Mathematical Analysis and its Contemporary Applications
Volume 5, Issue 4, 2023, 55–69
doi: 10.30495/maca.2023.2011241.1087
ISSN 2716-9898

Investigated on Neutrosophic 2-normed $\mathfrak{I}$-convergent double sequence spaces with bounded linear operator

M. Jeyaraman and S. Iswariya*

Abstract. The research here that we develop an operator of the bounded linear method generates certain Neutrosophic 2-normed double sequence spaces of $\mathfrak{I}$-convergent that are specifying their results. In addition, we search for certain fundamental topological as well as algebraic characteristics among these particular fields.

1. Introduction

Fuzzy topology has become one of the most essential and valuable methods for interacting with circumstances where conventional hypotheses fail. The newest improvement in fuzzy topology includes the concepts of Neutrosophic Normed (NN) space [12] along with Neutrosophic 2-Normed space (N2-NS). Unfortunately, there are some scenarios in which the usual standard fails to apply; hence, the idea of the Neutrosophic norm appears to be more feasible for these kinds of conditions; therefore, we may deal with problems like this by modelling the inexactness of the norm in certain environments.

Sal'at and others [13], Khan et al. [7, 8, 9], Tripathy and Hazarika [16], as well as several more researchers in the future, investigated it. Das et al. [3] have researched the idea of I in addition to double sequences of I-convergence within R. To begin with, according to an application of Mursaleen et al.'s [11], Khan and

2020 *Mathematics Subject Classification.* 46A70, 54A20, 54A40.
Key words and phrases. Bounded linear operator, Filter, Ideal, $\mathfrak{I}$-convergence, Neutrosophic 2-normed spaces
*Corresponding author

others investigated statistical convergence and the idea of double sequence of I-convergence within intuitionistic fuzzy normed spaces, while Mursaleen and Lohani improved I-convergence as well as I-Cauchy over sequence in *N2-NS*.

In 1998, Smarandache [14] developed the ideas of neutrosophic logic in addition to the Neutrosophic Set [NS]. Jeyaraman, Ramachandran, and Shakila [6] established theorems of approximate fixed points in 2022 regarding weak contractions in Neutrosophic Normed Spaces [NNS]. Statistical Δ^m convergence in *NNS* was recently presented by Jeyaraman and Jenifer [5].

2. Preliminaries

Definition 2.1. Let $\mathfrak{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal and an *N-2-NS* $(\Xi, \dot{\mu}, \ddot{\nu}, \dddot{\tau}, *, \Delta, \circledast)$. After that $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{ij}})$ sequence is said to have an $\mathfrak{I}$-Convergent towards $\mathfrak{L} \in \Xi$ in relate within a *NN* $(\dot{\mu}, \ddot{\nu}, \dddot{\tau})_2$, when for all $\varepsilon > 0$ as well as $\mathfrak{t} > 0$, the set

$$\{(\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}, \breve{p}; \mathfrak{t}) \leqslant 1 - \varepsilon \text{ or } \ddot{\nu}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}, \breve{p}; \mathfrak{t}) \geqslant \varepsilon \text{ and } \dddot{\tau}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}, \breve{p}; \mathfrak{t}) \geqslant \varepsilon\} \in \mathfrak{I}.$$

Here, the present part $\mathfrak{L}$ is referred to as the $\mathfrak{I}$-limit among the $(\mathfrak{x}_{\mathfrak{ij}})$ sequence in regard for the *NN* $(\dot{\mu}, \ddot{\nu}, \dddot{\tau})_2$ therefore we write $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x}_{\mathfrak{ij}} = \mathfrak{L}$.

Definition 2.2. Let $(\Xi, \dot{\mu}, \ddot{\nu}, \dddot{\tau}, *, \Delta, \circledast)$ be an *N-2-NS*. After that $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{ij}})$ sequence is said to be a $\mathfrak{I}$- Cauchy sequence in relate within a *NN* $(\dot{\mu}, \ddot{\nu}, \dddot{\tau})_2$, when for all $\varepsilon > 0$ as well as $\mathfrak{t} > 0$, the set

$$\{(\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{x}_{\mathfrak{mn}}, \breve{p}; \mathfrak{t}) \leqslant 1 - \varepsilon \text{ or } \ddot{\nu}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{x}_{\mathfrak{mn}}, \breve{p}; \mathfrak{t}) \geqslant \varepsilon \text{ and } \dddot{\tau}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{x}_{\mathfrak{mn}}, \breve{p}; \mathfrak{t}) \geqslant \varepsilon\} \in \mathfrak{I}.$$

3. Main Results

We introduce the double sequence spaces in the following section:

$${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}}) = \left\{ (\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\ell_\infty : \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \hat{\mathfrak{h}}; \mathfrak{t}) \leqslant 1 - \varepsilon \text{ or} \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \hat{\mathfrak{h}}; \mathfrak{t}) \geqslant \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \hat{\mathfrak{h}}; \mathfrak{t}) \geqslant \varepsilon \end{array} \right\} \in \mathfrak{I} \right\};$$

$${}_2\mathfrak{S}^{\mathfrak{I}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}}) = \left\{ (\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\ell_\infty : \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}), \hat{\mathfrak{h}}; \mathfrak{t}) \leqslant 1 - \varepsilon \text{ or} \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}), \hat{\mathfrak{h}}; \mathfrak{t}) \geqslant \varepsilon \text{ and } \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}), \hat{\mathfrak{h}}; \mathfrak{t}) \geqslant \varepsilon \end{array} \right\} \in \mathfrak{I} \right\}.$$

Take $\mathfrak{x} \in \Xi, \breve{r} \in (0,1)$ along with for every $\mathfrak{t} > 0$, after that a set becomes

$${}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{ (\hat{\mathfrak{h}}_{\mathfrak{ij}}) \in {}_2\ell_\infty : \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{h}}_{\mathfrak{ij}}), \breve{p}; \mathfrak{t}) > 1 - \breve{r} \text{ or} \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{h}}_{\mathfrak{ij}}), \breve{p}; \mathfrak{t}) < \breve{r} \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{h}}_{\mathfrak{ij}}), \breve{p}; \mathfrak{t}) < \breve{r} \end{array} \right\} \in \mathfrak{I} \right\}$$

is known as that open ball has a center of $\mathfrak{x}$ with a radius of $\breve{r}$ relative to $\mathfrak{t}$.

