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Solution of an infinite system of fractional differential equations in tempered sequence space

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Abstract. In this article, we study an infinite system of fractional differential equations involving a generalized Caputo-Fabrizio fractional operator. By using Darbo's fixed point theorem and the concept of measure of noncompactness, we establish the existence of a solution for the proposed system in tempered sequence space. Suitable examples are given to strengthen our article. At the end, we give an iterative algorithm using the homotopy perturbation method and Adomian decomposition method to solve our given example with high accuracy.

1. Introduction

Fractional calculus is a mathematical field that confines the study of derivatives and integrals of arbitrary order. The beginning of fractional calculus was done in the seventeenth century when Leibniz first proposed the notion of a derivative with an order of $x = \frac{1}{2}$ in his letter to L'Hospital in 1695. This historical marks the early origins of fractional calculus, as documented in references [**24**, **26**, **33**]. Since its inception, fractional calculus has preserved contributions from distinguished mathematicians including Abel, Laurent, Laplace, Fourier, Weyl, Riemann, Liouville, and Euler. These distinguished individuals have played significant roles in advancing the field throughout its history. The studies of fractional operators have given rise to numerous definitions, and the theories of fractional calculus have been advanced

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by prominent mathematicians, including Caputo, Leibniz, Riemann, Grunwald, Liouville, and Letnikov. For further in-depth information, one can refer to relevant references [9, 10, 11, 22]. This branch of mathematics perceives utility in simulating manifold, physical and engineering phenomena, including but not limited to electromagnetics, fluid mechanics, and signal processing. To advance fractional calculus, many researchers have dedicated their efforts to establishing solutions for nonlinear differential equations that involve multiple fractional differential operators, like Hilfer, Riemann-Liouville, Caputo, and others. For more details, see [4, 20, 29, 35]. In order to overcome the constraints of the existing operators, Caputo and Fabrizio [12] proposed an innovative definition of fractional derivative that terminates the presence of a singular kernel

$$^{CF}\mathcal{D}^{\delta} f(\tau) = \frac{1}{1-\delta} \int_{0}^{\tau} exp\left(\frac{-\delta(\tau-\nu)}{1-\delta}\right) f^{'}(\nu) d\nu, \ \tau \geq 0, \tag{1}$$

where $0 < \delta < 1$. This new formulation provides a solution to the problem, providing a more effective approach for modeling fractional calculus in various applications. The introduction of the Caputo-Fabrizio fractional derivative has provoked extended interest among researchers in exploring FDE. This is essentially due to its special characteristic of possessing a non-singular kernel. This approach has garnered attention for its ability to effectively model a wide range of phenomena, including fractional dynamics [36], radiotherapy for cancer cells using fractional derivatives [17], interactions between immune and tumour cells in immunogenetic tumours [18], as well as processes demonstrate manifold memory effects. Losada and Nieto [23] considered the following three types of Caputo-Fabrizio FDE (2), (3) and (4). They established the existence and uniqueness solution of the following FDE

$$\begin{aligned} {}^{CF}\mathcal{D}^{\delta} u(\tau) &= \gamma(\tau), \ \tau \geq 0, \ 0 < \delta < 1, \\ u(0) &= u_0 \in \mathbb{R}, \end{aligned} \tag{2}$$

$$\begin{aligned} {}^{CF}\mathcal{D}^{\delta} u(\tau) &= \lambda u(\tau) + \gamma(\tau), \ \tau \geq 0, \ \lambda \in \mathbb{R}, \\ u(0) &= u_0 \in \mathbb{R}, \end{aligned} \tag{3}$$

$$\begin{aligned} {}^{CF}\mathcal{D}^{\delta} u(\tau) &= \phi(\tau, u(\tau)), \ \tau \geq 0, \ 0 < \delta < 1, \\ u(0) &= u_0 \in \mathbb{R}. \end{aligned} \tag{4}$$

where $\mathbb{R}$ denotes the set of real numbers, $\gamma, \ \phi$ are continuous functions on $[0, \infty)$, and $[0, T] \times \mathbb{R}$, $T > 0$, respectively, and $u(\tau)$ is the solution of the corresponding equation.

In 2020, Alshabanat et al. [3] proposed a new fractional operator that is a generalization of Caputo-Fabrizio fractional operator, and this new formula contains exponential and trigonometric functions, permitting for a wider range of applications.

Definition 1.1. [3] The fractional differential operator of order $(\delta+n)$ having kernel which contains the exponential and trigonometric functions of the function $u \in C^{n+1}[0,\infty)$ is defined by

$$\left(\mathcal{D}^{\delta+n}_{0,h,k}u\right)(\tau) = \left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\int\limits_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\cos\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u^{(n+1)}(\nu)d\nu,\ \tau>0, \tag{5}$$

where $h>0,\ k\geq 0,\ 0<\delta<1,\ n\in\mathbb{N}\cup\{0\},\ u\in C^{n+1}[0,\infty)$.

Remark 1.2. [3] If we take $h=1$ and $k=0$ in Definition 1.1, then we have

$$\left(\mathcal{D}^{\delta+n}_{0,1,0}u\right)(\tau) = \frac{1}{1-\delta}\int\limits_0^\tau exp\left(\frac{-\delta(\tau-\nu)}{1-\delta}\right)f^{'}(\nu)d\nu,\ \tau\geq 0 = {}^{CF}\left(\mathcal{D}^{\delta+n}u\right)(\tau)\ \tau>0,$$

which is the Caputo-Fabrizio fractional operator ${}^{CF}\mathcal{D}^{\delta+n}$ of order $(\delta+n)$.

Alshabanat et al. [3] studied the existence and uniqueness solution of the following linear and nonlinear FDE by taking the generalized Caputo-Fabrizio fractional derivative defined by Definition 1.1

$$\begin{gathered}\left(\mathcal{D}^{\delta}_{0,h,k}u\right)(\tau) = \gamma(\tau),\ 0<\tau<T,\\ u(0)=u_0\in\mathbb{R},\end{gathered} \tag{6}$$

and

$$\begin{gathered}\left(\mathcal{D}^{\delta}_{0,h,k}u\right)(\tau) = \phi(\tau,u(\tau)),\ 0<\tau<T,\\ u(0)=u_0\in\mathbb{R},\end{gathered} \tag{7}$$

where, $0<\delta<1$ and $T>0$ and γ is a continuous functions on $[0,T]$ and ϕ is a continuous functions on $([0,T]\times\mathbb{R})$.
Motivated by the article [3], we consider the following infinite system of nonlinear FDE of order $0<\delta<1$ as

