Mathematical Analysis and its Contemporary Applications
Volume 6, Issue 2, 2024, 17–31
doi: 10.30495/maca.2024.2029404.1100
ISSN 2716-9898

Existence and stability results for discrete fractional three-point boundary value problems

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Abstract. In this study, firstly we obtain the existing results for the following three-point boundary value problem

$$-\nabla^{\mu+1}y(t) = Ay(t) + f(t, y(t)),\ t \in \mathbb{N}_{a+2}^{b}$$
$$y(a+1) = 0, \quad y(b) = \delta\nabla y(\nu)$$

$0 < \mu < 1$, $A : \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R}) \longrightarrow \mathbb{R}$, $f : \mathbb{N}_{a+2}^{b} \times \mathbb{R} \longrightarrow \mathbb{R}$, by using the Brouwer fixed point theorem and the Banach fixed point theorem. Furthermore, we have established the stability of this problem in the sense of Hyers and Ulam. Examples are given which illustrate the effectiveness of the theoretical results.

1. Introduction

The study of discrete fractional differential equations has garnered considerable attention in recent years due to its relevance in modelling phenomena with memory effects and long-range dependencies [1, 8, 17, 18, 19, 20]. In particular, the investigation of discrete fractional three-point boundary value problems (BVPs) stands as a crucial endeavour in understanding the dynamics of discrete systems with fractional derivatives.

2020 *Mathematics Subject Classification.* Primary: 34B15; Secondary: 39A10.

Key words and phrases. Existence of solutions, Discrete fractional equations, Ulam-Hyers stability.

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This paper delves into exploring the existence of solutions for discrete fractional three-point BVPs, shedding light on fundamental aspects of discrete fractional calculus. The formulation of such problems involves discrete fractional derivatives, which capture the fractional order dynamics of the system, coupled with boundary conditions specified at three distinct points. The intricate interplay between discretization and fractional calculus adds layers of complexity, analyzing these problems both challenging and intellectually rewarding.

The significance of studying discrete fractional three-point BVPs lies in their applicability across various fields, including mathematical physics, engineering, and computational sciences. These problems arise in diverse contexts, such as numerical approximations of fractional differential equations and the modelling of discrete systems exhibiting memory effects.

In this research, we aim to establish conditions for the existence of solutions to discrete fractional three-point BVPs using advanced mathematical techniques and theoretical analyses. By leveraging concepts from discrete fractional calculus and boundary value problem theory, we seek to provide insights into the behaviour of solutions and the underlying dynamics of the systems under consideration [**2**, **3**, **4**, **5**, **6**, **7**, **10**, **11**, **12**, **14**, **16**, **20**, **21**, **22**].

Furthermore, the outcomes of this study hold potential implications for practical applications, ranging from signal processing and image analysis to population dynamics and control theory. The exploration of discrete fractional three-point BVPs not only advances the theoretical understanding of fractional calculus but also paves the way for innovative approaches to addressing real-world challenges.

In the realm of functional equations theory, stability analysis stands out as a crucial and intriguing area of research. Stability, a fundamental property in mathematical analysis, holds paramount importance across numerous fields of engineering and science. Ulam-Hyers stability for differential equations was initialed in the 1940s by Ulam and Hyers. Roughly speaking, the Ulam-Hyers stability for a differential equation is the answer to the question whether there is an exact solution near an approximate solution to the differential equation. So it is obviously important for the study of numerical and approximate solutions and real world applications of differential equations. For this reason, numerous researchers have explored various aspects of Ulam-Hyers stability, addressing problems related to fractional integrals and fractional differential equations. These investigations have employed a multitude of techniques, as documented in [**9**, **13**, **15**, **23**, **24**, **25**] and the accompanying references. Nevertheless, there are only a few works on the stability of fractional differential equations.

Our goal is to obtain the existence and stability results of solutions for the following DFBVP (1)-(2), as shown

$$-\nabla^{\mu+1}y(t) = Ay(t) + f(t, y(t)),\ t \in \mathbb{N}_{a+2}^{b} \tag{1}$$

$$y(a+1) = 0,\ y(b) = \delta\nabla y(\nu), \tag{2}$$

where $0 < \mu < 1,\ A : \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R}) \longrightarrow \mathbb{R},\ f : \mathbb{N}_{a+2}^{b} \times \mathbb{R} \longrightarrow \mathbb{R}$.

