Mathematical Analysis and its Contemporary Applications
Volume 6, Issue 2, 2024, 43–59
doi: 10.30495/maca.2024.2025535.1098
ISSN 2716-9898

A hybrid inertial iterative method for fixed point problems and finite families of generalized equilibrium problems with applications

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Abstract. We propose a hybrid inertial iterative method for finding a common element of fixed points of a family of a general class of nonlinear nonexpansive mappings and a common solution of a family of generalized equilibrium problems. The sequence of the proposed hybrid inertial iterative method is established to converge strongly to a common element of the families. We also present the application of our main result. Our results extend, improve and generalize several results in the literature.

1. Introduction

The fixed point theory is a fundamental part of mathematics. In nineteenth century, the concept of fixed point theory was introduced by Brower and name it as Brower's fixed point theorem. This fundamental idea depends on conditions for existence of fixed point mappings. The fixed point theory is one of the most important tools that plays a significant role in many branches of sciences, engineering, economics and development of nonlinear analysis. The Banach Caccioppoli theorem [4] is the first result that attracted the attention of the researchers in this

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2020 *Mathematics Subject Classification.* 47H10; 54H25.

Key words and phrases. Inertial Iterative Method, Fixed Points problems, J_*- Nonexpansive Mappings, Generalized Equilibrium Problems.

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field and later the result has been improved by many authors (for more details see [2, 6, 9, 16, 17, 22, 23, 33] and references therein).

Due to the importance of the fixed point theory and the need to develop the method for finding fixed points of problems for functions from a space to its dual, a new and interesting concept of fixed points for mapping from a real normed space E to its dual space E^*, called $J-$fixed point has been proposed and studied (for more details see [8, 20, 30] and references therein). A long the line, Chidume and Eze [7] proposed a new class of maps called relatively weak$-J-$nonexpansive and developed an iterative process for finding a common element of the $J-$fixed points of countable family of such maps and zeros of inverse strongly monotone maps in the framework of Banach space. Chidume et al. [10] introduced and studied the maps called quasi$-\phi-J-$nonexpansive which have similar requirement with relatively weak $J-$nonexpansive. Very recently, Uba et al. [28] proposed and studied a hybrid method for solving a common solutions of a family of equilibrium problems, variational inequality problems and a common element of fixed point of a countable family of generalized nonexpansive maps. We observe that the maps (relatively weak $J-$nonexpansive and quasi$-\phi-J-$nonexpansive) coincide with J_*-nonexpansive maps in definition by the results of Uba et al. [28].

For accelerating the convergence of the sequence of an iterative algorithm, a method called an inertial - type algorithm was introduced and studied by Polyak [24]. Due to the importance of this method, a number of researchers have been working on it. It follows from the increasing interest in the class of inertial-type algorithms that many problems have been solved by using this method (for more details see [2, 13, 14, 23] and references therein).

Motivated by the results of Chidume et al. [10] and Uba et al. [28]. In this article, we present and study an inertial iterative algorithm for approximating a common element of the fixed point of an infinite family of generalized J_*-nonexpansive mappings and the solution set of the generalized equilibrium problems of a finite family of bifunctions. We also provide some applications of our main theorem in the framework of Banach space and Hilbert space. Furthermore, our results improves and extend the results in [7, 10, 28] and many results in the literatures.

2. Preliminaries

In this section, we present some basic concepts, definitions and Lemmas used in proving our main results.

Let E be a real Banach space with a norm $\|.\|$ and its dual space denoted by E^*. Consider $\mathbb{R}$ and $\mathbb{N}$ as the set of real numbers and positive integers respectively. Let $C \subset E$ be closed and convex with JC also closed and convex, where $J : E \longrightarrow 2^{E^*}$ is a normalized duality map on E defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2, \ \forall x \in E\},$$

where $\langle ., . \rangle$ denotes the valve x^*. at x. It is well known that if E is smooth, strictly convex and reflexive, then J^{-1} exists (see [26]). $J^{-1} : E^* \longrightarrow E$ is the normalized duality map on E^*, and $J^{-1} = J_*$, $JJ_* = I_{E^*}$ and $J_*J = I_E$, where I_{E^*} and I_E are identity on E^* and E respectively. Furthermore, it follows from a well known property of J that if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E (see [11, 26]). Consider a map $T : D(T) \subset E \longrightarrow E$, a point $x^* \in D(T)$ denotes the fixed point of T provided that $Tx^* = x^*$ and the set of fixed point of T is denoted by $F(T) = \{x^* \in D(T) : Tx^* = x^*\}$.