$$\left(\mathcal{D}^{\delta}_{0,h,k}u_n\right)(\tau) = f_n(\tau,u_n(\tau)),\ 0<\tau<T, \tag{8}$$

where $\left(\mathcal{D}^{\delta}_{0,h,k}u\right)$ is the generalized Caputo-Fabrizio fractional operator defined in [3] with initial condition $u_n(\tau)_{\tau=0}=u_n(0)=0,\ u_n\in C^1[0,T]$, and $f_n\in C^1([0,T]\times\mathbb{R})$ with $f_n(0,u_n(0))=0$. Here, we are concerned with the existence of a solution of the infinite system of nonlinear FDE (8). As a main tool, we use MNC to accomplish our aim. The MNC is defined by Kuratowski [21] in 1930. The concept of the MNC is used by various authors to explore the existence of solutions for infinite system of differential and integral equations. The contributions include the work of Mursaleen et al. [27], who established the existence of solutions for an infinite system of FDE in the spaces c_0 and l_p. Mursaleen and Mohiuddine [28] investigated the existence of solution of an infinite system of differential equations in the l_p space. Also, Das et al. [14] studied the existence of solutions of an infinite system of FDE in the tempered sequence space as well as Rabbani et al. [31] studied the existence of solutions for FDE in the tempered sequence space. These works serve as significant references for those who are interested in this particular area of research. In recent years, Mehravarana et al. [25] and Das et al. [15] have further studied this field by obtaining the existence of solutions of a system of FDE and a method of hybrid FDE, respectively, in the tempered sequence space.

In our study, we discussed some preliminaries of MNC with some important fixed point theorem and an important proposition in section 2. Next, the existence of a solution of the Eq. (8) using DFPT via MNC is discussed in tempered sequence space in section 3. In section 4 and 5 an example and an iterative algorithm are presented and discussed to understand the importance of our results. Finally, in section 6 we give the conclusion of this article.

2. Preliminaries

The notion of MNC is given in the research of Banás and Lecko [5] as follows.

Definition 2.1. Let $\mathbb{E}$ be a Banach space, then we define $\mathcal{A}_{\mathbb{E}}$ is the class of all nonempty bounded subsets of a Banach space $\mathbb{E}$ and $\mathcal{B}_{\mathbb{E}}$ is the set of all relatively compact sets of a Banach space $\mathbb{E}$. So, an MNC is a mapping $\beta : \mathcal{A}_{\mathbb{E}} \to \mathbb{R}_+$ satisfies the following conditions for all $\Upsilon,\ \Upsilon_1,\ \Upsilon_2 \in \mathcal{A}_{\mathbb{E}}$.

(I) The family $\ker \beta = \{\Upsilon \in \mathcal{A}_{\mathbb{E}} : \beta(\Upsilon) = 0\} \neq \emptyset$ and $\ker \beta \subset \mathcal{B}_{\mathbb{E}}$.

(II) $\Upsilon_1 \subset \Upsilon_2 \implies \beta(\Upsilon_1) \leq \beta(\Upsilon_2)$.

(III) $\beta(\bar{\Upsilon}) = \beta(\Upsilon)$, where $\bar{\Upsilon}$ is the closure of a nonempty bounded subset Υ of $\mathbb{E}$.

(IV) $\beta(Conv\Upsilon) = \beta(\Upsilon)$, where $Conv\Upsilon$ is the convex closure of a nonempty bounded subset Υ of $\mathbb{E}$.

(V) $\beta(k\Upsilon_1 + (1-k)\Upsilon_2) \leq k\beta(\Upsilon_1) + (1-k)\beta(\Upsilon_2)$ for $k \in [0,1]$.

(VI) If $\Upsilon_n \in \mathcal{A}_{\mathbb{E}},\ \Upsilon_{n+1} \subset \Upsilon_n,\ \Upsilon_n = \bar{\Upsilon}_n$, for $n = 1,2,3,...$ and $\lim_{n\to\infty} \beta(\Upsilon_n) = 0$, then $\Upsilon_\infty = \bigcap_{n=1}^{\infty} \Upsilon_n \neq \emptyset$ and precompact.

Remark 2.2. Since $\beta(\Upsilon_\infty) = \beta\left(\bigcap_{n=1}^{\infty} \Upsilon_n\right) \leq \beta(\Upsilon_n),\ \beta(\Upsilon_\infty) = 0,\ \Upsilon_\infty \in ker\beta$.

Definition 2.3. [5] Let Q be an element of metric space (Υ, d). Then the Hausdorff MNC $H(\Upsilon)$ is the infimum of the set of all real $\epsilon > 0$ such that Q covered by a finite number of balls of radii strictly less than δ, that is

$$H(\Upsilon)=\inf\left\{\delta>0: Q\subset\bigcup_{i=1}^{n}\bar{B}(y_i,r_i), y_i\in Q, r_i<\delta,\ (i=1,2,3,...,n), n\in\mathbb{N}\right\},$$

where $\bar{B}(y_i,r_i)$ is the closed ball of radius r_i centered at $y_i\in Q$.

Now, we are concerned with certain sequence spaces which are associated with the ℓ_p spaces. Let us define the set

$$P=\{\rho=(\rho_k):0<\rho_1\leq\rho_k\leq\rho_{k+1},(k+1)\rho_k\geq\rho_{k+1}\}.$$

In 1960, Sargent [34] introduced a space, where $J(s)$ denotes the set of all sequences that can be obtained by rearranging the elements of s. For $\rho\in P$, and $\rho_0=0$

$$h(\rho)=\left\{s=(s_n):\|s\|_{h(\rho)}=\sup_{v\in J(s)}\left(\sum_{n=1}^{\infty}|v_n|\Lambda\rho_n\right)<\infty\right\},$$

where $\Lambda\rho=\Lambda\rho_n=\rho_n-\rho_{n-1}$. Also, if $\rho_k=1$ then $h(\rho)=\ell_\infty$ and if $\rho_k=k$ then $h(\rho)=\ell_1$.