We organize the rest of this paper as follows. In Preliminaries, we will present some definitions and background results. For sake of convenience, we will also state the fixed point theorems. Considering the Green's function for this DFBVP, we will give some existence results in Section 3. Section 4 is devoted to show a generalized stability. The paper is ended by an example illustrating our results.

2. Preliminaries

In this section, we collect some basic definitions and lemmas for manipulating discrete fractional operators. For any real number β, let $N_\beta = \{\beta, \beta+1, \beta+ ...\}$ and we define $t^{\overline{k}} = \dfrac{\Gamma(t+k)}{\Gamma(t)}$ for any $t, k \in \mathbb{R}$. If $n \in \mathbb{N}$ then $t^{\overline{n}} := t(t+1)\cdots(t+n-1)$.

Lemma 2.1. *Let n and N be nonnegative integers. Then*

$$\frac{\Gamma(-n)}{\Gamma(-N)} = (-1)^{N-n}\frac{N!}{n!}.$$

Also, if t is a nonpositive integer and $t+r$ is not a nonpositive integer, then

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)} = 0.$$

Theorem 2.2. *The following equality hold*

$$\nabla(t+a)^{\overline{n}} = n(t+a)^{\overline{n-1}}, \quad t \in \mathbb{R}$$

for values of $n \in \mathbb{N}$ and $a \in \mathbb{R}$.

Definition 2.1. We define the nabla Taylor monomials, $H_n(t,a),\ n \in \mathbb{N}_0$ by $H_0(t,a) = 1,\ t \in \mathbb{N}_a$ and

$$H_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!},\ t \in \mathbb{N}_{a-n+1}$$

for $n \in \mathbb{N}_0$.

Definition 2.2. Let $\mu \neq -1, -2, ...$ we define μ-th order nabla fractional Taylor monomial, $H_\mu(t,a)$, by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} \tag{3}$$

whenever the right hand side of the equation (3) makes sense.

Definition 2.3. Let $f : \mathbb{N}_{a+1} \longrightarrow \mathbb{R}$ be given and $\mu \in \mathbb{R}^+$, then

$$\nabla_a^{-\mu} f(t) := \int_a^t H_{\mu-1}(t, \rho(s)) f(s) \nabla s$$

for $t \in \mathbb{N}_a$, where by convention $\nabla_a^{\mu}(a) = 0$.

Definition 2.4. Let $f : \mathbb{N}_{a-N+1} \longrightarrow \mathbb{R}$, $\mu \in \mathbb{R}^+$ and $N - 1 < \mu \leq N$. Then we define μ-th nabla fractional difference $\nabla_a^{\mu} f(t)$ by

$$\nabla_a^{\mu} f(t) := \nabla^N \nabla_a^{-(N-\mu)} f(t).$$

Definition 2.5. Assume $f : \mathbb{N}_{a-N+1} \longrightarrow \mathbb{R}$, $\mu \in \mathbb{R}^+$ and $N - 1 < \mu \leq N$. Then we define μ-th Caputo nabla fractional difference of f is defined by

$$\nabla_{a^*}^{\mu} f(t) := \nabla_a^{-(N-\mu)} \nabla^N f(t), \ t \in \mathbb{N}_{a+1}$$

where $N := \lceil \mu \rceil$.

Theorem 2.3. *Assume $\mu > 0$ and $N - 1 < \mu \leq N$. Then a general solution of $\nabla_a^{\mu} x(t) = 0$ is given by*

$$x(t) = c_1 H_{\mu-1}(t, a) + c_2 H_{\mu-2}(t, a) + ... + c_N H_{\mu-N}(t, a),$$

for $c_k \in \mathbb{R}$ and $t \in \mathbb{N}_a$.

The following fixed point theorems are fundamental and important to the proof of our main results.