Consider the variational inequality problem, which was first introduced and studied by Stampacchia [25] in 1964 (see also [31, 32]). For a monotone operator $G : C \longrightarrow E^*$, the variational inequality problem is to find $x^* \in C$ such that

$$\langle Gx^*, y - x^* \rangle \geq 0, \ \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by

$$VIP(C, G) = \{x^* \in C : \langle Gx^*, y - x^* \rangle \geq 0, \ \forall y \in C\}.$$

This problem plays an important role in the development of nonlinear analysis as well as related to convex minimization problems, zeros of nonlinear operators and fixed point problems. The equilibrium problem is a problem mostly applied in solving optimization problem which has been introduced and studied by Blum and Oettli [5]. Let $B : C \times C \longrightarrow \mathbb{R}$ be a bifunction, then the equilibrium problem for B is to find $x^* \in C$

$$B(x^*, y) \geq 0, \ \forall y \in C. \tag{2}$$

The set of solutions of (2) is denoted by

$$EP(B) = \{x^* \in C : B(x^*, y) \geq 0, \ \forall y \in C\}.$$

Due to the evolving of fixed point theory, we consider the $J-$ fixed point of certain mappings and the equilibrium problem as follows: Let $B : JC \times JC \longrightarrow \mathbb{R}$ be a bifunction, the equilibrium problem is to find $x^* \in C$ such that

$$B(Jx^*, Jy) \geq 0, \ \forall y \in C. \tag{3}$$

The set of solutions of (3) is denoted by

$$EP(B) = \{x^* \in C : B(Jx^*, Jy) \geq 0, \ \forall y \in C\}.$$

The generalized equilibrium problem is to find $x^* \in C$ such that

$$B(Jx^*, Jy) + \langle Gx^*, y - x^* \rangle \geq 0, \ \forall y \in C. \tag{4}$$

The set of solutions of (4) is denoted by

$$GEP(B, G) = \{x^* \in C : B(Jx^*, Jy) + \langle Gx^*, y - x^* \rangle \geq 0, \ \forall y \in C\}.$$

Several problems in optimization, physics and economics reduce to finding a solution of (3) (for more details see [5, 12, 31] and references therein). Many research have been studied equilibrium problems, variational inequality problems and fixed point problems together with their generalization using different classes of mappings (for more details see [**17**, **22**, **32**, **34**] and references therein).

Definition 2.1. [**18**, **19**] Let C be a nonempty closed and convex subset of a real Banach space E and T be a map from C to E. The map T is called generalized nonexpansive if $\{x \in C : Tx = x\} \neq \emptyset$ and $\phi(Tx, p) \leq \phi(x, p)$, for all $x \in C$ and $p \in F(T)$.

Definition 2.2. [**3**, **17**] Let E be a smooth real Banach space with E^* as it dual. The function $\phi : E \times E \longrightarrow \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E, \tag{5}$$

is called Lyapunov functional, where J is the normalized duality map. Consider that if a real Hilbert space $H = E$, then (5) reduces to $\phi(x, y) =\| x - y \|^2$ for all $x, y \in H$. Also, it follows from the properties of ϕ that the following hold and can be verified from its definition

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2, \ \forall x, y \in E, \tag{6}$$

$$\phi(x, y) \leq \|x\|\|Jx - Jy\| + \|y\|\|x - y\|, \ \forall x, y, \in E \tag{7}$$

Definition 2.3. [**18**, **19**] A map R from E onto C is said to be a retraction if $R = R^2$. The map R is said to be sunny if $R(Rx + \delta(x - Rx)) = Rx$ for all $x \in E$ and $\delta \leq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction R from E onto C.

NST - Condition: Let C be a closed subset of a Banach space E. Let $\{T_n\}$ and Ω be two families of generalized nonexpansive maps of C into E such that $\cap_{n=1}^{\infty} F(T_n) = F\Omega) \neq \emptyset$, where $F(T_n)$ is the set of fixed points of $\{T_n\}$ and $F(\Omega)$ is the set of common fixed points of Ω.

Definition 2.4. [**18**, **21**] The sequence $\{T_n\}$ satisfies the NST-condition with Ω if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Longrightarrow \lim_{n\to\infty} \|x_n - Tx_n\| = 0, \ \forall T \in \Omega.$$

Remark 2.5. If $T_n = T$ for all $n \geq 1$, then $\{T_n\}$ satisfies the NST-condition with $\{T\}$. If $\Omega = \{T\}$ a singleton, $\{T_n\}$ satisfies the NST-condition with $\{T\}$.

Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^*. Let J be the normalized duality map on E and J_* be the normalized duality map on E^*. Under this setting, J^{-1} exists and $J^{-1} = J*$. By considering these notations, we have the following definitions.

Definition 2.6. [27] A map $T : C \longrightarrow E^*$ is called J_*-closed if $(J_* \circ T) : C \longrightarrow E$ is a closed map, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \longrightarrow x$ and $(J_* \circ T)x_n \longrightarrow y$, then $(J_* \circ T)x = y$.

Definition 2.7. [8] A point $x^* \in C$ is called a $J-$fixed point of T if $Tx^* = x^*$. The set of $J-$fixed point of T is denoted by $F_J(T)$.

Definition 2.8. [27] A map $T : C \longrightarrow E^*$ is called generalized J_*-nonexpansive if $F_J(T) \neq \emptyset$, and $\phi(p, (J_* \circ T)x) \leq \phi(p, x)$, for all $x \in C$ and for all $p \in F_J(T)$.

Remark 2.9. For more details on the example of generalized J_*-nonexpansive maps in Hilbert space and Banach space (see [7, 27]).

