

Approximate multi-Euler-Lagrange additive mappings in intuitionistic fuzzy stability

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ABSTRACT. The main aim of this study is to prove the stability and hyperstability of multi-Euler-Lagrange additive mappings in the setting of intuitionistic fuzzy normed spaces by an alternative fixed point theorem.

1. Introduction

In 1965, Zadeh [31] introduced the notion of fuzzy sets, a powerful handset for modelling uncertainty and vagueness in various problems arising in science and engineering. The concept of intuitionistic fuzzy normed spaces was initially introduced by Saadati and Park in [24]. Then, Saadati et al. obtained a modified case of intuitionistic fuzzy normed spaces by improving the separation condition and strengthening some conditions in the definition of [25].

The concept of stability for functional equations arises when we replace the functional equation with an inequality which acts as a perturbation of the equation. In 1940, Ulam [26] asked the question concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [15] for the linear functional equation of Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [20] for linear mappings by

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considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference with a general control function in the spirit of Rassias approach. Next, many of mathematicians were attracted and motivated to investigate the stability problems of functional equations in various spaces.

Recall that a mapping A is called *Euler-Lagrange additive* if it satisfies the equation

$$A(ax + by) + A(bx + ay) = (a + b)[A(x) + A(y)], \quad (1)$$

where $a, b \in \mathbb{R} \setminus \{0\}$ are fixed with $a + b \neq 0, \pm 1$. In fact, J. M. Rassias [18, 19] introduced and investigated the stability problem of Ulam for (1). Next, Xu extended the definition above to several variables mappings [28].

It is worth mentioning that the fixed point theorems have been considered for various mappings and functional equations in [4], [5], [9], [11] and [25]. In other words, Cădariu and Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. In fact, they employed this fixed point theorem in the investigation of the Cauchy functional equation [12] and for the quadratic functional equation [11]. The generalized Hyers-Ulam stability of different functional equations in intuitionistic fuzzy normed spaces has been studied by several authors; see for example, [2], [3], [6], [7], [22], [23] and [29].

In this paper, we study some stability results concerning multi-Euler-Lagrange additive mappings in the setting of intuitionistic fuzzy normed spaces by a fixed point alternative method. Furthermore, we show that every multi-Euler-Lagrange additive mapping under some conditions can be hyperstable.

2. Definitions and Preliminaries

In this section, we restate the usual terminology, notations and conventions of the theory of intuitionistic fuzzy normed space, as in [17], [21], [22] and [23].

Let \leq_L be an order relation on the set $L = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2, x_1 + x_2 \leq 1\}$ defined by

$$(x_1, x_2) \leq_L (y_1, y_2) \iff x_1 \leq y_1, y_2 \leq x_2$$

for all $(x_1, x_2), (y_1, y_2) \in L$. It is easy to check that the pair (L, \leq_L) is a complete lattice (see also [21]). We denote the units of L by $0_L = (0, 1)$ and $1_L = (1, 0)$.

Definition 2.1. Let U be a non-empty set called the universe. An L -fuzzy set in U is defined as a mapping $\mathcal{F} : U \rightarrow L$. For each u in U , $\mathcal{F}(u)$ represents the degree (in L) to which u is an element of \mathcal{F} . An intuitionistic fuzzy set $\mathcal{F}_{\mu, \nu}$ in a universal set U is an object $\mathcal{F}_{\mu, \nu} = \{(\mu_{\mathcal{F}}(u), \nu_{\mathcal{F}}(u)) : u \in U\}$, where $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ belong to $[0, 1]$ for all $u \in U$ with $\mu_{\mathcal{F}}(u) + \nu_{\mathcal{F}}(u) \leq 1$. The numbers $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{F}_{\mu, \nu}$.

Definition 2.2. A triangular norm (t -norm) on L is a mapping $\mathcal{T} : L \times L \longrightarrow L$ satisfying the following conditions:

- (i) $\mathcal{T}(x, 1_L) = x$ (boundary condition) $(x \in L)$;
- (ii) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ (commutativity) $(x, y \in L)$;
- (iii) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ (associativity) $(x, y, z \in L)$;
- (iv) $x_1 \leq_L y_1$ and $x_2 \leq_L y_2 \implies \mathcal{T}(x_1, x_2) \leq_L \mathcal{T}(y_1, y_2)$ (monotonicity) $(x_1, x_2, y_1, y_2 \in L)$.