Theorem 3.1. *If there is a sequence* $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ *and then* ${}_2\mathfrak{S}^{\mathfrak{I}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ *is* $\mathfrak{I}$*-convergent with relate to the N2-N* $(\dot{\mu}, \ddot{\nu}, \dddot{\tau})_2$*, and then it is an unique limit.*

Proof. Let us assume that $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x} = \mathfrak{L}_1$ and $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x} = \mathfrak{L}_2$. Take $\breve{r} > 0$ which yields $(1-\breve{r}) * (1-\breve{r}) > 1-\varepsilon, \breve{r} \, \Delta \, \breve{r} < \varepsilon$ and $\breve{r} \circledast \breve{r} < \varepsilon$, given $\varepsilon > 0$. Then, determine the subsequent sets as follows for any $\mathfrak{t} > 0$:

$$
\begin{aligned}
{}_2\mathfrak{K}_{(\dot{\mu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left\{(\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \leqslant 1-\breve{r}\right\}, \\
{}_2\mathfrak{K}_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left\{(\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \leqslant 1-\breve{r}\right\}, \\
{}_2\mathfrak{K}_{(\ddot{\nu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left\{(\mathfrak{i},\mathfrak{j}) : \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\}, \\
{}_2\mathfrak{K}_{(\ddot{\nu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left\{(\mathfrak{i},\mathfrak{j}) : \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\}, \\
{}_2\mathfrak{K}_{(\dddot{\tau},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left\{(\mathfrak{i},\mathfrak{j}) : \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\}, \\
{}_2\mathfrak{K}_{(\dddot{\tau},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left\{(\mathfrak{i},\mathfrak{j}) : \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\}.
\end{aligned}
$$

Since $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x} = \mathfrak{L}_1$, we have

$$
{}_2\mathfrak{K}_{(\dot{\mu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}),\ {}_2\mathfrak{K}_{(\ddot{\nu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \text{ and } {}_2\mathfrak{K}_{(\dddot{\tau},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{I}.
$$

In addition, by applying $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x} = \mathfrak{L}_2$, we get

$$
{}_2\mathfrak{K}_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}),\ {}_2\mathfrak{K}_{(\ddot{\nu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \text{ and } {}_2\mathfrak{K}_{(\dddot{\tau},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{I}.
$$

Let us now

$$
\begin{aligned}
{}_2\mathfrak{K}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) = &\left({}_2\mathfrak{K}_{(\dot{\mu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cup {}_2\mathfrak{K}_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})\right) \\
&\cap \left({}_2\mathfrak{K}_{(\ddot{\nu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cup {}_2\mathfrak{K}_{(\ddot{\nu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})\right) \\
&\cap \left({}_2\mathfrak{K}_{(\dddot{\tau},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cup {}_2\mathfrak{K}_{(\dddot{\tau},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})\right) \in \mathfrak{I},
\end{aligned}
$$

afterwards, we observe that ${}_2\mathfrak{K}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{I}$. This suggests that the complement is ${}_2\mathfrak{K}^c_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{F}(\mathfrak{I})$. We get three possible cases, if $(\mathfrak{i},\mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$. That is,

$(\mathfrak{i},\mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\dot{\mu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cap {}_2\mathfrak{K}^c_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$ or $(\mathfrak{i},\mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\ddot{\nu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cap {}_2\mathfrak{K}^c_{(\ddot{\nu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$

and $(\mathfrak{i},\mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\dddot{\tau},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cap {}_2\mathfrak{K}^c_{(\dddot{\tau},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$. Let we first consider that $(\mathfrak{i},\mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\dot{\mu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cap {}_2\mathfrak{K}^c_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$. After, we have

$$
\begin{aligned}
\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{L}_1 - \mathfrak{L}_2), \breve{p}; \mathfrak{t}) &\geqslant \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) * \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \\
&> (1-\breve{r}) * (1-\breve{r}) > 1-\varepsilon.
\end{aligned}
$$

As a result that $\varepsilon > 0$ is arbitrarily in nature, consider $\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{L}_1 - \mathfrak{L}_2), \breve{p}; \mathfrak{t}) = 1$ for all $\mathfrak{t} > 0$, which yields $\mathfrak{L}_1 = \mathfrak{L}_2$, according to $\dot{\mu}(\mathfrak{x}, \breve{p}; \mathfrak{t}) > 0$ for every $\mathfrak{t} > 0$ as well as bounded linear operator $\tilde{\mathfrak{V}}$ (BLO). Let the second part, we write that if $(\mathfrak{i}, \mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\ddot{\nu},1)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) \cap {}_2\mathfrak{K}^c_{(\ddot{\nu},2)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}})$,

$$\ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{L}_1 - \mathfrak{L}_2), \breve{p}; \mathfrak{t}) \leqslant \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \Delta\, \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) < \breve{r} \,\Delta\, \breve{r} < \varepsilon.$$

Therefore, we have $\ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{L}_1 - \mathfrak{L}_2), \breve{p}; t) = 0$, for all $\mathfrak{t} > 0$, which suggest that $\mathfrak{L}_1 = \mathfrak{L}_2$, the fact that $\ddot{\nu}(\mathfrak{x}, \breve{p}; \mathfrak{t}) > 0$ for any $\mathfrak{t} > 0$ and a *BLO* $\mathfrak{V}$. Consider the another hand, when $(\mathfrak{i}, \mathfrak{j}) \in {}_2\mathfrak{K}^c_{(\dddot{\tau},1)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) \cap {}_2\mathfrak{K}^c_{(\dddot{\tau},2)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}})$, then we write