In 2017, Banás and Krajewska [6] give a new direction to an existence space by introducing a fix non-increasing real sequence $\alpha{=}(\alpha_i)_{i=1}^{\infty}$ is known as a tempering sequence. Assume $h^\alpha(\rho)$ be the space of all real or complex sequences $u{=}(u_i)_{i=1}^{\infty}$ such that $\alpha u=(\alpha_m u_m)\in h(\rho)$, and $h^\alpha(\rho)$ forms a Banach space with the norm

$$\|u\|_{h^\alpha(\rho)}=\|\alpha u\|_{h(\rho)}=\sup_{v\in J(u)}\left(\sum_{n=1}^{\infty}\alpha_n|v_n|\Lambda\rho_n\right).$$

Now, consider $G:h^\alpha(\rho)\to h(\rho)$ is a mapping defined by

$$G(s)=G((s_n)_{n=1}^{\infty})=(\alpha_n s_n)_{n=1}^{\infty}=(\alpha s),$$

where $s=(s_n)_{n=1}^{\infty}\in h^\alpha(\rho)$ and $(\alpha_n s_n)_{n=1}^{\infty}=\alpha s\in h(\rho)$. For any $a=(a_n)_{n=1}^{\infty}$ and $b=(b_n)_{n=1}^{\infty}\in h^\alpha(\rho)$, we have

$$\begin{aligned}\|G(a)-G(b)\|_{h(\rho)}&=\|(\alpha_n a_n)_{n=1}^{\infty}-(\alpha_n b_n)_{n=1}^{\infty}\|_{h(\rho)}\\&=\|\alpha a-\alpha b\|_{h(\rho)}\\&=\sup_{v\in J(\alpha(a-b))}\left(\sum_{n=1}^{\infty}|v_n|\Lambda\rho_n\right)\\&=\sup_{w\in J(a-b)}\left(\sum_{n=1}^{\infty}\alpha_n|w_n|\Lambda\rho_n\right)\\&=\|a-b\|_{h^\alpha(\rho)},\end{aligned}$$

where for any sequence v in $J(\alpha(a-b))$ can be obtained as the product of α and a sequence w in $J(a-b)$. Since condition $\|G(a)-G(b)\|_{h(\rho)}=\|a-b\|_{h^\alpha(\rho)}$ holds.

Hence, the spaces $h^\alpha(\rho)$ and $h(\rho)$ are isometric to each other. The Hausdorff MNC in Banach spaces $h(\rho)$ and $h^\alpha(\rho)$ are as follows. The Hausdorff MNC $H_{h(\rho)}$ for a nonempty and bounded set $\mathcal{B}$ is determined by the formula (see[28])

$$H_{h(\rho)}(\mathcal{B}) = \lim_{n\to\infty}\left[\sup_{s\in\mathcal{B}}\left(\sup_{v\in J(s)}\left(\sum_{m=n}^{\infty}|v_m|\Lambda\rho_n\right)\right)\right]. \tag{9}$$

Since $h^\alpha(\rho)$ and $h(\rho)$ are isometric to each other, the Hausdorff MNC $H_{h^\alpha(\rho)}$ for the nonempty and bounded set $\mathcal{B}$ is defined by the formula

$$H_{h^\alpha(\rho)}(\mathcal{B}^\alpha) = \lim_{n\to\infty}\left[\sup_{s\in\mathcal{B}^\alpha}\left(\sup_{w\in J(s)}\left(\sum_{m=n}^{\infty}\alpha_m|w_m|\Lambda\rho_n\right)\right)\right]. \tag{10}$$

Let $C(I,h^\alpha(\rho))$ be collection of all continuous functions defined on the interval $I=[0,J]$ for some $J>0$, and have a value on the space $h^\alpha(\rho)$ and the norm is defined as

$$\|u\|_{C(I,h^\alpha(\rho))} = \sup_{q\in I}\|u(q)\|_{h^\alpha(\rho)},$$

where $u(q)=(u(q))_{j=1}^{\infty}\in h^\alpha(\rho)$. Now, we present a fixed point theorem along with definitions that are used for establishing and proving our results.

Theorem 2.1. [2] A mapping $G:\Upsilon\to\Upsilon$ which is compact and continuous has a fixed point, where Υ is a nonempty convex closed subset of a Banach space $\mathbb{E}$.

Theorem 2.2. [13] A continuous mapping $G:\Upsilon\to\Upsilon$ satisfying

$$\beta\,(GD)\le k\ \beta(D),$$

for any set D of Υ, where k is constant, $k\in[0,1)$, and β is an MNC. Then the mapping G has a fixed point in Υ. This theorem is known as DFPT.

Definition 2.4. [3] The fractional operator of order $\delta+n$ for the function $u\in C^1[0,\infty)$ having a non-singular kernel is defined as

$$\left(\mathcal{D}^{\delta+n}_{0,h,k}u\right)(\tau) = \left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\cos\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u^{(n+1)}(\nu)d\nu,\ \tau>0, \tag{11}$$

where $h>0,\ k\ge 0,\ 0<\delta<1,\ n\in\mathbb{N}\cup 0$ and $u\in C^1[0,\infty)$ are given. Similarly, we define the fractional operator of order $\delta+n$ for the functions $u_n\in C^1[0,\infty)$ having a non-singular kernel as

$$\left(\mathcal{D}^{\delta+n}_{0,h,k}u_n\right)(\tau) = \left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\cos\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u_n^{(n+1)}(\nu)d\nu,\ \tau>0, \tag{12}$$

where $h>0,\ k\ge 0,\ 0<\delta<1,\ n\in\mathbb{N}\cup\{0\}$ and $u_n\in C^1[0,\infty)$ are given.

Definition 2.5. [3] Let $h > 0$, $k \geq 0$, $0 < \delta < 1$, and $f \in C[0,T]$. The fractional integral of order $0 < \delta < 1$ for a function f having a non-singular kernel is defined as

$$\left(\mathcal{I}^{\delta}_{0,h,k} f\right)(\tau) = \frac{h(1-\delta)}{h^2+k^2} f(\tau) + \delta \left(\int\limits_0^{\tau} f(\nu) d\nu - \frac{k^2}{h^2+k^2} \int\limits_0^{\tau} e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f(\nu) d\tau \right),$$

where $0 < \tau < T < \infty$ *and* $\left(\mathcal{I}^{\delta}_{0,h,k} f\right)(0) = 0$.

Similarly, we define a fractional integral operator of order $0 < \delta < 1$ for the functions f_n having a non-singular kernel is defined as

$$\begin{aligned}\left(\mathcal{I}^{\delta}_{0,h,k} f_n\right)(\tau) = &\frac{h(1-\delta)}{h^2+k^2} f_n(\tau, u_n(\tau)) \\ &+ \delta \left(\int\limits_0^{\tau} f_n(\nu, u_n(\nu)) d\nu - \frac{k^2}{h^2+k^2} \int\limits_0^{\tau} e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f_n(\nu, u_n(\nu)) d\nu \right), \quad (13)\end{aligned}$$

where $f_n \in C^1([0,T]\times\mathbb{R}),\ u_n \in C^1([0,T]),\ 0 < \tau < T < \infty$ *with* $\left(\mathcal{I}^{\delta}_{0,h,k} f_n\right)(0) = 0$.