Theorem 2.4. *(Banach Fixed Point Theorem) Let A be a contraction mapping from a closed subset K of a Banach space X into K. Then there exists a unique fixed point x in K such that $T(x) = x$.*

Theorem 2.5. *(Brouwer Fixed Point Theorem) Let K be a nonempty compact (closed and bounded) convex set in a Banach space and $A : K \to K$ is a continuous self mapping. Then A has (at least) one fixed point in K.*

3. Main Results

In this section, we prove the existence of solutions of the DFBVP (1)-(2) by using Theorem 2.4 and Theorem 2.5. To prove the main results, we will employ following lemma and theorems.

Lemma 3.1. *The homogeneous boundary value problem $-\nabla_a^{\mu+1} y(t) = 0$, $t \in \mathbb{N}_{a+2}^b$ with the boundary condition (2) has only the trivial solution if and only if*

$$-1 + \sum_{\tau=a+2}^{b} \frac{(\tau - a)^{\overline{\mu-1}}}{\Gamma(\mu)} - \delta \frac{(\nu - a)^{\overline{\mu-1}}}{\Gamma(\mu)} \neq 0.$$

PROOF. If $-\nabla_a^\mu(\nabla y)(t) = 0$, then it follows from Theorem 2.3 that

$$\nabla y(t) = c_1 \frac{(t-a)^{\overline{\mu-1}}}{\Gamma(\mu)}$$

$$y(t) - y(a) = \sum_{\tau=a+1}^{t} c_1 \frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)}.$$

If $y(a) = c_0$, then $y(t) = c_1 \sum_{\tau=a+1}^{t} \frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)} + c_0$, where $(\tau-a)^{\overline{\mu-1}} = \frac{\Gamma(\tau-a+\mu-1)}{\Gamma(\tau-a)}$. Thus we get

$$\begin{aligned} y(a+1) &= c_1 \frac{\Gamma(a+1-a+\mu-1)}{\Gamma(a+1-a)\Gamma(\mu)} + c_0 \\ &= c_1 \frac{\Gamma(\mu)}{\Gamma(1)\Gamma(\mu)} + c_0 = c_1 + c_0 \end{aligned}$$

and

$$\begin{aligned} \nabla y(a+1) &= c_1 \frac{\Gamma(a+1-a+\mu-1)}{\Gamma(\mu)\Gamma(a+1-a)} \\ &= c_1 \frac{\Gamma(\mu)}{\Gamma(\mu)} = c_1. \end{aligned}$$

Since $y(a+1) = 0$, we have

$$c_1 + c_0 = 0.$$

Similarly, using the second boundary condition $\gamma y(b) + \delta \nabla y(\nu) = 0$, we get

$$c_1 \left(\sum_{\tau=a+1}^{b} \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)} - \delta \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)} \right) + c_0 = 0.$$

Let we consider the system for c_0 and c_1 and we define the number

$$\Delta := -1 + \sum_{\tau=a+1}^{b} \frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)} - \delta \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)}. \tag{4}$$

Hence the solution $y(t)$ is nontrivial (i.e. c_0 and c_1 are not both equal to zero) if and only if $\Delta \neq 0$. □

Theorem 3.2. *Assume that Δ, as defined by (4), is not zero. Then the Green's function for the DFBVP (1-2) is given by*

$$G(t,s) = \begin{cases} G_1(t,s), & s \leq \nu, \\ G_2(t,s), & s \geq \nu+1, \end{cases}$$

where

$$G_1(t,s)=\begin{cases}-x(t,\rho(s))+\dfrac{1}{\Delta}\left(x(t,a)-1\right)\left(x(b,\rho(s))-\delta\dfrac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}\right), & s\le t,\\ \dfrac{1}{\Delta}\left(x(t,a)-1\right)\left(x(b,\rho(s))-\delta\dfrac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}\right), & s\ge t+1,\end{cases}$$

and

$$G_2(t,s)=\begin{cases}-x(t,\rho(s))+\dfrac{1}{\Delta}\left(x(t,a)-1\right)x(b,\rho(s)), & s\le t,\\ \dfrac{1}{\Delta}\left(x(t,a)-1\right)x(b,\rho(s)), & s\ge t+1.\end{cases}$$

such that $x(t,\rho(s))=\sum_{\tau=s}^{t}\dfrac{(\tau-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}$.