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{L}_1 - \mathfrak{L}_2), \breve{p}; \mathfrak{t}) \leqslant \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}) \circledast \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}) < \breve{r} \circledast \breve{r} < \varepsilon.$$

Therefore, we get $\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{L}_1 - \mathfrak{L}_2), \breve{p}; \mathfrak{t}) = 0$, for all $\mathfrak{t} > 0$, which implies that $\mathfrak{L}_1 = \mathfrak{L}_2$, since $\dddot{\tau}(\mathfrak{x}, \breve{p}; \mathfrak{t}) > 0$ for every $\mathfrak{t} > 0$ where $\tilde{\mathfrak{V}}$ is a linear bounded operator. Hence, we obtain the conclusion that the limit is unique in every case. Thus, the theorem is now fully proven. □

Theorem 3.2. *Let $\tilde{\mathfrak{V}}$ bounded linear operator that defines $\chi(\tilde{\mathfrak{V}})$ be an N-2-NS and let $\mathfrak{I}$ become an admissible ideal. And following that*

(i) *if $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x}_{\mathrm{ij}} = \mathfrak{L}_1$ and $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{y}_{\mathrm{ij}} = \mathfrak{L}_2$, then $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim(\mathfrak{x}_{\mathrm{ij}} + \mathfrak{y}_{\mathrm{ij}}) = \mathfrak{L}_1 + \mathfrak{L}_2$*
(ii) *if $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x}_{\mathrm{ij}} = \mathfrak{L}$ then $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \alpha\mathfrak{x}_{\mathrm{ij}} = \alpha\mathfrak{L}$*

where α acts as a scalar and $\chi = {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ along with ${}_2\mathfrak{S}^{\mathfrak{I}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$.

Proof. (i) Let $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x}_{\mathrm{ij}} = \mathfrak{L}_1$ and $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{y}_{\mathrm{ij}} = \mathfrak{L}_2$. Select $\breve{r} > 0$ in which case $(1 - \breve{r}) * (1 - \breve{r}) > 1 - \varepsilon, \breve{r} \,\Delta\, \breve{r} < \varepsilon$ and $\breve{r} \circledast \breve{r} < \varepsilon$ with given $\varepsilon > 0$. Define the subsequent sets as follows for all $\mathfrak{t} > 0$:

$${}_2\mathfrak{K}_{(\dot{\mu},1)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{(\mathfrak{i}, \mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \leqslant 1 - \breve{r}\right\},$$

$${}_2\mathfrak{K}_{(\dot{\mu},2)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{(\mathfrak{i}, \mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{y}_{\mathrm{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \leqslant 1 - \breve{r}\right\},$$

$${}_2\mathfrak{K}_{(\ddot{\nu},1)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{(\mathfrak{i}, \mathfrak{j}) : \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\},$$

$${}_2\mathfrak{K}_{(\ddot{\nu},2)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{(\mathfrak{i}, \mathfrak{j}) : \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{y}_{\mathrm{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\}, \text{ and}$$

$${}_2\mathfrak{K}_{(\dddot{\tau},1)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{(\mathfrak{i}, \mathfrak{j}) : \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\},$$

$${}_2\mathfrak{K}_{(\dddot{\tau},2)_2}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{(\mathfrak{i}, \mathfrak{j}) : \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{y}_{\mathrm{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \breve{r}\right\}$$

Since $\mathfrak{J}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x}_{\mathfrak{ij}} = \mathfrak{L}_1$, we have

$$_2\mathfrak{K}_{(\dot{\mu},1)_2}(\breve{r},t)(\tilde{\mathfrak{V}}),\ _2\mathfrak{K}_{(\ddot{\nu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \text{ and } _2\mathfrak{K}_{(\dddot{\tau},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{J}.$$

In addition, by applying $\mathfrak{J}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{y}_{\mathfrak{ij}} = \mathfrak{L}_2$, we have

$$_2\mathfrak{K}_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}),\ _2\mathfrak{K}_{(\ddot{\nu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \text{ and } _2\mathfrak{K}_{(\dddot{\tau},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{J}.$$

Let us now

$$\begin{aligned} _2\mathfrak{K}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) &= \left(_2\mathfrak{K}_{(\dot{\mu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cup\ _2\mathfrak{K}_{(\dot{\mu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})\right) \\ &\cap \left(_2\mathfrak{K}_{(\ddot{\nu},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cup\ _2\mathfrak{K}_{(\ddot{\nu},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})\right) \\ &\cap \left(_2\mathfrak{K}_{(\dddot{\tau},1)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \cup\ _2\mathfrak{K}_{(\dddot{\tau},2)_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})\right) \in \mathfrak{J}, \end{aligned}$$

this indicates that non empty set $_2\mathfrak{K}^c_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$ within $\mathfrak{F}(\mathfrak{J})$. At this point, we must demonstrate that

$$_2\mathfrak{K}^c_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \subset \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} + \hat{\mathfrak{y}}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{L}_1 + \mathfrak{L}_2), \breve{p}; \mathfrak{t}) > 1 - \varepsilon, \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} + \hat{\mathfrak{y}}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{L}_1 + \mathfrak{L}_2), \breve{p}; \mathfrak{t}) < \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} + \hat{\mathfrak{y}}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{L}_1 + \mathfrak{L}_2), \breve{p}; \mathfrak{t}) < \varepsilon \end{array} \right\}.$$

If $(\mathfrak{i},\mathfrak{j}) \in\ _2\mathfrak{K}^c_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$, then we get

$$\begin{aligned} &\dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) > 1 - \breve{r}, \dot{\mu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) > 1 - \breve{r}, \\ &\ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) < \breve{r}, \ddot{\nu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) < \breve{r}, \text{ and} \\ &\dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) < \breve{r}, \dddot{\tau}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) < \breve{r}. \end{aligned}$$