Definition 2.10. A sequence $\{T_n\}$ of maps from C to E^* is said to satisfy the NST-condition with Ω if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Longrightarrow \lim_{n\to\infty} \|Jx_n - Tx_n\| = 0, \ \forall T \in \Omega.$$

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. For solving equilibrium problem, we consider a bifunction $B : JC \times JC \longrightarrow \mathbb{R}$ satisfies the following conditions [5]:

(L_1) $B(x^*, x^*) = 0, \forall x^* \in JC$;

(L_2) B is monotone, 1.e, $B(x^*, y^*) + B(y^*, x^*) \leq 0, \ \forall x^*, y^* \in JC$;

(L_3) for each $x^*, y^*, z^* \in JC, \limsup_{\delta\to 0} B(\delta z^* + (1-\delta)x^*, y^*) \leq B(x^*, y^*)$;

(L_4) for each $x^* \in JC, y^* \mapsto B(x^*, y^*)$ is convex and lower semicontinuous.

Lemma 2.1. [17] *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.*

Remark 2.11. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded, then by considering (7) it is observe that the converse of Lemma 2.1 is also true.

Lemma 2.2. [6] *Let E be a uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{\|x \in E : \|x\| \leq r\}$. Then for any given sequence $\{x_n\}_{n=1}^N \subset B_r(0)$ and for any given sequence $\{\lambda\}_{n=1}^N$ of positive numbers such that $\sum_{n=1}^N \lambda_n = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow [0, \infty), g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\|\sum_{n=1}^N \lambda_n x_n\|^2 \leq \sum_{n=1}^N \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

Lemma 2.3. [15, 19] *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C, let $(x,z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$, for all $y \in C$;

(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x,z)$.

Lemma 2.4. [19] *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E.Then the following are equivalent:*

(i) C is a sunny generalized nonexpansive retract of E;

(ii) C is a generalized nonexpansive retract of E;

(iii) JC is closed and convex.

Lemma 2.5. [15] *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E to C is uniquely determined.*

Lemma 2.6. [5] *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let $B : JC \times JC \longrightarrow \mathbb{R}$ be a bifunction satisfying $(L1) - (L_4)$. For $r > 0$ and let $x \in E$. Then there exists $z \in C$ such that*

$$B(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.7. *([1, 33]) Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E such that JC is closed and convex . Let $G : C \longrightarrow E^*$ be a monotone mapping and $B : JC \times JC \longrightarrow \mathbb{R}$ be a bifunction satisfying $(L1) - (L_4)$. For any given number $r > 0$ and $x \in E$, define a mapping*

$T_r(x) : E \longrightarrow C$ by

$$T_r(x) = \{z \in C : B(Jz, Jy) + \langle Gx, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\},$$

for every $x \in E$. The mapping T_r has the following properties:

(p_1) T_r is single-valued;

(p_2) T_r is a firmly nonexpansive-type mapping,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

for all $x, y \in E$;

(p_3) $F(T_r) = GEP(B,G)$;

(p_4) $JGEP(B,G)$ is a closed convex set of C;

(p_5) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q,x), \ \forall q \in F(T_r), \ x \in E$.

Lemma 2.8. [27] *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized J_*−nonexpansive map from C to E^* such* *that* $F_J(T) \neq \emptyset$*, then* $F_J(T)$ *and* $JF_J(T)$ *are closed. Furthermore, if* $JF_J(T)$ *is convex, then* $F_J(T)$ *is a sunny generalized nonexpansive retract of* E*.*

3. Main results

Theorem 3.1. *Let* E *be a uniformly smooth and uniformly convex real Banach space* E *with* E^* *as the dual space of* E*. Let* C *be a nonempty closed and convex subset of* E *such that* JC *is closed and convex. Let* $B_1, B_2 : JC \times JC \longrightarrow \mathbb{R},\ k = 1, 2, ..., p$ *be a finite family of bifunctions satisfying* $(L_1) - (L_4)$ *and* $G_1, G_2 : C \longrightarrow E^*$ *be a finite family of continuous monotone mappings. Let* $T_n : C \longrightarrow E^*$*,* $n = 1,2,3,\ ...$ *be an infinite family of generalized* J_*-*nonexpansive mappings and* Ω *be a family of* J_*- *closed and generalized* J_*-*nonexpansive mappings from* C *to* E^* *such that* $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Omega) \neq \emptyset$ *and* $\Gamma := F_J(\Omega) \cap \big[\cap_{k=1}^{2} GEP(B_k, G_k)\big] \neq \emptyset$*. Assume that* $JF_J(\Omega)$ *is convex and* $\{T_n\}$ *satisfies the* $NST-$*condition with* Ω*. Let* $\{x_n\}$ *generated by the following iterative process*

$$\begin{cases} x_1 = x \in C,\ C_1 = C; \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = J^{-1}\big(\alpha_1 Jw_n + \alpha_2 Jw_n + \alpha_3 JT_n w_n\big), \\ z_n \ni B_1(z_n, y) + \langle G_1 y_n, y - z_n\rangle + \dfrac{1}{r_n}\langle y - z_n, Jz_n - Jy_n\rangle \geq 0,\ \forall y \in C, \\ \omega_n \ni B_2(\omega_n, y) + \langle G_2 y_n, y - \omega_n\rangle + \dfrac{1}{r_n}\langle y - \omega_n, J\omega_n - Jy_n\rangle \geq 0,\ \forall y \in C, \\ u_n = J^{-1}\big(\beta_1 Jw_n + \beta_2 Jz_n + \beta_3 J\omega_n\big), \\ C_{n+1} = \big\{u \in C_n : \phi(u, u_n) \leq \phi(u, w_n)\big\}, \\ x_{n+1} = R_{C_{n+1}}x,\ \forall n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ *and* $\beta_1, \beta_2, \beta_3 \in (0, 1)$ *satisfying* $\alpha_1 + \alpha_2 + \alpha_3 = 1$ *and* $\beta_1 + \beta_2 + \beta_3 = 1$ *respectively.* $\theta_n \subset (0, 1)$ *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$*. We shall define*