A t -norm \mathcal{T} on L is said to be *continuous* if, for any $x, y \in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y , respectively, $\lim_{n \rightarrow \infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y)$. For $x = (x_1, x_2), y = (y_1, y_2) \in L$, $\mathcal{T}(x, y) = (x_1 y_1, \min\{x_2 + y_2, 1\})$ and $\mathcal{M}(x, y) = (\min\{x_1, y_1\}, \max\{x_2, y_2\})$ are continuous t -norm [27].

Here, we define a sequence \mathcal{T}^n , recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, x^{(1)}, \dots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, x^{(1)}, \dots, x^{(n)}), x^{(n+1)})$$

for all $n \geq 2$ and $x^{(j)} \in L$.

Definition 2.3. A negator on L is a decreasing mapping $\mathfrak{N} : L \longrightarrow L$ satisfying $\mathfrak{N}(0_L) = 1_L$ and $\mathfrak{N}(1_L) = 0_L$. If $\mathfrak{N}(\mathfrak{N}(x)) = x$, for all $x \in L$, then \mathfrak{N} is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $\mathcal{N} : L \longrightarrow L$ satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. The standard negator on $[0, 1]$ is defined by $\mathcal{N}_s(x) = 1 - x$ for all $x \in [0, 1]$.

The following definitions of an intuitionistic fuzzy normed space are taken from [22].

Definition 2.4. Let $\mathcal{L} = (L, \leq_L)$. Suppose that X is a vector space, \mathcal{T} is a continuous t -norm on L and \mathcal{P} is an L -fuzzy set on $X \times (0, \infty)$ satisfying the following conditions:

- (i) $0 <_L \mathcal{P}(x, t)$;
- (ii) $\mathcal{P}(x, t) = 1_L$ if and only if $x = 0$;
- (iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
- (iv) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
- (v) The map $\mathcal{P}(x, \cdot) : (0, \infty) \longrightarrow L$ is continuous;
- (vi) $\lim_{t \rightarrow 0} \mathcal{P}(x, t) = 0_L$ and $\lim_{t \rightarrow \infty} \mathcal{P}(x, t) = 1_L$;

for all $x, y \in X$ and all $t, s > 0$. Then, the triple $(X, \mathcal{P}, \mathcal{T})$ is called an L -fuzzy normed space. In this case \mathcal{P} is called \mathcal{L} -fuzzy norm (briefly, L -fuzzy norm). If $\mathcal{P} = \mathcal{P}_{\mu, \nu}$ is an intuitionistic fuzzy set, then the triple $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be an intuitionistic fuzzy normed space (briefly, IFN-space). In this case, $\mathcal{P}_{\mu, \nu}$ is called an intuitionistic fuzzy norm on X ; some examples of IFN-spaces are provided in [27] and [29].

Note that, if \mathcal{P} is an L -fuzzy norm on X , then the following statements hold:

- (i) $\mathcal{P}(x, t)$ is non-decreasing with respect to t for all $x \in X$;
- (ii) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in X$ and $t > 0$.

Example 2.5. [27] Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(x, y) = (x_1 y_1, \min\{x_2 + y_2, 1\})$ for all $x = (x_1, x_2), y = (y_1, y_2) \in L$ and μ, ν be membership and non-membership degree, respectively, of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu(x, t), \nu(x, t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right) \quad (t \in \mathbb{R}_+).$$

Then, $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is an IFN-space.

Definition 2.6. Let $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ be an IFN-space.

- (1) A sequence $\{x_n\}$ in $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be *convergent* to a point x if $\mathcal{P}_{\mu, \nu}(x_n - x, t) \rightarrow 1_L$ as $n \rightarrow \infty$ for all $t > 0$;
- (2) A sequence $\{x_n\}$ in $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for every $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer N such that $(N_s(\epsilon), \epsilon) \leq_L \mathcal{P}_{\mu, \nu}(x_n - x_m, t)$ for all $m, n > N$, where N_s is the standard negator.;
- (3) $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is convergent to a point in $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

3. Intuitionistic fuzzy stability of multi-Euler-Lagrange additive mappings

We start this subsection by a definition, has been presented by Xu in [28].