Therefore

$$\begin{aligned} \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} + \hat{\mathfrak{y}}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{L}_1 + \mathfrak{L}_2), \breve{p}; \mathfrak{t}) &\geqslant \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) * \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{y}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \\ &> (1 - \breve{r}) * (1 - \breve{r}) > 1 - \varepsilon, \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} + \hat{\mathfrak{y}}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{L}_1 + \mathfrak{L}_2), \breve{p}; \mathfrak{t}) &\leqslant \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \Delta\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{y}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \\ &< \breve{r}\ \Delta\ \breve{r} < \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} + \hat{\mathfrak{y}}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{L}_1 + \mathfrak{L}_2), \breve{p}; \mathfrak{t}) &\leqslant \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}} - \mathfrak{L}_1), \breve{p}; \frac{\mathfrak{t}}{2}\right) \circledast \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{y}_{\mathfrak{ij}} - \mathfrak{L}_2), \breve{p}; \frac{\mathfrak{t}}{2}\right) \\ &< \breve{r} \circledast \breve{r} < \varepsilon. \end{aligned}$$

This shows that

$$ {}_2\mathfrak{K}^{c}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \subset \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}+\hat{\mathfrak{y}}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{L}_1+\mathfrak{L}_2),\breve{p};\mathfrak{t}) > 1-\varepsilon, \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}+\hat{\mathfrak{y}}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{L}_1+\mathfrak{L}_2),\breve{p};\mathfrak{t}) < \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}+\hat{\mathfrak{y}}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{L}_1+\mathfrak{L}_2),\breve{p};\mathfrak{t}) < \varepsilon \end{array} \right\}. $$

Since ${}_2\mathfrak{K}^{c}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{F}(\mathfrak{I})$, we have $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}-\lim(\mathfrak{x}_{\mathfrak{ij}}+\hat{\mathfrak{y}}_{\mathfrak{ij}}) = \mathfrak{L}_1+\mathfrak{L}_2$. As a result *BLO* $\tilde{\mathfrak{V}}$.

(ii) For $\alpha = 0$, that is obvious. By let $\alpha \neq 0$. When a given $\varepsilon > 0$ in addition $\mathfrak{t} > 0$,

$$ \tilde{\mathfrak{V}}(\varepsilon) = \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) > 1-\varepsilon, \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \end{array} \right\} \in \mathfrak{F}(\mathfrak{I}). $$

It provides sufficient proof that for every $\varepsilon > 0$ along with $\mathfrak{t} > 0$,

$$ \tilde{\mathfrak{V}}(\varepsilon) \subset \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) > 1-\varepsilon, \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \end{array} \right\}. \tag{1} $$

Let us say $(\mathfrak{i},\mathfrak{j}) \in \tilde{\mathfrak{V}}(\varepsilon)$. And then we get

$$ \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) > 1-\varepsilon, \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \text{ and } \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon. $$

So, we have

$$ \begin{aligned} \dot{\mu}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) &= \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\frac{\mathfrak{t}}{|\alpha|}\right) \geqslant \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) * \dot{\mu}\left(0,\breve{p};\frac{\mathfrak{t}}{|\alpha|}-t\right) \\ &= \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) * 1 = \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) > 1-\epsilon. \end{aligned} $$

Furthermore,

$$ \begin{aligned} \ddot{\nu}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) &= \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\frac{\mathfrak{t}}{|\alpha|}\right) \leqslant \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) \ \Delta\ \ddot{\nu}\left(0,\breve{p};\frac{\mathfrak{t}}{|\alpha|}-t\right) \\ &= \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) \ \Delta\ 0 = \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) < \epsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) &= \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\frac{\mathfrak{t}}{|\alpha|}\right) \leqslant \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) \circledast \dddot{\tau}\left(0,\breve{p};\frac{\mathfrak{t}}{|\alpha|}-t\right) \\ &= \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) \circledast 0 = \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}-\mathfrak{L}),\breve{p};\mathfrak{t}) < \epsilon. \end{aligned} $$

Hence, we obtain

$$ \tilde{\mathfrak{V}}(\varepsilon) \subset \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) > 1-\varepsilon, \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\alpha\mathfrak{x}_{\mathfrak{ij}}-\alpha\mathfrak{L}),\breve{p};\mathfrak{t}) < \varepsilon \end{array} \right\}, $$

and we conclude from (3.1) that is $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}-\lim \alpha\mathfrak{x}_{\mathfrak{ij}} = \alpha\mathfrak{L}$. □

Theorem 3.3. *${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ and ${}_2\mathfrak{S}^{\mathfrak{I}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ both are linear spaces.*

PROOF. Let's demonstrate to obtain space ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$. We can prove the other space in a similar manner. Letting $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{ij}}), \hat{\mathfrak{y}} = (\hat{\mathfrak{y}}_{\mathfrak{ij}}) \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ along with α, β be scalars. After that we get, for an assigned $\varepsilon > 0$,

$$\mathfrak{A}_1 = \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) \leqslant 1-\varepsilon \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) \geqslant \varepsilon \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) \geqslant \varepsilon \end{array} \right\} \in \mathfrak{I};$$

$$\mathfrak{A}_2 = \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) \leqslant 1-\varepsilon \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) \geqslant \varepsilon \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) \geqslant \varepsilon \end{array} \right\} \in \mathfrak{I}.$$

$$\mathfrak{A}_1^c = \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) > 1-\varepsilon \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) < \varepsilon \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) < \varepsilon \end{array} \right\} \in \mathfrak{F}(\mathfrak{I});$$

$$\mathfrak{A}_2^c = \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) > 1-\varepsilon \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) < \varepsilon \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) < \varepsilon \end{array} \right\} \in \mathfrak{F}(\mathfrak{I});$$

Establish that $\mathfrak{A}_3 = \mathfrak{A}_1 \cup \mathfrak{A}_2$ set, which means a way $\mathfrak{A}_3 \in \mathfrak{I}$. Therefore a non-empty set $\mathfrak{A}_3^c$ in $\mathfrak{F}(\mathfrak{I})$. For each $(\mathfrak{x}_{\mathfrak{ij}}), (\hat{\mathfrak{y}}_{\mathfrak{ij}}) \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$, we will demonstrate

$$\mathfrak{A}_3^c \subset \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) > 1-\varepsilon, \\ \ddot{\nu}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) < \varepsilon \text{ and} \\ \dddot{\tau}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) < \varepsilon \end{array} \right\}.$$