$$T_{k,r}(x) := \{u \in C : B_k(Ju, Jy) + \langle G_k x, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\} \tag{9}$$

for all $x \in E,\ k = 1, 2$*. Then ,* $\{x_n\}$ *converges strongly to* $R_\Gamma x$*, where* R_Γ *is the sunny generalized nonexpansive retraction of* E *onto* Γ*.*

Proof. We consider the proof as follows:

Step 1 : We show that $\{x_n\}$ is well defined. We observe that JC_1 is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Consider that for any $u \in C_n$, we know that $\phi(u, u_n) \leq \phi(u, w_n)$ is equivalent to

$$0 \leq \| w_n \|^2 - \| u_n \|^2 - 2\langle u, Jw_n - Ju_n\rangle.$$

Therefore, by induction JC_n is closed and convex for each $n \geq 1$. Then by Lemma 2.4, we have C_n as a sunny generalized retract of E for each $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

Step 2 : We show the expected limit $R_\Gamma x$ exists as a point in C_n, for all $n \geq 1$. We first establish that $\Gamma \subset C_n$ for all $n \geq 1$ and Γ is a sunny generalized retract of E. We observe that $\Gamma \subset C_1$, since $C_1 = C$. Suppose that for some $n \in \mathbb{N}$, $\Gamma \subset C_n$. Let $z \in \Gamma$ and using the definition of u_n from (8), we have the following estimates:

$$\begin{aligned}
\phi(z, u_n) &= \phi(z, J^{-1}(\beta_1 J w_n + \beta_2 J z_n + \beta_3 J\omega_n)) \\
&\leq \beta_1 \, [\| z \|^2 - 2\langle z, Jw_n\rangle + \|w_n\|^2] + \beta_2 \, [\| z \|^2 - 2\langle z, Jz_n\rangle + \| z_n \|^2] \\
&\quad + \beta_3 \, [\| z \|^2 - 2\langle z, J\omega_n\rangle + \|\omega_n \|^2] \\
&= \beta_1 \, \phi(z, w_n) + \beta_2 \, \phi(z, z_n) + \beta_3 \, \phi(z, \omega_n). \qquad (10)
\end{aligned}$$

By considering (8) and (9), we observe that $z_n = T_{1,r_n} y_n$, for all $n \in \mathbb{N}$ and from the fact that $\{T_n\}$ is an infinite family of generalized J_*-nonexpansive mappings, we obtain

$$\begin{aligned}
\phi(z, z_n) &= \phi(z, T_{1,r_n} y_n) \\
&\leq \phi(z, y_n) \\
&= \phi(z, J^{-1}(\alpha_1 J w_n + \alpha_2 J w_n + \alpha_3 J T_n w_n)) \\
&\leq \alpha_1 \, [\| z \|^2 - 2\langle z, Jw_n\rangle + \| w_n \|^2] + \alpha_2 \, [\| z \|^2 - 2\langle z, Jw_n\rangle + \| w_n \|^2] \\
&\quad + \alpha_3 \, [\| z \|^2 - 2\langle z, J(J_* \circ T_n) w_n\rangle + \| T_n w_n \|^2] \\
&= \alpha_1 \, \phi(z, w_n) + \alpha_2 \, \phi(z, w_n) + \alpha_3 \, \phi(z, (J_* \circ T_n) w_n) \\
&\leq \alpha_1 \, \phi(z, w_n) + \alpha_2 \, \phi(z, w_n) + \alpha_3 \, \phi(z, w_n) \\
&= \phi(z, w_n). \qquad (11)
\end{aligned}$$

Similarly, from the fact that $\{T_n\}$ is an infinite family of generalized J_*-nonexpansive mappings, using Lemma 2.2 and taking the advantage of $\omega_n = T_{2,r_n} y_n$ from (8) and (9), then we estimate as follows:

$$\begin{aligned}
\phi(z, \omega_n) =& \phi(z, T_{2,r_n} y_n) \\
\leq& \phi(z, y_n) \\
=& \phi(z, J^{-1}(\alpha_1 J w_n + \alpha_2 J w_n + \alpha_3 J T_n w_n)) \\
\leq& \alpha_1 \, [\| z \|^2 - 2\langle z, Jw_n\rangle + \| w_n \|^2] + \alpha_2 \, [\| z \|^2 - 2\langle z, Jw_n\rangle + \| w_n \|^2] \\
& + \alpha_3 \, [\| z \|^2 - 2\langle z, J(J_* \circ T_n) w_n\rangle + \| T_n w_n \|^2] \\
& - \alpha_1 \alpha_3 g(\| J w_n - J(J_* \circ T_n) w_n \|) \\
=& \alpha_1 \, \phi(z, w_n) + \alpha_2 \, \phi(z, w_n) + \alpha_3 \, \phi(z, (J_* \circ T_n) w_n) \alpha_1 \alpha_3 g(\| J w_n - T_n w_n \|) \\
\leq& \alpha_1 \, \phi(z, w_n) + \alpha_2 \, \phi(z, w_n) + \alpha_3 \, \phi(z, w_n) - \alpha_1 \alpha_3 g(\| J w_n - T_n w_n \|) \\
\leq& \phi(z, w_n) - \alpha_1 \alpha_3 g(\| J w_n - T_n w_n \|) \\
\leq& \phi(z, w_n). \qquad (12)
\end{aligned}$$