Proposition 2.3. *The problem given by Eq. (8) is equivalent to the following system of integral equations*

$$u_n(\tau) = \left(\mathcal{I}^{\delta}_{0,h,k} f_n(.,u_n(.))\right)(\tau),\ 0 \leq \tau \leq T, \quad (14)$$

i.e., the problem of infinite system of Eq. (8) and Eq. (14) have the same solution.

Proof. Suppose $u_n \in C^1[0,T]$ is the solution of Eq. (8), then we have

$$\left(\mathcal{D}^{\delta}_{0,h,k} u_n\right)'(\tau) = f'_n(\tau, u_n(\tau)),\ 0 < \tau < T. \quad (15)$$

By using Eq. (12), we obtain

$$
\begin{aligned}
&\left(\mathcal{D}^{\delta}_{0,h,k}u_n\right)'(\tau)\\
&=\left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\left\{u_n'(\tau)+\int_0^{\tau}\frac{d}{d\tau}\left(e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\cos\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)\right)u_n'(\nu)d\nu\right\}\\
&=\left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)u_n'(\tau)-\left(\frac{1}{1-\delta}\right)\left(\frac{h\delta}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\times\\
&\int_0^{\tau}e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\cos\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u_n'(\nu)d\nu-\left(\frac{1}{1-\delta}\right)\left(\frac{k\delta}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\times\\
&\int_0^{\tau}e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\sin\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u_n'(\nu)d\nu\\
&=\left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)u_n'(\tau)-\left(\frac{h\delta}{1-\delta}\right)f_n(\tau,u_n(\tau))\\
&-\left(\frac{1}{1-\delta}\right)\left(\frac{k\delta}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)\Upsilon(\tau),
\end{aligned}
\tag{16}
$$

where

$$
\Upsilon(\tau)=\int_0^{\tau}e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\sin\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u_n'(\nu)d\nu. \tag{17}
$$

Differentiating with respect to τ of the Eq. (17), we have

$$
\begin{aligned}
\Upsilon'(\tau)&=\int_0^{\tau}\frac{d}{d\tau}\left(e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\sin\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)\right)u_n'(\nu)d\nu\\
&=-\left(\frac{h\delta}{1-\delta}\right)\int_0^{\tau}e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\sin\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u_n'(\nu)d\nu\\
&\quad+\left(\frac{k\delta}{1-\delta}\right)\int_0^{\tau}e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}\cos\left(\frac{k\delta(\tau-\nu)}{1-\delta}\right)u_n'(\nu)d\nu\\
&=-\left(\frac{h\delta}{1-\delta}\right)\Upsilon(\tau)+\left(\frac{hk\delta}{h^2+k^2}\right)f_n(\tau,u_n(\tau)).
\end{aligned}
\tag{18}
$$

Solving Eq. (18) by using $\Upsilon(0)=0$, we get

$$
\Upsilon(\tau)=\left(\frac{hk\delta}{h^2+k^2}\right)\int_0^{\tau}e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}f_n(\nu,u_n(\nu))d\nu.
$$

Hence, from Eq. (16) we have

$$\left(\mathcal{D}^{\delta}_{0,h,k}u_n\right)'(\tau) = \left(\frac{1}{1-\delta}\right)\left(\frac{h^2+k^2}{h}\right)u_n'(\tau) - \left(\frac{h\delta}{1-\delta}\right)f_n(\tau,u_n(\tau)) - \left(\frac{k\delta}{1-\delta}\right)^2\int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}f_n(\nu,u_n(\nu))d\nu. \tag{19}$$

Now by using the Eq. (15) and Eq. (19) we have

$$u_n'(\tau) = \frac{h(1-\delta)}{h^2+k^2}f_n'(\tau,u_n(\tau)) + \frac{h^2\delta}{h^2+k^2}f_n(\tau,u_n(\tau)) + \left(\frac{hk^2\delta^2}{(1-\delta)(h^2+k^2)}\right)\int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}f_n(\nu,u_n(\nu))d\nu.$$

Integrating above equality from $\nu=0$ to $\nu=\tau$ and using $u_n(\tau)_{\tau=0}=u_n(0)$ and $f_n(0,u_n(0))=0$, we obtain

$$u_n(\tau)-u_n(0) = \frac{h(1-\delta)}{h^2+k^2}f_n(\tau,u_n(\tau)) + \frac{h^2\delta}{h^2+k^2}\int_0^\tau f_n(\nu,u_n(\nu))d\nu + \left(\frac{hk^2\delta^2}{(1-\delta)(h^2+k^2)}\right)\int_0^\tau\int_0^\sigma e^{\frac{-h\delta(\sigma-\nu)}{1-\delta}}f_n(\nu,u_n(\nu))d\nu d\sigma. \tag{20}$$

To use the Fubini's theorem, we have

$$\begin{aligned}&\int_0^\tau\int_0^\sigma e^{\frac{-h\delta(\sigma-\nu)}{1-\delta}}f_n(\nu,u_n(\nu))d\nu d\sigma\\ &= \int_0^\tau f_n(\nu,u_n(\nu))e^{\frac{-h\delta\tau}{1-\delta}}\left(\int_\nu^\tau e^{\frac{-h\delta\sigma}{1-\delta}}d\sigma\right)d\nu\\ &= \left(\frac{1-\delta}{h\delta}\right)\int_0^\tau f_n(\nu,u_n(\nu))d\nu - \left(\frac{1-\delta}{h\delta}\right)\int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}}f_n(\nu,u_n(\nu))d\nu.\end{aligned} \tag{21}$$

Hence, from Eq. (20) and Eq. (21), we have

$$u_n(\tau) = \left(\mathcal{I}^{\delta}_{0,h,k}f_n(.,u_n(.))\right)(\tau).$$

Since $f_n(0,u_n(0))=0$ and $u_n(\tau)_{\tau=0}=u_n(0)=0$, u_n is the solution of the Eq. (14).

Assuming that u_n satisfies Eq. (14), it is obvious that $u_n\in C^1[0,T]$. Moreover, since $f_n(0,u_n(0))=0$, we have $u_n(0)=0$. Also, an easy calculation show that $\left(\mathcal{D}^{\delta}_{0,h,k}u_n\right)(\tau)=f_n(\tau,u_n(\tau))$ for $0<\tau<T$. Hence, it can be deduced that u_n possesses a solution to Eq. (8). $\square$

Now, we reach at a position to demonstrate our main result, i.e., the existence of the solution of an infinite system of FDE (8) in the sequence space $C(I, h^{\alpha}(\rho))$ using the DFPT.