Let us take $(\mathfrak{m},\mathfrak{n}) \in \mathfrak{A}_3^c$. In that case

$$\dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) > 1-\varepsilon \quad \text{or}$$
$$\ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) < \varepsilon \quad \text{and}$$
$$\dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2|\alpha|}\right) < \varepsilon,$$

and

$$\begin{aligned} &\dot{\mu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{mn}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) > 1-\varepsilon \quad \text{or} \\ &\ddot{\nu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{mn}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) < \varepsilon \quad \text{and} \\ &\dddot{\tau}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{mn}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2|\beta|}\right) < \varepsilon. \end{aligned}$$

We have

$$\begin{aligned} &\dot{\mu}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{mn}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) \\ &\geqslant \dot{\mu}\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \alpha\mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2}\right) * \dot{\mu}\left(\beta\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \beta\mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2}\right) \\ &= \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2}|\alpha|\right) * \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2}|\beta|\right) \\ &> (1-\varepsilon) * (1-\varepsilon) = 1-\varepsilon, \\ &\ddot{\nu}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{mn}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) \\ &\leqslant \ddot{\nu}\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \alpha\mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2}\right) \Delta\, \ddot{\nu}\left(\beta\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \beta\mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2}\right) \\ &= \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2}|\alpha|\right) \Delta\, \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2}|\beta|\right) \\ &< \varepsilon\, \Delta\, \varepsilon = \varepsilon \, \text{and} \\ &\dddot{\tau}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{mn}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) \\ &\leqslant \dddot{\tau}\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \alpha\mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2}\right) \circledast \dddot{\tau}\left(\beta\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \beta\mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2}\right) \\ &= \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_1, \breve{p}; \frac{\mathfrak{t}}{2}|\alpha|\right) \circledast \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}_2, \breve{p}; \frac{\mathfrak{t}}{2}|\beta|\right) \\ &< \varepsilon \circledast \varepsilon = \varepsilon. \end{aligned}$$

This implies that

$$\mathfrak{A}_3^c \subset \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) > 1-\varepsilon, \\ \ddot{\nu}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) < \varepsilon \, \text{and} \\ \dddot{\tau}\left(\left(\alpha\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) + \beta\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathfrak{ij}})\right) - (\alpha\mathfrak{L}_1 + \beta\mathfrak{L}_2), \breve{p}; \mathfrak{t}\right) < \varepsilon \end{array} \right\}.$$

Hence, the space ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ which is linear. □

Theorem 3.4. *Every open ball* ${}_2\mathfrak{B}_{\mathfrak{r}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$ *is an open set in* ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$.

Proof. Consider the open ball ${}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$, which has a centre at $\mathfrak{x}$ with a radius of $\breve{r}$ in relate to $\mathfrak{t}$. It becomes

$${}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{ \hat{\mathfrak{y}} = (\hat{\mathfrak{y}}_{\mathrm{ij}}) \in {}_2\ell_\infty : \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \mathfrak{B}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}\right) > 1-\breve{r} \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}\right) < \breve{r} \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}\right) < \breve{r} \end{array} \right\} \in \mathfrak{I} \right\}.$$

Consider $\hat{\mathfrak{y}} \in {}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$, after that

$$\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}) > 1-\breve{r}, \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}) < \breve{r}$$

and

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}) < \breve{r}.$$

As a result $\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}) > 1-\breve{r}$, there exists $\mathfrak{t}_0 \in (0,\mathfrak{t})$ which means

$$\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}_0) > 1-\breve{r}, \quad \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}_0) < \breve{r}$$

and

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{y}_{\mathrm{ij}}),\breve{p};\mathfrak{t}_0) < \breve{r}.$$

Putting $\breve{r}_0 = \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\mathfrak{t}_0)$, we have $\breve{r}_0 > 1-\breve{r}$, that $\mathfrak{s} \in (0,1)$ exist which yields $\breve{r}_0 > 1-\mathfrak{s} > 1-\breve{r}$. We possess $\breve{r}_0 > 1-\mathfrak{s}$, to obtain $\breve{r}_1,\breve{r}_2,\breve{r}_3 \in (0,1)$ which yields

$$\breve{r}_0 * \breve{r}_1 > 1-\mathfrak{s}, (1-\breve{r}_0) \; \Delta \; (1-\breve{r}_2) \leqslant \mathfrak{s}$$

and

$$(1-\breve{r}_0) \circledast (1-\breve{r}_3) \leqslant \mathfrak{s}.$$

Adding $\breve{r}_4 = \max\{\breve{r}_1,\breve{r}_2,\breve{r}_3\}$. Let the ball ${}_2\mathfrak{B}_{\hat{\mathfrak{y}}}(1-\breve{r}_3,\mathfrak{t}-\mathfrak{t}_0)(\tilde{\mathfrak{V}})$, let us demonstrate for this

$${}_2\mathfrak{B}_{\hat{\mathfrak{y}}}(1-\breve{r}_4,\mathfrak{t}-\mathfrak{t}_0)(\tilde{\mathfrak{V}}) \supset {}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}).$$

Let $\mathfrak{w} = (\mathfrak{w}_{\mathrm{ij}}) \in {}_2\mathfrak{B}_{\hat{\mathfrak{y}}}(1-\breve{r}_4,\mathfrak{t}-\mathfrak{t}_0)(\tilde{\mathfrak{V}})$, after that

$$\begin{aligned} &\dot{\mu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}-\mathfrak{t}_0\right) > \breve{r}_4, \\ &\ddot{\nu}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}-\mathfrak{t}_0\right) < 1-\breve{r}_4 \quad \text{and} \\ &\dddot{\tau}\left(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}-\mathfrak{t}_0\right) < 1-\breve{r}_4. \end{aligned}$$