Definition 3.1. A mapping $f : V^n \rightarrow W$ is called multi-Euler-Lagrange additive if it satisfies the Euler-Lagrange additive equation (1) in each of their n arguments, namely,

$$\begin{aligned} & [f(v_1, \dots, v_{i-1}, a_i v_i + b_i v'_i, v_{i+1}, \dots, v_n) + f(v_1, \dots, v_{i-1}, b_i v_i + a_i v'_i, \dots, v_n)] \\ & = (a_i + b_i) [f(v_1, \dots, v_{i-1}, v_i, \dots, v_n) + f(v_1, \dots, v_{i-1}, v'_i, \dots, v_n)], \end{aligned}$$

where $a_j, b_j \in \mathbb{R} \setminus \{0\}$ are fixed with $a_j + b_j \neq 0, \pm 1$.

In the sequel, consider $a_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ such that $a_{1j} + a_{2j} \neq 0$, where $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$. We write $a_i^{[n]}$ simply a_i when there is no ambiguity. For $x_1, x_2 \in V^n$ and a_1, a_2 as in the above, consider the following notations:

$$B_j = \sum_{i=1}^2 a_{ij} x_{ij} \quad \text{and} \quad B'_j = \sum_{i=1}^2 a_{3-i,j} x_{ij}, \quad (2)$$

where $j \in \{1, \dots, n\}$. Here, we bring a definition.

Definition 3.2. [8] We say a mapping $f : V^n \rightarrow W$

(i) satisfies (has) the *linear condition* in the j th variable if

$$f(z_1, \dots, z_{j-1}, a^* z_j, z_{j+1}, \dots, z_n) = a^* f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n),$$

for all $z_1, \dots, z_n \in V^n$, where $a^* \in \{a_{1j}, a_{2j}, a_{1j} + a_{2j}\}$;

(ii) has *zero condition* if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.

Remark 3.1. It is clear that if a mapping $f : V^n \rightarrow W$ satisfies the linear condition in the j th variable then it has zero condition in the same variable. Therefore, if f has the linear condition in each variable, then it has zero condition. It is used from this fact to prove the upcoming result which was proved in [8].

Theorem 3.2. [8] *For a mapping $f : V^n \rightarrow W$, the following assertions are equivalent:*

- (i) f is multi-Euler-Lagrange additive;
- (ii) f satisfies equation

$$\sum_{\substack{\mathfrak{B}_j \in \{B_j, B'_j\} \\ j \in \{1, \dots, n\}}} f(\mathfrak{B}_1, \dots, \mathfrak{B}_n) = \prod_{j=1}^n (a_{1j} + a_{2j}) \sum_{l_1, \dots, l_n \in \{1, 2\}} f(x_{l_1 1}, \dots, x_{l_n n}) \quad (3)$$

and the linear condition in each variable.

Let $a, b \in \mathbb{R} \setminus \{0\}$ be fixed with $a + b \neq 0, \pm 1$. If we put $a_{1j} = a$ and $a_{2j} = b$ in (3) for all $j \in \{1, \dots, n\}$, then this equation converts to the equation

$$\sum_{t_1, \dots, t_n \in \{(a, b), (b, a)\}} f(\mathfrak{B}_1^{t_1}, \dots, \mathfrak{B}_n^{t_n}) = m^n \sum_{l_1, \dots, l_n \in \{1, 2\}} f(x_{l_1 1}, \dots, x_{l_n n}), \quad (4)$$

where $m = a + b$, $\mathfrak{B}_j^{(a, b)} = ax_{1j} + bx_{2j}$ and $\mathfrak{B}_j^{(b, a)} = bx_{1j} + ax_{2j}$, whereas $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$.

In this section, we prove the Hyers-Ulam stability of multi-Euler-Lagrange additive mappings in intuitionistic fuzzy normed spaces, based on the next theorem that is a result of fixed point theory [13] and play a fundamental role to obtain our aim.