Proof. Suppose that $y(t)$ is a solution of the non-homogeneous boundary value problem $-\nabla_a^{\mu}(\nabla y)(t)=h(t)$, $t\in\mathbb{N}_{a+2}^{b}$ with the boundary condition (2). Then $x(t)=(\nabla y)(t)$ solves the initial value problem

$$\begin{aligned}-\nabla_a^{\mu}x(t)&=h(t)\\ x(a+1)&=y(a+1)-y(a).\end{aligned}$$

The solution of this initial value problem has the form

$$x(t)=-\nabla_a^{-\mu}h(t)+c_1\frac{(t-a)^{\overline{\mu-1}}}{\Gamma(\mu)}.$$

Hence we have that

$$\nabla y(t)=-\sum_{s=a+1}^{t}\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}h(s)+c_1\frac{(t-a)^{\overline{\mu-1}}}{\Gamma(\mu)}.$$

We sum both sides from $a+1$ to t to get

$$y(t)-y(a)=-\sum_{\tau=a+1}^{t}\left(\sum_{s=a+1}^{\tau}\frac{(\tau-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}h(s)-c_1\frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)}\right).$$

Letting $y(a)=c_1$ and interchanging sums, we obtain

$$\begin{aligned}y(t)&=-\sum_{s=a+1}^{t}\sum_{\tau=s}^{t}\frac{(\tau-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}h(s)+c_1\sum_{\tau=a+1}^{t}\frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)}+c_0\\ &=-\sum_{\tau=a+1}^{t}x(t,\rho(s))h(s)+c_1x(t,a)+c_0.\end{aligned}$$

Now, we will use the boundary value conditions to obtain formulas for the constants c_0 and c_1. Since

$$\begin{aligned} y(a+1) &= -x(a+1,a)h(a+1) + c_1 x(a+1,a) + c_0 \\ &= c_0 - h(a+1) + c_1 \end{aligned}$$

and

$$\nabla y(a+1) = c_1 - h(a+1),$$

we have that

$$c_0 - h(a+1) + c_1 = 0.$$

Since $h : \mathbb{N}_{a+2}^{b} \longrightarrow \mathbb{R}$, we extend the domain of h by letting $h(a+1) = 0$. Rewriting this equation to collect the terms involving c_0 and c_1, we obtain

$$c_0 + c_1 = 0. \tag{5}$$

Since

$$y(b) = -\sum_{s=a+1}^{b} x(b,\rho(s))h(s) + c_1 x(b,a) + c_0$$

and

$$\nabla y(\nu) = -\sum_{s=a+1}^{\nu} \frac{(\nu-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} h(s) + c_1 \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)},$$

we have that

$$\begin{aligned} &\left(-\sum_{s=a+1}^{b} x(b,\rho(s))h(s) + c_1 x(b,a) + c_0 \right) \\ &- \delta \left(-\sum_{s=a+1}^{\nu} \frac{(\nu-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} h(s) + c_1 \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)} \right) = 0 \end{aligned}$$

and so

$$c_1 \left(x(b,a) - \delta \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)} \right) + c_0 = \sum_{s=a+1}^{b} x(b,\rho(s))h(s) - \delta \sum_{s=a+1}^{\nu} \frac{(\nu-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} h(s) \tag{6}$$

From (5), we find

$$c_1 = -c_0$$

and substituting this in to (6), we obtain

$$c_1 \left(x(b,a) - \delta \frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)} - 1 \right) = \sum_{s=a+1}^{b} x(b,\rho(s))h(s) - \delta \sum_{s=a+1}^{\nu} \frac{(\nu-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} h(s)$$

and so

$$c_1\left(-1+\sum_{\tau=a+1}^{b}\frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)}-\delta\frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)}\right)=\sum_{s=a+1}^{b}x(b,\rho(s))h(s)$$
$$-\delta\sum_{s=a+1}^{\nu}\frac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}h(s),$$

thus we get

$$c_1=\frac{1}{\Delta}\left(\sum_{s=a+1}^{b}x(b,\rho(s))h(s)-\delta\sum_{s=a+1}^{b}\frac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}h(s)\right).$$

Using this, we have

$$c_0=-\frac{1}{\Delta}\left(\sum_{s=a+1}^{b}x(b,\rho(s))h(s)-\delta\sum_{s=a+1}^{\nu}\frac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}h(s)\right).$$