Therefore

$$\begin{aligned}\dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}\right) &\geqslant \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}_0) * \dot{\mu}(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}-\mathfrak{t}_0)\\ &\geqslant (\breve{r}_0 * \breve{r}_4) \geqslant (\breve{r}_0 * \breve{r}_1) \geqslant (1-\mathfrak{s}) \geqslant (1-\breve{r}),\\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}) &\leqslant \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}_0)\ \Delta\ \ddot{\nu}(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}-\mathfrak{t}_0)\\ &\leqslant (1-\breve{r}_0)\ \Delta\ (1-\breve{r}_4) \leqslant (1-\breve{r}_0)\ \Delta\ (1-\breve{r}_2) \leqslant \mathfrak{s} \leqslant \breve{r} \text{ and}\\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}) &\leqslant \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}}),\breve{p};\mathfrak{t}_0) \circledast \dddot{\tau}(\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}_{\mathrm{ij}})-\tilde{\mathfrak{V}}(\mathfrak{w}_{\mathrm{ij}}),\breve{p};\mathfrak{t}-\mathfrak{t}_0)\\ &\leqslant (1-\breve{r}_0) \circledast (1-\breve{r}_4) \leqslant (1-\breve{r}_0) \circledast (1-\breve{r}_3) \leqslant \mathfrak{s} \leqslant \breve{r}.\end{aligned}$$

Consequently, $\mathfrak{w} = (\mathfrak{w}_{\mathrm{ij}}) \in {}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$, thereby hence the result

$${}_2\mathfrak{B}_{\hat{\mathfrak{y}}}(1-\breve{r}_4,\mathfrak{t}-\mathfrak{t}_0)(\tilde{\mathfrak{V}}) \subset {}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}).$$

□

Remark 3.5. ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ *is an N-2-NS. Describe*

$${}_2\mathfrak{T}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}}) = \Big\{\mathfrak{A} \subset {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}}) : \textit{there are } \mathfrak{t} > 0 \textit{ along with } \breve{r} \in (0,1) \textit{ which means } {}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \subset \mathfrak{A} \textit{ exists for each } \mathfrak{x} \in \mathfrak{A}\Big\}.$$

Then, ${}_2\mathfrak{T}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ *is a topology on* ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$.

Theorem 3.6. ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ *and* ${}_2\mathfrak{S}^{\mathfrak{I}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ *Hausdorff spaces.*

Proof. Let we demonstrate that outcome for ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$. For ${}_2\mathfrak{S}^{\mathfrak{I}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$, its proof proceeds in a similar manner. Consider $\mathfrak{x},\hat{\mathfrak{y}} \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ where we have $\mathfrak{x} \neq \hat{\mathfrak{y}}$. After that

$$0 < \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}),\breve{p};\mathfrak{t}) < 1, 0 < \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}),\breve{p};\mathfrak{t}) < 1$$

and

$$0 < \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}),\breve{p};\mathfrak{t}) < 1.$$

Using $\breve{r}_1 = \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}),\breve{p};\mathfrak{t}), \breve{r}_2 = \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}),\breve{p};\mathfrak{t}), \breve{r}_3 = \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\hat{\mathfrak{y}}),\breve{p};\mathfrak{t})$ and $\breve{r} = \max\{\breve{r}_1, 1-\breve{r}_2, 1-\breve{r}_3\}$. There exists $\breve{r}_4, \breve{r}_5$ and $\breve{r}_6$ for each $\breve{r}_0 \in (\breve{r},1)$ which corresponds to $\breve{r}_4 * \breve{r}_4 \geqslant \breve{r}_0$, $(1-\breve{r}_5)\ \Delta\ (1-\breve{r}_5) \leqslant (1-\breve{r}_0)$ and $(1-\breve{r}_6) \circledast (1-\breve{r}_6) \leqslant (1-\breve{r}_0)$. Adding $\breve{r}_7 = \max\{\breve{r}_4,\breve{r}_5,\breve{r}_6\}$ and for the open balls ${}_2\mathfrak{B}_{\mathfrak{x}}\left(1-\breve{r}_7,\frac{\mathfrak{t}}{2}\right)$ as well as ${}_2\mathfrak{B}_{\hat{\mathfrak{y}}}\left(1-\breve{r}_7,\frac{\mathfrak{t}}{2}\right)$. Then it is evident that ${}_2\mathfrak{B}^c_{\mathfrak{x}}\left(1-\breve{r}_7,\frac{\mathfrak{t}}{2}\right) \cap {}_2\mathfrak{B}^c_{\hat{\mathfrak{y}}}\left(1-\breve{r}_7,\frac{\mathfrak{t}}{2}\right) = \varnothing$.

For if there exists $\mathfrak{w} \in {}_2\mathfrak{B}^c_{\mathfrak{x}}\left(1-\breve{r}_7, \frac{\mathfrak{t}}{2}\right) \cap {}_2\mathfrak{B}^c_{\mathfrak{y}}\left(1-\breve{r}_7, \frac{\mathfrak{t}}{2}\right)$, then

$$\begin{aligned}\breve{r}_1 = \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\mathfrak{t}) &\geqslant \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(z),\breve{p};\frac{\mathfrak{t}}{2}\right) * \dot{\mu}\left(\tilde{\mathfrak{V}}(z)-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\frac{\mathfrak{t}}{2}\right)\\ &\geqslant \breve{r}_7 * \breve{r}_7 \geqslant \breve{r}_4 * \breve{r}_4 \geqslant \breve{r}_0 > \breve{r}_1,\\ \breve{r}_2 = \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\mathfrak{t}) &\leqslant \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\frac{\mathfrak{t}}{2}\right) \Delta\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(z)-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\frac{\mathfrak{t}}{2}\right)\\ &\leqslant (1-\breve{r}_7)\ \Delta\ (1-\breve{r}_7) \leqslant (1-\breve{r}_5)\ \Delta\ (1-\breve{r}_5) \leqslant (1-\breve{r}_0) < \breve{r}_2\end{aligned}$$

and

$$\begin{aligned}\breve{r}_3 = \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\mathfrak{t}) &\leqslant \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x})-\tilde{\mathfrak{V}}(\breve{p}),\breve{p};\frac{\mathfrak{t}}{2}\right) \circledast \dddot{\tau}\left(\tilde{\mathfrak{V}}(z)-\tilde{\mathfrak{V}}(\mathfrak{y}),\breve{p};\frac{\mathfrak{t}}{2}\right)\\ &\leqslant (1-\breve{r}_7) \circledast (1-\breve{r}_7) \leqslant (1-\breve{r}_6) \circledast (1-\breve{r}_6) \leqslant (1-\breve{r}_0) < \breve{r}_3.\end{aligned}$$