Theorem 3.3. (The fixed point alternative theorem) *Let (X, d) be a complete generalized metric space and $\mathcal{J} : X \rightarrow X$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $a \in X$, either $d(\mathcal{J}^n(a), \mathcal{J}^{n+1}(a)) = \infty$ for all $n \geq 0$, or there exists a $n_0 \in \mathbb{N}$ such that*

- (i) $d(\mathcal{J}^n(a), \mathcal{J}^{n+1}(a)) < \infty$ for all $n \geq n_0$;
- (ii) the sequence $\{\mathcal{J}^n(a)\}$ is convergent to a fixed point b^* of \mathcal{J} ;
- (iii) b^* is the unique fixed point of \mathcal{J} in the set $X_1 = \{b \in X : d(\mathcal{J}^{n_0}(a), b) < \infty\}$;
- (iv) $d(b, b^*) \leq \frac{1}{1-L} d(b, \mathcal{J}(b))$ for all $b \in X_1$.

Here and subsequently, we assume that all t -norms are as

$$\mathcal{T}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}),$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L$.

Given a mapping $f : V^n \rightarrow W$, for simplicity, we use the notation

$$\mathfrak{D}f(x_1, x_2) := \sum_{t_1, \dots, t_n \in \{(a,b), (b,a)\}} f(\mathfrak{B}_1^{t_1}, \dots, \mathfrak{B}_n^{t_n}) - m^n \sum_{l_1, \dots, l_n \in \{1,2\}} f(x_{l_1}, \dots, x_{l_n}),$$

for all $x_1, x_2 \in V^n$ in which $x_i = (x_{i1}, \dots, x_{in})$. With this notation, we have the next stability result for the functional equation (4).

Theorem 3.4. *Let $\beta \in \{1, -1\}$ be fixed and $\lambda \in \mathbb{R}_+$ with $\lambda \neq m^n$. Let V be a linear space and $(Z, \mathcal{P}'_{\mu, \nu}, \mathcal{T}')$ be an intuitionistic fuzzy normed space. Suppose that $\phi : V^n \times V^n \rightarrow Z$ is a mapping fulfilling*

$$\mathcal{P}'_{\mu, \nu}(\lambda^\beta \phi(x_1, x_2), t) \leq_L \mathcal{P}'_{\mu, \nu}(\phi(m^\beta x_1, m^\beta x_2), t), \quad (5)$$

for all $x_1, x_2 \in V^n$ and $t > 0$. If $(W, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is a complete intuitionistic fuzzy normed space and $f : V^n \rightarrow W$ is a mapping satisfying

$$\mathcal{P}'_{\mu, \nu}(\phi(x_1, x_2), t) \leq_L \mathcal{P}_{\mu, \nu}(\mathfrak{D}f(x_1, x_2), t), \quad (6)$$

for all $x_1, x_2 \in V^n$ and $t > 0$, then there exists a solution $\mathcal{A} : V^n \rightarrow W$ of (3) such that

$$\mathcal{P}'_{\mu, \nu}(\phi(x, 0), 2^n |m^n - \lambda|t) \leq_L \mathcal{P}_{\mu, \nu}(f(x) - \mathcal{A}(x), t) \quad (7)$$

for all $x \in X$ and $t > 0$. Moreover, if \mathcal{A} has the linear condition in each variable, then it is a unique multi-Euler-Lagrange additive mapping.

PROOF. For $\beta = 1$ and $\beta = -1$, we regard $\lambda < m^n$ and $\lambda > m^n$, respectively. Putting $x = x_1 = x_2$ in (6), we have

$$\mathcal{P}'_{\mu, \nu}(\phi(x, 0), t) \leq_L \mathcal{P}_{\mu, \nu}(2^n f(mx) - (2m)^n f(x), t), \quad (8)$$

for all $x \in V^n$ and $t > 0$. It follows from (8) that

$$\mathcal{P}'_{\mu, \nu} \left(\phi \left(m^{\frac{\beta-1}{2}} x, 0 \right), 2^n m^{\frac{n(\beta+1)}{2}} t \right) \leq_L \mathcal{P}_{\mu, \nu} \left(\frac{1}{m^{n\beta}} f(m^\beta x) - f(x), t \right), \quad (9)$$

for all $x \in V^n$ and $t > 0$. Consider the set $\Omega = \{g : V^n \rightarrow W\}$ and introduce the generalized metric on Ω as follows:

$$d(g, h) := \inf \left\{ c \in (0, \infty) : \right.$$

$$\left. \mathcal{P}'_{\mu, \nu} \left(\phi \left(m^{\frac{\beta-1}{2}} x, 0 \right), 2^n t \right) \leq_L \mathcal{P}_{\mu, \nu}(g(x) - h(x), ct), \forall x \in V^n, t > 0 \right\},$$