3. Existence of solution of Eq. (8) in sequence space $C(I, h^{\alpha}(\rho))$

To demonstrate the existence of solution for Eq. (8), and we introduce the following assumptions:

(*i*) The functions $G_n : I \times C(I, h^{\alpha}(\rho)) \to \mathbb{R}$ $(n \in \mathbb{N})$ and operator $G : C(I, h^{\alpha}(\rho)) \to C(I, h^{\alpha}(\rho))$ is defined as

$$(\tau, u(\tau)) \to Gu(\tau) = (G_n(\tau, u(\tau)))_{n=1}^{\infty},$$

where

$$\begin{aligned} Gu_n(\tau) = {} & \frac{h(1-\delta)}{h^2+k^2} f_n(\tau, u_n(\tau)) \\ & + \delta \left(\int_0^{\tau} f_n(\nu, u_n(\nu)) d\nu - \frac{k^2}{h^2+k^2} \int_0^{\tau} e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f_n(\nu, u_n(\nu)) d\nu \right). \end{aligned}$$

Besides that, the class $((Gu)(\tau))_{\tau \in I}$ is equicontinuous at all points of $C(I, h^{\alpha}(\rho))$.

(*ii*) The continuous functions $f_n : I \times \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ satisfying $|f_n(\tau, u_n(\tau))| \leq \psi_n(\tau)|u_n(\tau)|$, where ψ_n are continuous functions on I and $(\alpha_n \psi_n(\tau))$ is equibounded sequence on I.

Write $\psi = \sup_{n \in \mathbb{N}, \tau \in I} \{\psi_n(\tau)\}$, $M_\delta = \frac{h(1-\delta)}{h^2+k^2}$ and $N_\delta = \frac{k^2\delta}{h^2+k^2}$.

(*iii*) Assume that $\psi(M_\delta + N_\delta T + \delta T) < 1$.

Theorem 3.1. If an infinite system of FDE (8) follows the assumptions $(i)-(iii)$, then Eq. (8) possesses at least one solution $u(\tau) = (u_n(\tau))_{n=1}^{\infty} \in C(I, h^{\alpha}(\rho))$ for every $\tau \in I$.

Proof. Suppose that $J(u(\tau))$ is the set of sequences which are rearrangements of $u(\tau)$ and assume that if $v(\tau) \in J(u(\tau))$, then $\sum_{m=1}^{\infty} \alpha_m |v_m(\tau)| \Lambda s_m \leq L$, where $L > 0$, and $u(\tau) = (u_n(\tau))_{n=1}^{\infty} \in h^{\alpha}(\rho)$ for every $\tau \in I = [0, T]$ with $T > 0$.

From Eq. (14), we have the following expression for any $\tau \in I = [0, T]$

$$\begin{aligned}
\|u(\tau)\|_{h^\alpha(\rho)} =& \|\alpha u(\tau)\|_{h(\rho)} \\
=& \sup_{v \in J(u(\tau))} \left[\sum_{m=1}^{\infty} \alpha_m |v_m(\tau)| \Lambda s_m \right] \\
=& \sup_{v \in J(u(\tau))} \left[\sum_{m=1}^{\infty} \alpha_m \left(\left| \frac{h(1-\delta)}{h^2+k^2} f_m(\tau, v_m(\tau)) \right.\right.\right. \\
& \left.\left.\left. + \delta \int_0^\tau f_m(\nu, v_m(\nu)) d\nu - \frac{\delta k^2}{h^2+k^2} \int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f_m(\nu, v_m(\nu)) d\nu \right| \right) \Lambda s_m \right] \\
\leq& \sup_{v \in J(u(\tau))} \sum_{m=1}^{\infty} \alpha_m \left| \frac{h(1-\delta)}{h^2+k^2} f_m(\tau, v_m(\tau)) \right| \Lambda s_m \\
& + \sup_{v \in J(u(\tau))} \delta \sum_{m=1}^{\infty} \alpha_m \left| \int_0^\tau f_m(\nu, v_m(\nu)) d\nu \right| \Lambda s_m \\
& + \sup_{v \in J(u(\tau))} \delta \sum_{m=1}^{\infty} \alpha_m \left| \frac{k^2}{h^2+k^2} \int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f_m(\nu, v_m(\nu)) d\nu \right| \Lambda s_m \\
\leq& \frac{h(1-\delta)}{h^2+k^2} \sup_{v \in J(u(\tau))} \sum_{m=1}^{\infty} \alpha_m |f_m(\tau, v_m(\tau))| \Lambda s_m \\
& + \delta \sup_{v \in J(u(\tau))} \sum_{m=1}^{\infty} \alpha_m \int_0^\tau |f_m(\nu, v_m(\nu))| d\nu \Lambda s_m \\
& + \delta \frac{k^2}{h^2+k^2} \sup_{v \in J(u(\tau))} \sum_{m=1}^{\infty} \alpha_m \int_0^\tau \left| e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f_m(\nu, v_m(\nu)) \right| d\nu \Lambda s_m \\
\leq& M_\delta \sup_{v \in J(u(\tau))} \sum_{m=1}^{\infty} \alpha_m |\psi_m(\tau) v_m(\tau)| \Lambda s_m \\
& + \sup_{v \in J(u(\tau))} \left\{ \delta \int_0^\tau \left(\sum_{m=1}^{\infty} \alpha_m |\psi_m(\nu) v_m(\nu)| \Lambda s_m \right) d\nu \right. \\
& \left. + N_\delta \int_0^\tau \left(\sum_{m=1}^{\infty} \alpha_m |\psi_m(\nu) v_m(\nu)| \Lambda s_m \right) d\nu \right\} \\
\leq& M_\delta \psi \|u(\tau)\|_{h^\alpha(\rho)} + (\delta\psi + N_\delta \psi) L \int_0^\tau d\nu \\
\leq& M_\delta \psi \|u(\tau)\|_{h^\alpha(\rho)} + (\delta\psi + N_\delta \psi) L T,
\end{aligned}$$

i.e.,

$$\|u(\tau)\|_{h^\alpha(\rho)} \leq \frac{(\delta\psi + N_\delta\psi)LT}{1 - M_\delta\psi} = \theta(say).$$

Let $\bar{B} = \{u(\tau) \in C(I, h^\alpha(\rho)) : \ \|u\|_{C(I,h^\alpha(\rho))} \leq \theta\}$. Then $\bar{B}$ is closed, convex and bounded. Consider $S = (S_n)$ be an operator on $C(I, h^\alpha(\rho))$ such that for any $\tau \in I$

$$(Su)(\tau) = \{(S_n u)(\tau)\}_{n=1}^\infty = \{G_n(\tau, u(\tau))\}_{n=1}^\infty,$$

where, $u(\tau) = (u_n(\tau))_{n=1}^\infty \in h^\alpha(\rho)$, and $u_n(\tau) \in C(I, \mathbb{R})$ by assumption (i). Using assumption (i), we get, for all $\tau \in I$

$$(Su)(\tau) \in h^\alpha(\rho) \ and \ \sup_{v \in J(u(\tau))} \left(\sum_{m=1}^\infty \alpha_m |(S_m u)(\tau)| \Lambda s_m \right) \leq \theta < \infty,$$

with $(S_n u)(0) = 0$. Since $\|(Su)(\tau)\|_{h^\alpha(\rho)} \leq \theta$, therefore $S : \bar{B} \to \bar{B}$ is self mapping. Hence, we can say that S is continuous operator on $C(I, \bar{B})$ by assumption (i).