Because $\Delta\neq 0$, both of these constants are well defined. Thus we have

$$\begin{aligned}
y(t)=&-\sum_{s=a+1}^{t}x(t,\rho(s))h(s)+\frac{1}{\Delta}\left(\sum_{s=a+1}^{b}x(b,\rho(s))h(s)-\delta\sum_{s=a+1}^{\nu}\frac{(\nu-s+1)^{\overline{\mu-1}}}{(\mu)}h(s)\right)x(t,a)\\
&-\frac{1}{\Delta}\left(\sum_{s=a+1}^{b}x(b,\rho(s))h(s)-\delta\sum_{s=a+1}^{\nu}\frac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}h(s)\right)\\
=&-\sum_{s=a+1}^{t}x(t,\rho(s))h(s)+\frac{1}{\Delta}\sum_{s=a+1}^{\nu}\left(x(t,a)-1\right)\left(x(b,\rho(s))-\delta\frac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}\right)h(s)\\
&+\frac{1}{\Delta}\sum_{s=\nu+1}^{b}\left(x(t,a)-1\right)\gamma x(b,\rho(s))h(s)\\
=&\sum_{s=a+1}^{b}G(t,s)h(s),
\end{aligned}$$

where

$$G(t,s)=\begin{cases}G_1(t,s), & s\leq\nu,\\ G_2(t,s), & s\geq\nu+1,\end{cases}$$

such that

$$G_1(t,s)=\begin{cases}-x(t,\rho(s))+\dfrac{1}{\Delta}\left(x(t,a)-1\right)\left(x(b,\rho(s))-\delta\dfrac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}\right), & s\leq t,\\ \dfrac{1}{\Delta}\left(x(t,a)-1\right)\left(x(b,\rho(s))-\delta\dfrac{(\nu-s+1)^{\overline{\mu-1}}}{\Gamma(\mu)}\right), & s\geq t+1,\end{cases}$$

and

$$G_2(t,s)=\begin{cases}-x(t,\rho(s))+\dfrac{1}{\Delta}\left(x(t,a)-1\right)\gamma x(b,\rho(s)), & s\le t,\\ \dfrac{1}{\Delta}\left(x(t,a)-1\right)x(b,\rho(s)), & s\ge t+1.\end{cases}$$

Here,

$$x(t,\rho(s))=x(t,s-1)=\sum_{\tau=s}^{t}\frac{(\tau-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}=\sum_{\tau=s}^{t}H_{\mu-1}(\tau,\rho(s))$$

and

$$\Delta=-1+\sum_{\tau=a+1}^{b}\frac{(\tau-a)^{\overline{\mu-1}}}{\Gamma(\mu)}-\delta\frac{(\nu-a)^{\overline{\mu-1}}}{\Gamma(\mu)}.$$

In other words,

$$G_1(t,s)=\begin{cases}-\displaystyle\sum_{\tau=s}^{t}H_{\mu-1}(\tau,\rho(s))+\frac{K(t)}{\Delta}\left(\gamma\sum_{\tau=s}^{b}H_{\mu-1}(\tau,\rho(s))-\delta H_{\mu-1}(\nu,\rho(s))\right), & s\le t,\\ \dfrac{K(t)}{\Delta}\left(\displaystyle\sum_{\tau=s}^{b}H_{\mu-1}(\tau,\rho(s))-\delta H_{\mu-1}(\nu,\rho(s))\right), & s\ge t+1,\end{cases} \tag{7}$$

and

$$G_2(t,s)=\begin{cases}-\displaystyle\sum_{\tau=s}^{t}H_{\mu-1}(\tau,\rho(s))+\frac{K(t)}{\Delta}H_{\mu-1}(b,\rho(s)), & s\le t,\\ \dfrac{K(t)}{\Delta}H_{\mu-1}(b,\rho(s)), & s\ge t+1,\end{cases}$$

where $K(t)=H_\mu(t,a)-1$. □

Lemma 3.3. *The boundary value problem (1)-(2) has a solution*

$$y(t)=\sum_{a+1}^{b}G(t,s)(Ay(s)+f(s,y(s))),$$

where G(t,s) is given in (7).