It contradicts this way. Hence ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ which is Hausdorff. □

Theorem 3.7. *${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ is a NNS and a topology ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ is on ${}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$. And then a sequence $(\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$, $\mathfrak{x}_{\mathfrak{ij}} \to \mathfrak{x}$ if and only if $\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 1$, $\ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 0$ and $\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 0$ as $\mathfrak{i},\mathfrak{j} \to \infty$.*

Proof. Fix that $\mathfrak{t}_0 > 0$. Assume that $\mathfrak{x}_{\mathfrak{ij}} \to \mathfrak{x}$. There are $\mathfrak{n}_0 \in \mathbb{N}$ exists in such a way that $(\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}})$ for every $\mathfrak{i},\mathfrak{j} \geqslant \mathfrak{n}_0$, for $\breve{r} \in (0,1)$,

$${}_2\mathfrak{B}_{\mathfrak{x}}(\breve{r},\mathfrak{t})(\tilde{\mathfrak{V}}) = \left\{\begin{array}{c}(\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \leqslant 1-\breve{r} \text{ or}\\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \geqslant \breve{r} \text{ and}\\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \geqslant \breve{r}\end{array}\right\} \in \mathfrak{I},$$

which means ${}_2\mathfrak{B}^c_{\mathfrak{x}}(\mathfrak{r},\mathfrak{t})(\tilde{\mathfrak{V}}) \in \mathfrak{F}(\mathfrak{I})$. After that $1-\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) < \breve{r}$, $\ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) < \breve{r}$, and $\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) < \breve{r}$. As a result,

$$\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 1, \quad \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 0$$

and

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 0 \quad \text{being } \mathfrak{i},\mathfrak{j} \to \infty.$$

In contrast, when according to each $\mathfrak{t} > 0$,

$$\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 1, \quad \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 0$$

and

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) \to 0 \text{ as } \mathfrak{i},\mathfrak{j} \to \infty,$$

after that for $\breve{r} \in (0,1)$, there are $\mathfrak{n}_0 \in \mathbb{N}$ exists that means the fact

$$1-\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) < \breve{r}, \quad \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) < \breve{r}$$

and

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}})-\tilde{\mathfrak{V}}(\mathfrak{x}),\breve{p};\mathfrak{t}) < \breve{r}$$

for all $\mathfrak{i}, \mathfrak{j} \geqslant \mathfrak{n}_0$. Thus, it implies

$$\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}), \breve{p}; \mathfrak{t}) > 1 - \breve{r} \quad \text{or} \quad \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}), \breve{p}; \mathfrak{t}) < \breve{r}$$

and

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}), \breve{p}; \mathfrak{t}) < \breve{r}$$

for all $\mathfrak{i}, \mathfrak{j} \geqslant \mathfrak{n}_0$. Thus $(\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\mathfrak{B}^c_{\mathfrak{x}}(\breve{r}, \mathfrak{t})(\tilde{\mathfrak{V}})$, for all $\mathfrak{i}, \mathfrak{j} \geqslant \mathfrak{n}$ and as a result $\mathfrak{x}_{\mathfrak{ij}} \to \mathfrak{x}$. $\square$

Theorem 3.8. *A $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ sequence is $\mathfrak{I}$-convergent if and only if it has an integer $\mathfrak{M} = \mathfrak{M}(\mathfrak{x}, \varepsilon, \mathfrak{t}), \mathfrak{N} = \mathfrak{N}(\mathfrak{x}, \varepsilon, \mathfrak{t})$ which means for all $\varepsilon > 0$ and $\mathfrak{t} > 0$*

$$\left\{ \begin{array}{c} (\mathfrak{M}, \mathfrak{N}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) > 1 - \varepsilon \quad or \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon \quad and \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}) < \varepsilon \end{array} \right\} \in \mathfrak{F}(\mathfrak{I}).$$

Proof. Assume that $\mathfrak{I}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2} - \lim \mathfrak{x} = \mathfrak{L}$ along with consider $\varepsilon > 0$ as well as $\mathfrak{t} > 0$. Select $\mathfrak{s} > 0$ for an assigned $\varepsilon > 0$, which means $(1-\varepsilon) * (1-\varepsilon) > 1 - \mathfrak{s}, \varepsilon \,\Delta\, \varepsilon < \mathfrak{s}$ and $\varepsilon \circledast \varepsilon < \mathfrak{s}$. After that, for each $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{ij}}) \in {}_2\mathfrak{S}^{\mathfrak{I}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$

$$\mathfrak{P} = \left\{ \begin{array}{c} (\mathfrak{i}, \mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) \leqslant 1 - \varepsilon \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \varepsilon \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) \geqslant \varepsilon \end{array} \right\} \in \mathfrak{I}.$$

It suggests that

$$\mathfrak{P}^c = \left\{ \begin{array}{c} (\mathfrak{i}, \mathfrak{j}) : \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) > 1 - \varepsilon \text{ or} \\ \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon \text{ and} \\ \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon \end{array} \right\} \in \mathfrak{F}(\mathfrak{I}).$$

Let us select $(\mathfrak{M}, \mathfrak{N}) \in \mathfrak{P}$ on the contrary. Afterwards

$$\dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) > 1 - \varepsilon \quad \text{or} \quad \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon$$

and

$$\dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}) - \mathfrak{L}, \breve{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon.$$