if there exists such constant c , and $d(g, h) = \infty$, otherwise. One can check that d is a complete metric. Define the mapping $\mathcal{J} : \Omega \rightarrow \Omega$ through $\mathcal{J}g(x) = \frac{1}{m^{n\beta}} g(m^\beta x)$

for all $x \in V^n$. Let $g, h \in \Omega$, and let δ be an arbitrary constant with $d(g, h) < \delta$. Then

$$\mathcal{P}'_{\mu, \nu} \left(\phi \left(m^{\frac{\beta-1}{2}} x, 0 \right), 2^n t \right) \leq_L \mathcal{P}_{\mu, \nu}(g(x) - h(x), \delta t)$$

for all $x \in V^n$ and $t > 0$. Thus

$$\begin{aligned} \mathcal{P}'_{\mu, \nu} \left(\phi \left(m^{\frac{\beta-1}{2}} x, 0 \right), 2^n \left(\frac{m^n}{\lambda} \right)^\beta t \right) &\leq_L \mathcal{P}_{\mu, \nu}(g(m^\beta x) - h(m^\beta x), m^{n\beta} \delta t) \\ &= \mathcal{P}_{\mu, \nu}(\mathcal{J}g(x) - \mathcal{J}h(x), \delta t) \end{aligned}$$

for all $x \in V^n$ and $t > 0$. Hence, $d(\mathcal{J}g, \mathcal{J}h) \leq \left(\frac{\lambda}{m^n}\right)^\beta d(g, h)$ for all $g, h \in \Omega$ and therefore \mathcal{J} is a strictly contractive mapping of Ω with the Lipschitz constant $\left(\frac{\lambda}{m^n}\right)^\beta$. It follows from (9) that $d(f, \mathcal{J}f) \leq \frac{1}{m^{\frac{(\beta+1)n}{2}}}$. By Theorem 3.3, there exists a mapping $\mathcal{A} : V^n \rightarrow W$ satisfying:

(i) \mathcal{A} is a unique fixed point of \mathcal{J} in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$, which satisfies

$$\mathcal{A}(x) = \frac{1}{m^{n\beta}} \mathcal{A}(m^\beta x) \quad (10)$$

for all $x \in V^n$. In other words, there exists a $c > 0$ with

$$\mathcal{P}'_{\mu, \nu} \left(\phi \left(m^{\frac{\beta-1}{2}} x, 0 \right), 2^n t \right) \leq_L \mathcal{P}_{\mu, \nu}(f(x) - \mathcal{Q}(x), ct)$$

for all $x \in V^n$ and $t > 0$.

(ii) It is clear that $d(\mathcal{J}^l f, \mathcal{A}) \rightarrow 0$ as $l \rightarrow \infty$ and so

$$\mathcal{A}(x) = \lim_{l \rightarrow \infty} \mathcal{J}^l f(x) = \lim_{l \rightarrow \infty} \frac{1}{m^{\beta l n}} f(m^{\beta l} x) \quad (11)$$

for all $x \in V^n$.

(iii) For every $f \in \Omega$, we have $d(f, \mathcal{A}) \leq \frac{1}{1 - \left(\frac{\lambda}{m^n}\right)^\beta} d(\mathcal{J}f, f)$. Since, $d(\mathcal{J}f, f) \leq \frac{1}{m^{\frac{(\beta+1)n}{2}}}$, we find $d(f, \mathcal{A}) \leq \frac{\lambda^{-\frac{1}{2}(\beta-1)}}{|m^n - \lambda|}$. The last inequality necessitates that

$$\mathcal{P}'_{\mu, \nu} \left(\phi \left(m^{\frac{\beta-1}{2}} x, 0 \right), 2^n t \right) \leq_L \mathcal{P}_{\mu, \nu} \left(f(x) - \mathcal{A}(x), \frac{\lambda^{-\frac{\beta-1}{2}}}{|m^n - \lambda|} t \right) \quad (12)$$

for all $x \in V^n$ and $t > 0$. Relations (5) and (12) show that inequality (7) holds. Interchanging (x_1, x_2) by $(m^{\beta l} x_1, m^{\beta l} x_2)$ in (6), we obtain