Lemma 3.4. *For all $t,s\in\mathbb{N}_{a+1}^{b}$, the Green's function $G(t,s)$ satisfies the following inequality*

$$|G(t,s)|\le M,$$

where $M=H_\mu(b,a)+\dfrac{H_\mu(b,a)-1}{\Delta}H_\mu(b,a).$

Proof. Because of $H_\mu(b,a) > H_{\mu-1}(b,a) > H_{\mu-1}(\nu,a)$ and $1 > \delta$, we get $H_\mu(b,a) > \delta H_{\mu-1}(\nu,a)$ and so $H_\mu(b,a) - \delta H_{\mu-1}(\nu,a) > 0$. Thus, for all $t,s \in \mathbb{N}_{a+1}^b$, we have

$$\begin{aligned}|G(t,s)| &\le H_\mu(t,\rho(s)) + \frac{H_\mu(t,a)-1}{\Delta} H_\mu(b,\rho(s)) \\ &\le H_\mu(b,a) + \frac{H_\mu(b,a)-1}{\Delta} H_\mu(b,a) \\ &= H_\mu(b,a)\left\{1 + \frac{H_\mu(b,a)-1}{\Delta}\right\}.\end{aligned}$$

□

Let we define $T : \mathbb{C}[a,b]_{\mathbb{N}_{a+2}} \to \mathbb{C}[a,b]_{\mathbb{N}_{a+2}}$ by

$$Ty(t) = \sum_{a+1}^{b} G(t,s)(Ay(s) + f(s,y(s))). \tag{8}$$

If $y \in (\mathbb{C}[a,b]_{\mathbb{N}_{a+2}}, \mathbb{R})$ then $Ty(t) \in (\mathbb{C}[a,b]_{\mathbb{N}_{a+2}}, \mathbb{R})$. Before stating the main results, we introduce the following assumptions;

(H1) For all $t \in \mathbb{N}_{a+2}^b$, $f(t,y)$ is a continuous function with respect to y, and there exists a constant $L \in \mathbb{R}^+$ such that

$$|f(t,y_1) - f(t,y_2)| \le L_1|y_1 - y_2|$$

(H2) For all $t \in \mathbb{N}_{a+2}^b$, *and* $y_1, y_2 \in \mathbb{R}$, there exists a constant $N \in \mathbb{N}^+$ such that

$$|Ay_1(t) - Ay_2(t)| \le N_1|y_1 - y_2|$$

(H3) For all $t \in \mathbb{N}_{a+2}^b$ and $y \in \mathbb{R}$, $f(t,y)$ is a continuous function with respect to y and there exists a constants $L_2 \in \mathbb{R}^+$ such that

$$\max_{(t,y)\in \mathbb{N}_{a+2}^b \times [-R,R]} |f(t,y)| \le L_2,$$

where $R \in \mathbb{R}^+$.

(H4) For all $t \in \mathbb{N}_{a+2}^b$ and $y \in \mathbb{R}$, $Ay(t)$ is a continuous function and there exists a constant $N_2 \in \mathbb{R}^+$ such that

$$|Ay(t)| \le \mathbb{N}_2.$$

Consider the Banach space

$$E = \{y : y(t) \in \mathbb{C}[a,b]_{\mathbb{N}_{a+2}}\}$$

with the norm

$$\|y\| = \sup_{t\in \mathbb{N}_{a+2}^b} \{|y(t)|\}.$$

Theorem 3.5. *Suppose the validity of the conditions* ***(H1)-(H2)****. If* $M(L_1 + N_1)(b-a) < 1$*, then the problem (1) and (2) has a unique solution.*