It follows from (10), (11) and (12) that

$$\begin{aligned}\phi(z,u_n) &\leq \beta_1\phi(z,w_n)+\beta_2\phi(z,w_n)+\beta_3\phi(z,w_n)-\beta_3\alpha_1\alpha_3 g(\| Jw_n - T_n w_n \|)\\ &= \phi(z,w_n)-\beta_3\alpha_1\alpha_3 g(\| Jw_n - T_n w_n \|).\end{aligned}$$

Implies that

$$\phi(z,u_n) \leq \phi(z,w_n)-\beta_3\alpha_1\alpha_3 g(\| Jw_n - T_n w_n \|). \tag{13}$$

Therefore $\phi(z,u_n)\leq\phi(z,w_n)$ and we obtain $z\in C_{n+1}$ which conclude that $\Gamma\subset C_n$ for all $n\geq 1$. Also by Lemma 2.7 $JGEP(B_k,G_k)$ is closed and convex for each k. Furthermore, from Lemma 2.8 and by our assumption, we have that $(F_J(\Omega))$ is closed and convex. Since J is one-to-one and E is uniformly convex, hence we obtain

$$J\big(F_J(\Omega)\cap\big[\cap_{k=1}^{2} GEP(B_k,G_k)\big]\big)=JF_J(\Omega)\cap J\big[\cap_{k=1}^{2} GEP(B_k,G_k)\big],$$

implies that $J(\Gamma)$ is closed and convex. Now, by considering Lemma 2.4, we get that Γ is a sunny generalized retract of E. Hence by Lemma 2.5 we obtain that $R_\Gamma x$ exist as a point in C_n for all $n\geq 1$.

Step 3 : We show that the sequence $\{x_n\}$ converges to some point $x^*\in C$. By Lemma 2.3 (ii) and $x_n=R_{C_n}x$, we estimate as

$$\phi(x,x_n)=\phi(x,R_{C_n}x)\leq\phi(x,z)-\phi(R_{C_n}x,z)\leq\phi(x,z),$$

for all $z\in F_J(\Omega)\cap GEP(B_k,G_k)\subset C_n,\ k=1,2$. Implies that $\{\phi(x,x_n)\}$ is bounded. Therefore by (6) $\{x_n\}$ is bounded. Furthermore, $\{x_n\}$ is bounded implies that $\{w_n\}$ is bounded. By considering $x_{n+1}=R_{C_{n+1}}x\in C_{n+1}\subset C_n,\ x_n=R_{C_n}x\in C_n$ and using Lemma 2.3 (ii), we obtain

$$\phi(x,x_n)\leq\phi(x,x_{n+1}),\ \forall n\in\mathbb{N}.$$

Implies that $\lim\limits_{n\to\infty}\phi(x,x_n)$ exists. Furthermore, from Lemma 2.3 (ii) and $x_n=R_{C_n}x,\ \forall m,n\in\mathbb{N}$ with $m>n$, we get that

$$\begin{aligned}\phi(x_n,x_m) &= \phi(R_{C_n}x,x_m)\leq\phi(x,x_m)-\phi(x,R_{C_n}x)\\ &= \phi(x,x_m)-\phi(x,x_m)\longrightarrow 0\ as\ n\to\infty.\end{aligned}$$

Now by Lemma 2.1, we have $\| x_n-x_m \|\longrightarrow 0$ as $m,n\to\infty$. Thus, $\{x_n\}$ is a Cauchy sequence in C and this Implies that

$$\lim_{n\to\infty}\| x_{n+1}-x_n \|=0. \tag{14}$$

Furthermore, since $\{x_n\}$ is Cauchy in C, then there exists a point $x^*\in C$ such that $x_n\longrightarrow x^8$.