Now, we define the Hausdorff MNC on the space $\bar{B} \subset C(I, h^\alpha(\rho))$ as follows. For any fixed $\tau \in I$, we have

$$\begin{aligned}
H_{h^\alpha(\rho)}(S\bar{B}) &= \lim_{n\to\infty} \left[\sup_{u(\tau)\in\bar{B}} \left(\sup_{v\in J(u(\tau))} \sum_{m\geq n} \alpha_m |v_m(\tau)| \Lambda s_m \right) \right] \\
&= \lim_{n\to\infty} \left[\sup_{u(\tau)\in\bar{B}} \sup_{v\in J(u(\tau))} \left\{ \sum_{m\geq n} \alpha_m \left| \frac{h(1-\delta)}{h^2+k^2} f_m(\tau, v_m(\tau)) \right.\right.\right. \\
&\quad \left.\left.\left. + \delta \int_0^\tau f_m(\nu, v_m(\nu)) d\nu - \delta \frac{k^2}{h^2+k^2} \int_0^\tau e^{\frac{-h\delta(\tau-\nu)}{1-\delta}} f_m(\nu, v_m(\nu)) d\nu \right| \Lambda s_m \right\} \right] \\
&\leq M_\delta\psi \lim_{n\to\infty} \left[\sup_{u(\tau)\in\bar{B}} \left(\sup_{v\in J(u(\tau))} \sum_{m\geq n} \alpha_n |v_n(\tau)| \Lambda s_n \right) \right] \\
&\quad + \delta\psi T \lim_{n\to\infty} \left[\sup_{u(\tau)\in\bar{B}} \left(\sup_{v\in J(u(\tau))} \sum_{m\geq n} \alpha_n |v_n(\tau)| \Lambda s_n \right) \right] \\
&\quad + N_\delta\psi T \lim_{n\to\infty} \left[\sup_{u(\tau)\in\bar{B}} \left(\sup_{v\in J(u(\tau))} \sum_{m\geq n} \alpha_n |v_n(\tau)| \Lambda s_n \right) \right] \\
&= \left(M_\delta\psi + \delta\psi T + N_\delta\psi T \right) H_{h^\alpha(\rho)}(\bar{B}).
\end{aligned}$$

Hence, we have

$$H_{h^\alpha(\rho)}(S\bar{B}) \leq \psi \left(M_\delta + \delta T + N_\delta T \right) H_{h^\alpha(\rho)}(\bar{B}).$$

As $\psi\left(M_\delta + \delta T + N_\delta T\right) < 1$ and S satisfies all the conditions of Theorem 2.2. Hence, S has a fixed point in $\bar{B}$. Therefore an infinite system of FDE (8) has a solution in $C(I, h^\alpha(\rho))$. $\square$

Now, we give an example of an infinite system of FDE of the type (8) to validate our results.

4. Example

Example 4.1. Consider an infinite system of Caputo-Fabrizio FDE

$$\left(\mathcal{D}^{\frac{1}{5}}_{0,1,0} u_n\right)(\tau) = \sum_{m \geq n} \frac{\sin(u_m(\tau))}{10 m^2 n^2 e^\tau}, \;\; \tau \in [0,1], \;\; u_n \in C(I, \mathbb{R}), \;\; n \in \mathbb{N} \tag{22}$$

with $u_n(0) = 0$.

Here, $f_n(\tau, u_n(\tau)) = \sum\limits_{m \geq n} \frac{\sin(u_m(\tau))}{10 m^2 n^2 e^\tau}$, $\tau \in [0,1] = I$ and for every $n \in \mathbb{N}$. But $u_n(\tau) \in C(I, \mathbb{R})$, then for every $n \in \mathbb{N}$, f_n's are continuous functions and for every $n \in \mathbb{N}$, we have

$$\left| f_n(\tau, u_n(\tau)) \right| = \left| \sum_{m \geq n} \frac{\sin(u_m(\tau))}{10 m^2 n^2 e^\tau} \right| \leq \sum_{m \geq n} \frac{|u_m(\tau)|}{10 m^2 n^2},$$

where $\psi_n(\tau) = \frac{\sum\limits_{m \geq n} \frac{1}{m^2}}{10 n^2}$. Let $\alpha_n = \frac{1}{n^3}$ and $s_n = n$. Then, we have $\psi = \frac{\pi^2}{60}$, $M_\delta = \frac{4}{5}$, and $N_\delta = 0$ and

$$\psi\left(M_\delta + \delta T + N_\delta T\right) = \frac{\pi^2}{60}\left(\frac{4}{5} + \frac{1}{5}\right) = 0.164 < 1.$$

Therefore, Theorem 3.1 figure out that an infinite system of FDE (22) possesses a solution in the sequence space $C(I, h^\alpha(\rho))$.

5. Homotopy perturbation and ADM to solve Example (4.1)

In this section, we approximated the following infinite system of an integral equation, which is equivalent to the infinite system of differential Eq. (22)

$$u_n(\tau) = \frac{4}{5} \sum_{m \geq n} \frac{\sin(u_m(\tau))}{10 m^2 n^2 e^\tau} + \frac{1}{5} \int_0^\tau \sum_{m \geq n} \frac{\sin(u_m(\nu))}{10 m^2 n^2 e^\nu} d\nu, \tag{23}$$

by a coupled semi-analytical method. This method is a combination of the mHPM and ADM. For more details about these methods and their application, one can see the references [**1**, **7**, **16**, **32**]. In this article, we generalize the mHPM to infinite functions and use the ADM for the simplification of nonlinear terms. In order to do so, we consider the nonlinear problem with boundary conditions of the following form

$$\begin{aligned} &A(u_1(\tau), u_2(\tau), ..., u_n(\tau), ...) - f(\tau, n) = 0 \ (\tau \in \Omega, n \in \mathbb{N}), \\ &B\left(u_i, \frac{\partial u_i}{\partial r}\right) = 0, \ (r \in \Gamma), \end{aligned} \tag{24}$$

where A is the general nonlinear operator and B is the boundary operator and f is an analytic function. By observing the work of [19] and [30], we convert the operator A into nonlinear operators N_1 and N_2 (sometimes N_1 and N_2 can be linear operators) and divide f into f_1 and f_2. Therefore, we can write Eq. (24) as

$$N_1(u_1(\tau), u_2(\tau), ..., u_n(\tau), ...) - f_1(\tau, n) + N_1(u_1(\tau), u_2(\tau), ..., u_n(\tau), ...) - f_2(\tau, n) = 0.$$

