, SSu)\Xi_2(SSu, Su) + \Xi_1(Su, SSu)\Xi_1(Su, Su)\right] \end{array} \right\}\\
= &\Xi_1^2(SSu, Tv).
\end{aligned}
$$

This implies that $SSu = Tv$. Therefore, $SSu = Tv = Su$. Next we claim that $TTv = Tv$. Putting $u = u, v = Tv$ in (C_3), we get

$$
\begin{aligned}
&[1 + p\Xi_1(Mu, B\text{Tv})]\, \Xi_1^2(Su, T\text{Tv})\\
\leq &p \max \left\{ \begin{array}{c} \frac{1}{2}\left[\Xi_1^2(Mu, Su)\Xi_1(B\,\text{Tv}, T\text{Tv}) + \Xi_1(Mu, Su)\Xi_1^2(B\,\text{Tv}, T\text{Tv})\right], \\ \Xi_1(Mu, Su)\Xi_2(Mu, T\text{Tv})\Xi_1(B\,\text{Tv}, Su), \\ \Xi_2(Mu, T\text{Tv})\Xi_1(B\text{Tv}, Su)\Xi_1(B\,\text{Tv}, T\text{Tv}) \end{array} \right\}\\
&+ m(Mu, B\text{Tv}) - \emptyset m(Mu, B\text{Tv})
\end{aligned}
$$

where,

$$
m(Mu, B\,\text{Tv}) = \max \left\{ \begin{array}{c} {\Xi_1}^2(Mu, B\,\text{Tv}), \Xi_1(Mu, Su)\Xi_1(B\,\text{Tv}, T\,\text{Tv}), \\ \Xi_2(Mu, T\,\text{Tv})\Xi_1(B\,\text{Tv}, Su), \\ \frac{1}{2}\left[\Xi_1(Mu, Su)\Xi_2(Mu, T\,\text{TVv})\right. \\ \left. + \Xi_1(B\,\text{Tv}, Su)\Xi_1(B\,\text{Tv}, T\,\text{Tv})\right] \end{array} \right\}.
$$

This implies that $Su = TTv$. Therefore, $TTv = Su = Tv$. Now we have, $SSu = MSu, Su = TTv = TSu$ and $Su = TTv = BTv = BSu$, since $Tv = Su$. Hence, $SSu = MSu = TSu = BSu = Su$, i.e., Su is a common fixed point of M, B, S and T. Uniqueness can be easily found. □

4. Conclusion

Our results generalized the result of [2],[12] and [17] using minimal Ξ-commuting mappings in setting of Modular Metric Spaces.

References

[1] M.M. Al-Thagafi and N. Shahzad, *Generalized I-nonexpansive self-maps and invariant approximations*, Acta Math. Sin., **24** (2008), 867–876.

[2] R.K. Bisht and N. Shahzad, *Faintly compatible mappings and common fixed points*, Fixed Point Theory Appl., **2013** (2013), 156.

[3] W. Chistyakov, *Metric modular spaces and their application*, Doklady Math., **73** (2006), 32–35.

[4] M. Chirasak, S.Wutiphol, and K. Poom, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl., **93** (2011).

[5] W. Chistyakov, *Modular metric spaces, I: Basic concepts*, Nonlinear Anal.: Theory Meth. Appl., **72** (2010), 1–14.

[6] W. Chistyakov, *Modular Metric Spaces Generated by F-Modular*, Folia Math., **14** (2008), 3–25.

[7] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Month., **83** (1976), 261.

[8] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., **9** (1986), 771–779.

[9] G. Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity*, Indian Pure Appl. Math., **29**(3) (1998), 227–238.

[10] M.A. Khamsi and W.M. Kozlowski, *Fixed Point Theory in Modular Function Spaces*, Springer International Publishing Switzerland, 2015.

[11] A. Padcharoen, D. Gopal, P. Chaipunya, and P. Kumam, *Fixed point and periodic point results for α-type F-contractions in modular metric spaces*, Fixed Point Theory Appl., **39** (2016).

[12] R.P. Pant and V, Pant, *Common fixed points of conditionally commuting maps*, Fixed Point Theory, **11**(1) (2010), 113–118.

[13] R.P. Pant, *R-weak commutativity and common fixed points of non-compatible maps*, Ganita, **49** (1998), 19–27.

[14] R.P. Pant, *Common fixed point theorems for contractive maps*, J. Math. Anal. Appl., **226** (1998), 251–258.

[15] R.P. Pant, *Non-compatible mappings and common fixed points*, Soochow J. Math., **26** (2000), 29–35.

[16] R. P. Pant, *A common fixed point theorem under a new condition*, Indian. J. Pure Appl. Math., **30**(2) (1999), 147–152.

[17] R.P. Pant, R.K. Bisht, *Occasionally weakly compatible mappings and fixed points*, Bull. Belg. Math. Soc. Simon Stevin, **19** (2012), 655–661.

[18] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Pub. Inst. Math., **32**(46) (1982), 149–153.

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