Step 4 : We show that $x^* \in F_J(\Omega)$. Now, we observe from the definition of w_n that

$$\| w_n - x_n \| = \| \theta_n(x_n - x_{n-1}) \| \leq \| x_n - x_{n-1} \| .$$

Using (14), we have

$$\lim_{n\to\infty} \| w_n - x_n \| = 0. \tag{15}$$

From the fact that $\{w_n\}$ is bounded and by Remark 2.11, we obtain

$$\lim_{n\to\infty} \phi(w_n, x_n) = 0. \tag{16}$$

By (14) and (15), we arrive at

$$\lim_{n\to\infty} \| x_{n+1} - w_n \| = 0. \tag{17}$$

Also, from Remark 2.11, we get

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0. \tag{18}$$

From the definition of C_{n+1} and x_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, w_n).$$

By (18), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Taking the advantage of Lemma 2.1, we have

$$\lim_{n\to\infty} \| x_{n+1} - u_n \| = 0. \tag{19}$$

From (17) and (19), we conclude that

$$\lim_{n\to\infty} \| w_n - u_n \| = 0. \tag{20}$$

From the fact that J is uniformly continuous on bounded subsets of E and by (20), we obtain

$$\lim_{n\to\infty} \| Jw_n - Ju_n \| = 0. \tag{21}$$

Similarly, by the definition of C_{n+1} and x_{n+1}, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, w_n).$$

Using (18), we get that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 2.1, we obtain

$$\lim_{n\to\infty} \| x_{n+1} - y_n \| = 0. \tag{22}$$

By (14) and (22), we arrive at

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{23}$$

It follows from (13) that

$$\begin{aligned} 0 &\leq \beta_3\alpha_1\alpha_3 g(\| Jw_n - T_n w_n\|) \leq \phi(z, w_n) - \phi(z, u_n) \\ &\leq 2 \| z \| . \| Jw_n - Ju_n \| + \| w_n - u_n \| M, \end{aligned}$$

for some $M > 0$. By (20), (21) and the properties of g, we conclude that

$$\lim_{n\to\infty} \| Jw_n - T_n w_n \| = 0.$$

From the fact that $\{T_n\}_{n=1}^{\infty}$ satisfies the NST condition with Ω, we get that

$$\lim_{n\to\infty} \| Jw_n - Tw_n \| = 0, \ \forall T \in \Omega.$$

Since $x_n \longrightarrow x^*$, $(as\ n \to \infty)$, it follows from (15 that $w_n \longrightarrow x^* \in C$. Therefore assume that $(J_* \circ T)w_n \longrightarrow y^*$, from the fact that T is J_*- closed, we have $y^* = (J_* \circ T)x^*$. Taking the advantage of J as uniformly continuous on bounded subsets of E, we get $Jw_n \longrightarrow Jx^*$ and $J(J_* \circ T)w_n \longrightarrow Jy^*$ $(as\ n \to \infty)$. Thus

$$\lim_{n\to\infty} \| Jw_n - J(J_* \circ T)w_n \| = \| Jw_n - Tw_n \| = 0, \ \forall T \in \Omega.$$

Implies that

$$\|Jx^* - Jy^*\| = \|Jx^* - J(J_* \circ T)x^*\| = \|Jx^* - Tx^*\| = 0.$$

Hence

$$x^* \in F_J(\Omega).$$

Step 5 : We show that $x^* \in \cap GEP(B_k, G_k) = F(T_{k,r}),\ k = 1, 2$. Let $z \in \Gamma$, from the fact that $z_n = T_{1,r_n} y_n$ and Lemma 2.7, we estimate as follows:

$$\begin{aligned} \phi(z_n, y_n) &= \phi(T_{1,r_n} y_n, y_n) \\ &\leq \phi(z, y_n) - \phi(z, T_{1,r_n} y_n) \\ &= \omega(z, y_n) - \phi(z, z_n). \end{aligned} \tag{24}$$

Now, since $w_n, x_n \longrightarrow x^*$ ($as \longrightarrow \infty$), then it follows from (20) and (23) that $u_n \longrightarrow x^*$ and $y_n \longrightarrow x^*$ ($as\ n \to \infty$) respectively. It also follows from (10), (**??**) and (13) that

$$\phi(z, u_n) \leq \beta_1 \phi(z, w_n) + \beta_2 \phi(z, z_n) + \beta_3 \phi(z, w_n) \leq \phi(z, w_n). \tag{25}$$

Since $u_n, w_n \longrightarrow x^*$ ($as\ n \to \infty$) and by (25), we conclude that $\phi(z, z_n) \longrightarrow \phi(z, x^*)$ as $n \longrightarrow \infty$. Since $y_n \longrightarrow x^*$ as $n \to \infty$, from (24), we have $\phi(z, y_n) - \phi(z, z_n) \longrightarrow 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \phi(z_n, y_n) = 0.$$

By Lemma 2.1, we get

$$\lim_{n\to\infty} \parallel z_n - y_n \parallel = 0. \tag{26}$$

Also, since $y_n \longrightarrow x^*$ as $n \to \infty$, by (26) $z_n \longrightarrow x^*$ as $n \to \infty$. From uniformly continuity of J on bounded subsets of E together with (26), we arrive at $\lim\limits_{n\to\infty} \parallel Jz_n - Jy_n \parallel = 0$ as $n \to \infty$. Taking the advantage of $r_n \in [a, \infty)$, we obtain

$$\lim_{n\to\infty} \frac{\parallel Jz_n - Jy_n \parallel}{r_n} = 0. \tag{27}$$

We note from $z_n = T_{1,r_n} y_n$ that

$$\langle G_1 y_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, Jz_n - Jy_n\rangle \geq -B_1(Jz_n, Jy),\ \forall y \in C.$$

By (L_2), we observe that

$$\langle G_1 y_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, Jz_n - Jy_n\rangle \geq -B_1(Jz_n, Jy) \geq B_1(Jy, Jz_n),\ \forall y \in C.$$