$$\begin{aligned} \mathcal{P}'_{\mu, \nu} \left(\phi(x_1, x_2), \frac{m^{n\beta l}}{\lambda^{\beta l}} t \right) &\leq_L \mathcal{P}'_{\mu, \nu}(\mathfrak{D}f(m^{\beta l} x_1, m^{\beta l} x_2), m^{\beta l n} t) \\ &\leq_L \mathcal{P}_{\mu, \nu} \left(\frac{1}{m^{\beta l n}} \mathfrak{D}f(m^{\beta l} x_1, m^{\beta l} x_2), t \right) \end{aligned}$$

for all $x_1, x_2 \in V^n$ and $t > 0$. Taking l tends to infinity and using (11), we see that \mathcal{A} is a solution of (3). If \mathcal{A} has the linear condition in each variable, then it is a

multi-Euler-Lagrange additive mapping by Theorem 3.2. The proof of uniqueness of solution is routine. \square

The following corollaries are the direct consequences of Theorem 3.4 concerning the stability of (6).

Corollary 3.5. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq n$. Let also V be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space, $(W, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space, and $z_0 \in Z$. If $f : V^n \rightarrow W$ is a mapping such that*

$$\mathcal{P}'_{\mu,\nu} \left(\sum_{i=1}^2 \sum_{j=1}^n \|x_{ij}\|^\alpha z_0, t \right) \leq_L \mathcal{P}_{\mu,\nu}(\mathfrak{D}f(x_1, x_2), t)$$

for all $x_1, x_2 \in V^n$ and $t > 0$, then there exists a solution $\mathcal{A} : V^n \rightarrow W$ of (3) such that

$$\mathcal{P}'_{\mu,\nu} \left(\sum_{j=1}^n \|x_{1j}\|^\alpha z_0, 2^n |m^n - m^\alpha| t \right) \leq_L \mathcal{P}_{\mu,\nu}(\mathcal{A}(x) - f(x), t)$$

for all $x = x_1 \in V^n$ and $t > 0$. In particular, if \mathcal{A} has the linear condition in each variable, then it is a unique multi-Euler-Lagrange additive mapping.

PROOF. Defining $\phi(x_1, x_2) := \sum_{i=1}^2 \sum_{j=1}^n \|x_{ij}\|^\alpha z_0$ and applying Theorem 3.4, we get the desired result. \square

For two sets X and Y , the set of all mappings from X to Y is denoted by Y^X . Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^A . We say that operator equation

$$\mathcal{F}_1\varphi(a_1, \dots, a_n) = \mathcal{F}_2\varphi(a_1, \dots, a_n) \quad (13)$$

is ψ -hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathcal{F}_1\varphi_0(a_1, \dots, a_n), \mathcal{F}_2\varphi_0(a_1, \dots, a_n)) \leq \psi(a_1, \dots, a_n), \quad a_1, \dots, a_n \in A,$$

fulfils (13); this definition is introduced in [10]. In other words, a functional equation \mathcal{F} is *hyperstable* if any mapping f satisfying the equation \mathcal{F} approximately is a true solution of \mathcal{F} .

Corollary 3.6. *Let $\alpha_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$ fulfill $\sum_{i=1}^2 \sum_{j=1}^n \alpha_{ij} \neq n$. Let also V be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space, $(W, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : V^n \rightarrow W$ is a mapping such that*

$$\mathcal{P}'_{\mu,\nu} \left(\prod_{i=1}^2 \prod_{j=1}^n \|x_{ij}\|^{\alpha_{ij}} z_0, t \right) \leq_L \mathcal{P}_{\mu,\nu}(\mathfrak{D}f(x_1, x_2), t)$$

for all $x_1, x_2 \in V^n$ and $t > 0$, then f is a solution of (3). Additionally, if f has the linear condition in each variable, then it is a multi-Euler-Lagrange additive mapping.

PROOF. Set $\phi(x_1, x_2) := \prod_{i=1}^2 \prod_{j=1}^n \|x_{ij}\|^{\alpha_{ij}} z_0$. It follows from Theorem 3.4 that there exists a unique multi-Euler-Lagrange additive mapping $\mathcal{A} : V^n \rightarrow W$ such that

$$\mathcal{P}_{\mu, \nu}(\mathcal{A}(x) - f(x), t) = 1$$

for all $x \in V^n$ and $t > 0$ and hence $f(x) = \mathcal{A}(x)$ for all $x \in V^n$. \square

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