Proof. Consider the operator $T : \mathbb{C}[a, b]_{\mathbb{N}_{a+2}} \to \mathbb{C}[a, b]_{\mathbb{N}_{a+2}}$ defined (8) are the solutions of the problem (1)-(2). Now consider for all $t \in \mathbb{N}_{a+2}^b$

$$\begin{aligned}|Ty_1(t) - Ty_2(t)| =& \left|\sum_{s=a+1}^{b} G(t,s)(Ay_1(s) + f(s, y_1(s))) - \sum_{s=a+1}^{b} G(t,s)(Ay_2(s) + f(s, y_2(s)))\right| \\ \leq& \sum_{s=a+1}^{b} |G(t,s)|(|Ay_1(s) - Ay_2(s)| + |f(s, y_1(s)) - f(s, y_2(s))|) \\ \leq& \sum_{s=a+1}^{b} |G(t,s)|(N_1 \|y_1 - y_2\| + L_1 \|y_1 - y_2\|) \\ \leq& M(b-a)(L_1 + N_1) \|y_1 - y_2\| .\end{aligned}$$

Thus, we get

$$\|Ty_1 - Ty_2\| \leq M(b-a)(L_1 + N_1) \|y_1 - y_2\| ,$$

where M is given Lemma 1.2. Thus, T is a contraction mapping on $\mathbb{C}[a, b]_{\mathbb{N}_{a+2}}$ with the contraction constant $M(L_1 + N_1)(b - a)$. By applying Banach Fixed Point Theorem, we can say that the operator T has a unique fixed point on $\mathbb{C}[a, b]_{\mathbb{N}_{a+2}}$ which implies the problem (1)-(2) has a unique solution in $\mathbb{C}[a, b]_{\mathbb{N}_{a+2}}$. □

Theorem 3.6. *Suppose that the conditions **(H3)-(H4)** are satisfied. Then the problem (1)-(2) has a solution, if there exists an $R \in \mathbb{R}^+$ such that $(N_2 + L_2)M(b - a) \leq R$*

Proof. Consider the set

$$\mathbb{B}_R = \{y \in \mathbb{C}[a, b]_{\mathbb{N}_{a+2}} \longrightarrow \mathbb{R}, \|y\| \leq R\}.$$

Clearly, $\mathbb{B}_R$ is a nonempty, compact, convex subset of $\mathbb{R}$. Let T be an operator as defined in (8). It is clear that T is a continuous operator. Now consider,

$$\begin{aligned}\|Ty(t)\| \leq& \sum_{s=a+1}^{b} |G(t,s)|(|Ay(s)| + |f(s, y(s)|) \\ \leq& (N_2 + L_2) \sum_{s=a+1}^{b} |G(t,s)| \\ \leq& (N_2 + L_2)M(b-a) \\ \leq& R.\end{aligned}$$

Thus, $\|Ty\| \leq R$ and $T : \mathbb{B}_R \longrightarrow \mathbb{B}_R$. It follows at once by Brouwer fixed point theorem, that exist a fixed point of T, such that $\|y\| \leq R$. □

4. Generalized Ulam Stabilities

In this section, we presents a theorem showing the problem (1)-(2) admits both Ulam-Hyers and generalized Ulam-Hyers stabilities. By using the sum

$$x(t) = \sum_{a+1}^{b} G(t,s)(Ax(s) + f(s, x(s))) \tag{9}$$

we discuss the Ulam Stability, here $x(t) \in (\mathbb{C}[a,b]_{\mathbb{N}_{a+2}}, \mathbb{R})$ and $A : \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R}) \longrightarrow \mathbb{R}, f : \mathbb{N}_{a+2}^{b} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Then we define the nonlinear continuous operator.

$$P : \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R}) \longrightarrow \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R})$$

as follows

$$Px(t) = -\nabla^{\mu+1}x(t) - Ax(t) - f(t, x(t)).$$

Definition 4.1. For each $\varepsilon > 0$ and for each solution $x(t)$ of (1), such that

$$\|Px\| \leq \varepsilon, \tag{10}$$

the problem (1), is said to be Ulam-Hyers stability if we can find a positive real number ν and a solution $y(t) \in (\mathbb{C}[a,b]_{\mathbb{N}_{a+2}}, \mathbb{R})$ of (1) satisfying the inequality

$$|y - x| \leq \nu\varepsilon^*$$

where ε^* is a positive real number depending on ε.

Definition 4.2. Let $m \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|y(t) - x(t)| \leq m(\varepsilon), \quad t \in \mathbb{N}_{a+2}^{b}.$$

Then the problem (1) is said to be generalized Ulam-Hyers stable.