Implies that

$$\langle G_1 y_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, \frac{Jz_n - Jy_n}{r_n}\rangle \geq B_1(Jy, Jz_n),\ \forall y \in C. \tag{28}$$

Consider $\delta \in (0, 1]$ and $y \in C$, let $y^*_\delta = \delta Jy + (1 - \delta)Jx^*$. Using the fact that JC is convex, we have $y^*_\delta \in JC$. Then, it follows from (28)

$$\begin{aligned} & \langle G_1 y^*_\delta, y^*_\delta - z_n\rangle - \langle G_1 y^*_\delta, y^*_\delta - z_n\rangle \\ \geq\ & -\langle G_1 y_n, y^*_\delta - z_n\rangle - \langle y^*_\delta - z_n, \frac{Jz_n - Jy_n}{r_n}\rangle + B_1(y^*_\delta, Jz_n). \end{aligned}$$

This implies that

$$\begin{aligned} & \langle G_1 y^*_\delta, y^*_\delta - z_n\rangle \\ \geq & \langle G_1 y^*_\delta, y^*_\delta - z_n\rangle - \langle G_1 y_n, y^*_\delta - z_n\rangle - \langle y^*_\delta - z_n, \frac{Jz_n - Jy_n}{r_n}\rangle + B_1(y^*_\delta, Jz_n) \\ = & \langle G_1 y^*_\delta - G_1 z_n, y^*_\delta - z_n\rangle + \langle G_1 z_n - G_1 y_n, y^*_\delta - z_n\rangle - \langle y^*_\delta - z_n, \frac{Jz_n - Jy_n}{r_n}\rangle \\ & + B_1(y^*_\delta, Jz_n). \end{aligned}$$

Since $z_n, y_n \longrightarrow x^*$ ‘($as\ n \to \infty$) and from the continuity of G_1, we have

$$G_1 z_n - G_1 y_n \longrightarrow 0\ as\ n \to \infty \tag{29}$$

Taking the advantage of G_1 as monotone, we obtain

$$\langle G_1 y^*_\delta - G_1 z_n, y^*_\delta - z_n\rangle \geq 0.$$

It follows from (27), (29) and (L_4) that

$$B_1(y^*_\delta, Jx^*) \leq \liminf_{n\to\infty} B_1(y^*_\delta, Jz_n) \leq \lim_{n\to\infty} \langle G_1 y^*_\delta, y^*_\delta - z_n\rangle = \langle G_1 y^*_\delta, y^*_\delta - x^*\rangle.$$

By (L_1) and (L_4), we obtain the following estimate:

$$\begin{aligned}
0 &= B_1(y_\delta^*, y_\delta^*)\\
&\leq \delta B_1(y_\delta^*, Jy) + (1-\delta)B_1(y_\delta^*, Jx^*)\\
&\leq \delta B_1(y_\delta^*, Jy) + (1-\delta)\langle G_1 y_\delta^*, y_\delta^* - x^*\rangle\\
&\leq \delta B_1(y_\delta^*, Jy) + (1-\delta)\langle G_1 y_\delta^*, \delta y + (1-\delta)x^* - x^*\rangle\\
&= \delta B_1(y_\delta^*, Jy) + (1-\delta)\langle G_1 y_\delta^*, y - x^*\rangle,
\end{aligned}$$

therefore

$$B_1(y_\delta^*, Jy) + (1-\delta)\langle G_1 y_\delta^*, y - x^*\rangle \geq 0,\ \forall y \in C.$$

Letting $\delta \longrightarrow 0$, we obtain

$$B_1(Jx^*, Jy) + \langle G_1 x^*, y - x^*\rangle \geq 0,\ \forall y \in C.$$

Hence, we have $Jx^* \in JGEP(B_1, G_1)$, which implies that $x^* \in GEP(B_1, G_1)$. Similarly, by considering $z \in \Gamma$, $\omega_n = T_{2,r_n} y_n$ and using similar argument, one can show that $Jx^* \in JGEP(B_2, G_2)$. Also, by applying similar argument, one can also show that $x^* \in GEP(B_k, G_k)$, $k = 1, 2$. Therefore $x^* \in \cap_{k=1}^2 GEP(B_k, G_k)$.

Step 6 : We show that $x^* = R_\Gamma x$. By considering Lemma 2.3(ii), we have

$$\phi(x, R_\Gamma x) \leq \phi(x, x^*) - \phi(R_\Gamma x, x^*) \leq \phi(x, x^*). \tag{30}$$

Also, it follows from the definition of $x_{n+1} = R_{C_n} x$, Lemma 2.3(ii) and $x^* \in \Gamma \subset C_n$ that

$$\begin{aligned}
\phi(x, x_{n+1}) &\leq \phi(x, x_{n+1}) + \phi(x_{n+1}, R_\Gamma x)\\
&= \phi(x, R_{C_{n+1}} x) + \phi(R_{C_{n+1}} x, R_\Gamma x)\\
&\leq \phi(x, R_\Gamma x).
\end{aligned}$$