Applying mHPM for an infinite functions, we have

$$\begin{cases} H(v_1(\tau), v_2(\tau), ..., v_n(\tau), .., p) & = N_1(v_1(\tau), v_2(\tau), ..., v_n(\tau), ...) - f_1(\tau, n) \\ & \quad +p(N_2(v_1(\tau), v_2(\tau), ..., v_n(\tau), ...)) - f_2(\tau, n) = 0 \\ (p \in [0,1]), & \end{cases} \tag{25}$$

where p is a perturbation parameter and v_i are the approximation of u_i for $i \in \mathbb{N}$. Variating perturbation parameter p from $p = 0$ to $p = 1$, we have

$$\begin{gathered} N_1(v_1(\tau), v_2(\tau), ..., v_n(\tau), ...) = f_1(\tau, n), \\ \vdots \\ A(v_1(\tau), v_2(\tau), ..., v_n(\tau), ...) - f(\tau, n) = 0. \end{gathered}$$

Therefore, we get the solution of Eq. (24) by putting $p = 1$ in the Eq. (25) and let the solution is in series form

$$\begin{cases} u_n(\tau) \approx v_n(\tau) = \sum\limits_{j=0}^{\infty} p^j v_{j,n}(\tau) \\ u_n(\tau) = \lim\limits_{p \to 1} v_n(\tau). \end{cases} \tag{26}$$

To solve the nonlinear infinite system of integral Eq. (23), we select the operators N_1 and N_2 and f as

$$\begin{cases} N_1(u_1(\tau), u_2(\tau), ..., u_n(\tau), ...) = u_n(\tau), \\ N_2(u_1(\tau), u_2(\tau), ..., u_n(\tau), ...) = -\frac{4}{5} \sum\limits_{m \geq n} \frac{\sin(u_m(\tau))}{10m^2n^2e^{\tau}} - \frac{1}{5} \int\limits_0^{\tau} \sum\limits_{m \geq n} \frac{\sin(u_m(\nu))}{10m^2n^2e^{\nu}} d\nu, \\ f(t, n) = 0. \end{cases} \tag{27}$$

Putting the values of Eq. (27) and Eq. (26) into the Eq. (25), we have

$$p\left(-\frac{4}{5}\sum_{m\geq n}\frac{\sin(u_m(\tau))}{10m^2n^2e^{\tau}}-\frac{1}{5}\int_0^{\tau}\sum_{m\geq n}\frac{\sin(u_m(\nu))}{10m^2n^2e^{\nu}}d\nu-f_2(\tau,n)\right)$$
$$+\sum_{j=0}^{\infty}p^j v_{j,n}(\tau)-f_1(\tau,n)=0. \tag{28}$$

To approximate the nonlinear term in Eq. (28), we apply the ADM of the form,

$$\sum_{m\geq n}\frac{\sin(u_m(\tau))}{10m^2n^2}=\sum_{j=0}^{\infty}p^j A_{j,n}(\tau), \tag{29}$$

where the Adomian polynomial is

$$A_{k,n}(\tau)=\frac{1}{k!}\left(\frac{d^k}{dp^k}\sum_{m\geq n}\frac{\sin\left(\sum\limits_{j=0}^{\infty}p^j v_{j,n}(\tau)\right)}{10m^2n^2}\right)_{p=0}.$$

Using Eq. (29) into the Eq. (28), we get

$$p\left(-\frac{4}{5}e^{-\tau}\sum_{j=0}^{\infty}p^jA_{j,n}(\tau)-\frac{1}{5}\int_0^{\tau}e^{-\nu}\sum_{j=0}^{\infty}p^jA_{j,n}(\nu)d\nu-f_2(\tau,n)\right)$$
$$+\sum_{j=0}^{\infty}p^j v_{j,n}(\tau)-f_1(\tau,n)=0. \tag{30}$$

Comparing the coefficient of p^{th}-powers of the Eq. (30), we have

$$p^0:\left(v_{0,n}(\tau)-f_1(\tau,n)\right),$$
$$p^1:\left(v_{1,n}(\tau)-\frac{4}{5}e^{-\tau}A_{0,n}(\tau)-\frac{1}{5}\int_0^{\tau}e^{-\nu}A_{0,n}(\nu)d\nu-f_2(\tau,n)\right),$$
$$p^k:\left(v_{k,n}(\tau)-\frac{4}{5}e^{-\tau}A_{k-1,n}(\tau)-\frac{1}{5}\int_0^{\tau}e^{-\nu}A_{k-1,n}(\nu)d\nu\right),\ \textit{where}\ (k\geq 2).$$

Since, Eq. (30) is equal to zero, the coefficients of p^{th} powers are equal to zero. Thus, we obtain an iterative algorithm to solve the Eq. (23) as follows.

Algorithm

$$v_{0,n}(\tau) = f_1(\tau, n),$$

$$v_{1,n}(\tau) = f_2(\tau, n) + \frac{4}{5}e^{-\tau}A_{0,n}(\tau) + \frac{1}{5}\int_0^\tau e^{-\nu}A_{0,n}(\nu)d\nu,$$

$$v_{k,n}(\tau) = \frac{4}{5}e^{-\tau}A_{k-1,n}(\tau) + \frac{1}{5}\int_0^\tau e^{-\nu}A_{k-1,n}(\nu)d\nu, \ where\ (k \geq 2).$$