Theorem 4.1. *Under assumptions (H1)-(H2), with the problem (1)-(2) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $y \in \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R})$ be a solution of (1)-(2) in the sense of Theorem 3.5. Let x be any solution satisfying (10). Lemma 3.3 implies the equivalence between the operators P and $T - Id$ (where Id is the identity operator) for every solution $x \in \mathbb{C}(\mathbb{N}_{a+2}^{b}, \mathbb{R})$ of (1)-(2) satisfying. Therefore, we deduce by the fixed point property of the operator T that

$$\begin{aligned}\big|y(t)-x(t)\big| =&\big|y(t)-Tx(t)+Tx(t)-x(t)\big|\\ =&\big|x(t)-Tx(t)+Tx(t)-Ty(t)\big|\\ \leq&\big|Tx(t)-Ty(t)\big|+\big|Tx(t)-x(t)\big|\\ \leq&\|Tx-Ty\|+\|Tx-x\|\\ <&M(b-a)(L_1+N_1)\,\|x-y\|+\|(T-I)x\|\\ =&M(b-a)(L_1+N_1)\,\|x-y\|+\|Px\|\\ \leq&M(b-a)(L_1+N_1)\,\|x-y\|+\varepsilon.\end{aligned}$$

Because $M(b-a)(L_1+N_1)<1$ and $\varepsilon>0$, we find

$$|x-y|\leq\frac{\varepsilon}{1-M(b-a)(L_1+N_1)}=m(\varepsilon).$$

Fixing $\varepsilon_*=\dfrac{\varepsilon}{1-M(b-a)(L_1+N_1)}$ and $\nu=1$, we obtain Ulam-Hyers stability condition. In adition, the generalized Ulam Hyers stability follows by taking $m(\varepsilon)=\dfrac{\varepsilon}{1-M(b-a)(L_1+N_1)}$. □

Example 4.1. *Consider the following DFBVP*

$$-\nabla_1^{\frac{7}{6}}y(t)=\exp(3-t^2)\arctan y(t)+\frac{\sqrt{t}}{10^4}\cos y(t),\qquad t\in\mathbb{N}_3^{21}\tag{11}$$

$$y(2)=0,\qquad y(21)=\frac{1}{5}\nabla y(4),\tag{12}$$

where $\delta=\frac{1}{5}$, $a=1$, $b=21$, $\nu=4$, $Ay(t)=\exp(3-t^2)\arctan y(t)$ *and* $f(t,y(t))=\dfrac{\sqrt{t}}{10^4}\cos y(t)$. *By a straightforward calculation, we see that* $M\cong 50$. *Since*

$$|f(t,y_1)-f(t,y_2)|\leq\frac{\sqrt{t}}{10^4}|\cos y_1(t)-\cos y_2(t)|\leq\frac{\sqrt{21}}{10^4}|y_1-y_2|$$

and

$$|Ay_1(t)-Ay_2(t)|\leq\exp(3-t^2)|\arctan y_1(t)-\arctan y_2(t)|\leq e^{-6}|y_1-y_2|,$$

conditions (H1) and (H2) are satisfied with $L_1=\dfrac{\sqrt{21}}{10^4}$ *and* $N_1=e^{-6}$.

Then the DFBVP (11)-(12) *has a unique solution, provided that* $M(L_1+N_1)(b-a)\cong 50(0.000458+0.000237)20=0.06<1$, *by using Theorem 3.5. Also the DFBVP* (11)-(12) *is both Ulam-Hyers and generalized Ulam-Hyers stable by Theorem 4.1.*

5. Conclusion

Our study has provided significant insights into the existence and stability of solutions for discrete fractional three-point boundary value problems. Through a rigorous analysis and application of various mathematical techniques, we have established several key results.

These findings shed light on the practical implications of the studied problems in various fields, including engineering, physics, and biology. The ability to accurately predict and control the behavior of discrete fractional systems is invaluable for designing efficient and robust solutions to real-world problems.

In summary, our research contributes to the advancement of knowledge in the field of discrete fractional calculus by providing rigorous theoretical results and practical insights into the behavior of three-point boundary value problems. We believe that our findings will inspire further research in this area and facilitate the development of innovative applications with significant societal impact.

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Received : May 2024
Accepted : July 2024