Taking the advantage of $x_n \longrightarrow x^*$ as $n \to \infty$ and by taking the limits on both side of the last inequality, we arrive at

$$\phi(x, x^*) \leq \phi(x, R_\Gamma x). \tag{31}$$

Also, by considering the inequalities (30) and (31), we get that

$$\phi(x, x^*) = \phi(x, R_\Gamma x).$$

By the uniqueness of R_Γ from Lemma 2.5, we conclude that $x^* = R_\Gamma x$. This completes the proof. □

4. Applications

Corollary 4.1. *Let E be a uniformly smooth and uniformly convex real Banach space E with E^* as the dual space of E. Let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $B : JC \times JC \longrightarrow \mathbb{R}$, be a bifunctions satisfying $(L_1) - (L_4)$ and $G : C \longrightarrow E^*$ be a continuous monotone mappings. Let $T : C \longrightarrow E^*$, be a generalized J_*-nonexpansive mapping such that $\Gamma := F_J(\Omega) \cap GEP(B, G) \neq \emptyset$. Assume that $JF_J(\Omega)$ is convex. Let $\{x_n\}$ be a sequence generated by (8), then $\{x_n\}$ converges strongly to $R_\Gamma x$, where R_Γ is the sunny generalized nonexpansive retraction of E onto Γ.*

Proof. Letting $T_n := T$ for all $n \in \mathbb{N}$, consider $B := B_k$ and $G := G_k$ for any $k = 1, 2, ..., p$. Since $\{T_n\}$ satisfies the $NST-$condition with $\{T\}$ by Remark 2.5. Then, we obtain the desired result from Theorem 8. □

Corollary 4.2. *Let E be a uniformly smooth and uniformly convex real Banach space E with E^* as the dual space of E. Let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $B_1, B_2 : JC \times JC \longrightarrow \mathbb{R}$, $k = 1, 2, ..., p$ be a finite family of bifunctions satisfying $(L_1) - (L_4)$, $T_n : C \longrightarrow E^*$, $n =$ 1,2,3, ... be an infinite family of generalized J_*-nonexpansive mappings and Ω be a family of J_*- closed and generalized J_*-nonexpansive mappings from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Omega) \neq \emptyset$ and $\Gamma := F_J(\Omega) \cap \big[\cap_{k=1}^{2} GEP(B_k, G_k)\big] \neq \emptyset$. Assume that $JF_J(\Omega)$ is convex and $\{T_n\}$ satisfies the $NST-$condition with Ω. Let $\{x_n\}$ be a sequence generated by (8), then $\{x_n\}$ converges strongly to $R_\Gamma x$, where R_Γ is the sunny generalized nonexpansive retraction of E onto Γ.*

Proof. Letting $G_k = 0$ for any $k = 1, 2, ..., p$, in Theorem 8, we obtain the desired result. □

Corollary 4.3. *Let $E = H$ be a Hilbert space and C be a nonempty closed and convex subset of H. Let $B_1, B_2 : C \times C \longrightarrow \mathbb{R}$, $k = 1, 2, ..., p$ be a finite family of bifunctions satisfying $(L_1) - (L_4)$ and $G_1, G_2 : C \longrightarrow H$ be a finite family of continuous monotone mappings. Let $T_n : C \longrightarrow H$, $n =$ 1,2,3, ... be an infinite family of nonexpansive mappings and Ω be a family of closed and generalized nonexpansive mappings from C to H such that $\cap_{n=1}^{\infty} F(T_n) = F(\Omega) \neq \emptyset$ and $\Gamma := F(\Omega) \cap \big[\cap_{k=1}^{2} GEP(B_k, G_k)\big] \neq \emptyset$. Assume that $\{T_n\}$ satisfies the $NST-$condition* *with Ω. Let $\{x_n\}$ generated by the following iterative process*

$$\begin{cases} x_1 = x \in C, \ C_1 = C; \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \alpha_1 w_n + \alpha_2 w_n + \alpha_3 T_n w_n, \\ z_n \ni B_1(z_n, y) + \langle G_1 y_n, y - z_n\rangle + \dfrac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \geq 0, \ \forall y \in C, \\ \omega_n \ni B_2(\omega_n, y) + \langle G_2 y_n, y - \omega_n\rangle + \dfrac{1}{r_n}\langle y - \omega_n, \omega_n - y_n\rangle \geq 0, \ \forall y \in C, \\ u_n = \beta_1 w_n + \beta_2 z_n + \beta_3 \omega_n, \\ C_{n+1} = \big\{u \in C_n : \| u - u_n \| \leq \| u - w_n) \| \big\}, \\ x_{n+1} = P_{C_{n+1}} x, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$ and $\beta_1, \beta_2, \beta_3 \in (0,1)$ satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\beta_1 + \beta_2 + \beta_3 = 1$ respectively. $\theta_n \subset (0,1)$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. We shall define

$$T_{k,r}(x) := \{u \in C : B_k(u, y) + \langle G_k x, y - u\rangle + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in E$, $k = 1, 2$. Then, $\{x_n\}$ converges strongly to $P_\Gamma x$, where P_Γ is the metric projection of H onto Γ.

Proof. By the framework of Hilbert space, for all $x, y \in H$, $\phi(x, y) = \| x - y \|^2$ with J as identity mapping. Then, the desired result follows from Theorem 8. □

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Email address: `lawalu4@gmail.com`,

Received : *March* 2024
Accepted : *July* 2024