For test purposes, we evaluate some terms of the sequence $\{u_1(\tau), u_2(\tau), ...\}$ by the above algorithm, where the Adomian polynomial is

$$A_{0,n}(\tau) = \sum_{m \geq n} \frac{\sin(v_{0,n}(\tau))}{10m^2n^2}.$$

Since $f(\tau, n) = 0$ in Eq. (23), we choose $f_1(\tau, n) = -f_2(\tau, n) = \frac{\pi}{2}$ and set $v_{0,n}(\tau) = f_1(\tau, n) = \frac{\pi}{2} = -f_2(\tau, n)$. Therefore, we have

$$\begin{cases} v_{0,n}(\tau) = \frac{\pi}{2}, \\ v_{1,n}(\tau) = -\frac{\pi}{2} + \frac{4}{5}e^{-\tau}A_{0,n}(\tau) + \frac{1}{5}\int_0^\tau e^{-\nu}A_{0,n}(\nu)d\nu, \end{cases} \tag{31}$$

and for $n = 1$ we get

$$\begin{cases} v_{0,1}(\tau) = \frac{\pi}{2}, \\ v_{1,1}(\tau) = -\frac{\pi}{2} + \frac{\pi^2}{75}e^{-\tau} + \frac{\pi^2}{300}(1 - e^{-\tau}). \end{cases}$$

To approximate the solution, we only consider first two terms of the series Eq. (26), therefore we have

$$u_1(\tau) = v_{0,1}(\tau) + v_{1,1}(\tau) = 0.0329 + 0.0987e^{-\tau}. \tag{32}$$

Similarly, we get some terms of the above series by using MATLAB (R2023a) version as follows

$$\begin{aligned} u_2(\tau) &= 0.0082 + 0.0247e^{-\tau}, \\ u_3(\tau) &= 8.7763e^{-04} + 0.0026e^{-\tau}, \\ u_4(\tau) &= 3.5478e^{-04} + 0.0011e^{-\tau}, \\ u_5(\tau) &= 2.2000e^{-04} + 6.6000e^{-04}e^{-\tau}. \end{aligned} \tag{33}$$

We find out some terms of the sequence $\{u_n(\tau)\}$, in Eq. (32) and Eq. (33) for $n = 1, 2, 3, 4, 5$. Moreover tracing them in the Figures 1 and 2, it shows that for any $0 \leq \tau \leq 1$, $u_1(\tau) < 1.4 \times 10^{-1}$, $u_2(\tau) < 1.3 \times 10^{-2}$, $u_3(\tau) < 3.6 \times 10^{-3}$,.., $u_5(\tau) < 9.0 \times 10^{-4}$. Thus, we conclude that for any $0 \leq \tau \leq 1$ and n large,

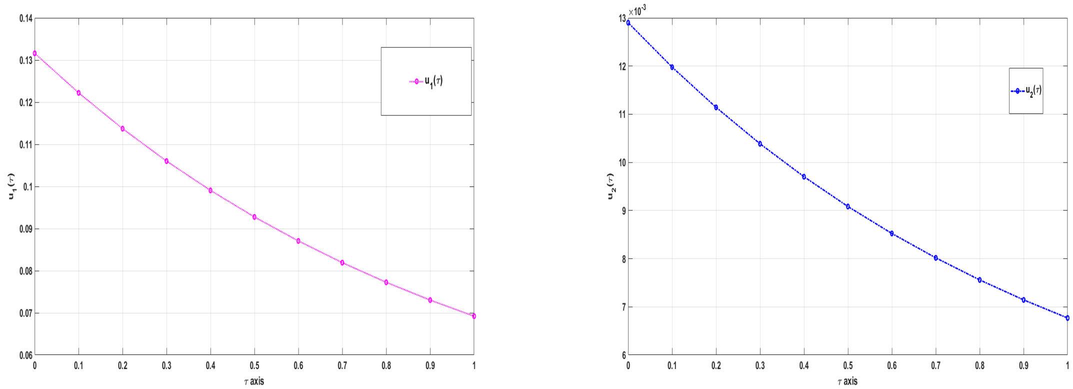

FIGURE 1. **The first and second terms of the sequence.**

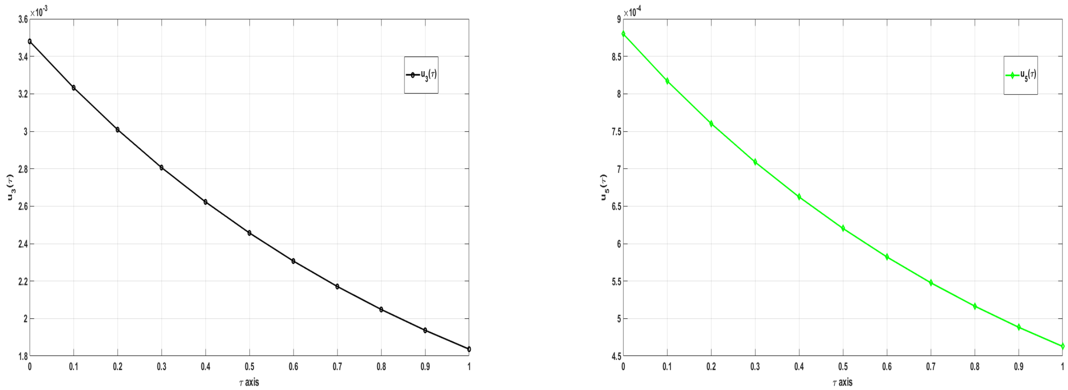

FIGURE 2. **The third and fifth terms of the sequence.**

$u_n(\tau) \to 0$. Figure 3 illustrated the better understanding of the convergence of the sequence $\{u_n(\tau)\}$.

6. Conclusion

In this article, we find an infinite system of FDE involving a generalized Caputo-Fabrizio fractional derivative with a kernel function having trigonometric and exponential functions. Using the DFPT, we established the existence of the solution of the given equation in the tempered sequence space $C(I, h^{\alpha}(\rho))$. Moreover, we provide an example and an algorithm based on mHPM and ADM methods to validate our obtained results and approximate the solution with high accuracy.

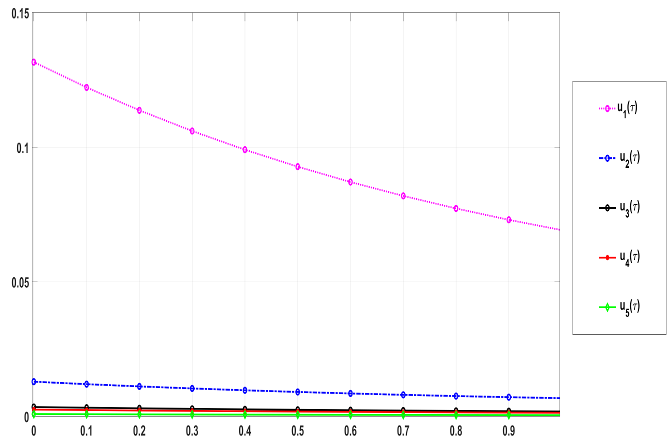

Figure 3. **Convergence of $u_n \to 0$.**

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Conflict of interest

All the authors declare that there is no conflict of interest.

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