We want now to demonstrate that an integer $\mathfrak{M} = \mathfrak{M}(\mathfrak{x}, \varepsilon, \mathfrak{t}), \mathfrak{N} = \mathfrak{N}(\mathfrak{x}, \varepsilon, \mathfrak{t})$ exist in such a way that

$$\left\{ \begin{array}{c} (\mathfrak{i}, \mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}), \breve{p}; \mathfrak{t}) \leqslant 1 - \mathfrak{s} \quad \text{or} \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}), \breve{p}; \mathfrak{t}) \geqslant \mathfrak{s} \quad \text{and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{M}\mathfrak{N}}), \breve{p}; \mathfrak{t}) \geqslant \mathfrak{s} \end{array} \right\} \in \mathfrak{I}.$$

In order to do this, declare according to each $\mathfrak{x} \in {}_2\mathfrak{S}^{\mathfrak{J}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$

$$\mathfrak{Q} = \left\{ \begin{array}{c} (\mathfrak{i},\mathfrak{j}) : \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) \leqslant 1 - \mathfrak{s} \text{ or} \\ \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) \geqslant \mathfrak{s} \text{ and} \\ \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{ij}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) \geqslant \mathfrak{s} \end{array} \right\} \in \mathfrak{J}.$$

We must now demonstrate that $\mathfrak{Q} \subset \mathfrak{P}$. Assume that a subset belonging to $\mathfrak{P}$ is not $\mathfrak{Q}$. After that there are $(\mathfrak{m},\mathfrak{n}) \in \mathfrak{Q}$ and $(\mathfrak{m},\mathfrak{n}) \notin \mathfrak{P}$ exists. Thus, we have $\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) \leqslant 1 - \mathfrak{s}$ or $\dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}) > 1 - \varepsilon$. In particular $\dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) > 1 - \varepsilon$. Therefore we get

$$\begin{aligned} 1 - \mathfrak{s} \geqslant \dot{\mu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) &\geqslant \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) * \dot{\mu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) \\ &\geqslant (1-\varepsilon) * (1-\varepsilon) > 1 - \mathfrak{s}, \end{aligned}$$

this cannot be possible. But on another hand

$$\ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) \geqslant s \quad \text{or} \quad \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}) < \varepsilon.$$

In particular $\ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon$. Therefore we have

$$\begin{aligned} \mathfrak{s} \leqslant \ddot{\nu}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) &\leqslant \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) \Delta\, \ddot{\nu}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) \\ &\leqslant \varepsilon \,\Delta\, \varepsilon < \mathfrak{s} \quad \text{and} \end{aligned}$$

$$\dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) \geqslant \mathfrak{s} \text{ or } \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}) < \epsilon.$$

In particular $\dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) < \varepsilon$. Therefore we have

$$\begin{aligned} \mathfrak{s} \leqslant \dddot{\tau}(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}), \check{p}; \mathfrak{t}) &\leqslant \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{mn}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) \circledast \dddot{\tau}\left(\tilde{\mathfrak{V}}(\mathfrak{x}_{\mathfrak{MN}}) - \mathfrak{L}, \check{p}; \frac{\mathfrak{t}}{2}\right) \\ &\leqslant \varepsilon \circledast \varepsilon < \mathfrak{s} \end{aligned}$$

it is impossible. Hence a result $\mathfrak{Q} \subset \mathfrak{P}.\mathfrak{P} \in \mathfrak{J}$ it suggest that $\mathfrak{Q} \in \mathfrak{J}$. □

4. Conclusions

In the present article, we propose and investigate a few fresh double sequence spaces derived from bounded linear operators concerning *N2-NS* through ideal convergence, namely ${}_2\mathfrak{S}^{\mathfrak{J}}_{(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$ and ${}_2\mathfrak{S}^{\mathfrak{J}}_{0(\dot{\mu},\ddot{\nu},\dddot{\tau})_2}(\tilde{\mathfrak{V}})$, for the purpose of demonstrating that a bounded linear operator in relation to *N2-NS* upholds certain of these spaces fundamental topological and algebraic characteristics The above concepts and outcomes that we emphasise in that work present a more general structure for dealing with the uncertainty, ambiguity, and convergence of double-sequence problems that arise within numerous scientific fields, including technology as well as research.

References

[1] S. Broumi, S. Mohanaselvi, T. Witczak, M. Talea, A. Bakali, and F. Smarandache, *Complex fermatean neutrosophic graph and application to decision making*, Decision Mak.: Appl. Manag. Engin., **6**(1) (2023), 474-501.

[2] S. Broumi, R. Sundareswaran, M. Shanmugapriya, A. Bakali, and M. Talea, *Theory and applications of Fermatean neutrosophic graphs*, Neutrosophic Sets Syst., **50** (2022), 248-286.

[3] P. Das, P. Kostyrko, W. Wilczynski, and P. Malik, *$\mathfrak{I}$ and $\mathfrak{I}^*$-convergence of double sequences*, Math. Slovaca, **58** (2008), 605-620.

[4] M. Jeyaraman, *Generalized Hyers-Ulam-Rassias stability in neutrosophic normed spaces*, Octogon Math. Mag., **30**(2) (2022), 773-792.

[5] M. Jeyaraman and P. Jenifer, *Statistical Δ^m-convergence in neutrosophic normed spacs*, J. Comput. Math., **7**(1) (2023), 46-60.

[6] M. Jeyaraman, A. Ramachandran, and V.B. Shakila, *Approximate fixed point theorems for weak contractions on neutrosophic normed spaces*, J. Comput. Math., **6**(1) (2022), 134-158.

[7] V.A. Khan, H.F. Ahmad, and M.F. Khan, *On some results in intuitionistic fuzzy ideal convergence double sequence spaces*, Adv. Differ. Equ., **2019**(1) (2019), 1-10.

[8] V.A. Khan, M. Ahmad, S.N. Hasan, and R. Ahmad, *$\mathfrak{I}$-convergent difference sequence spaces*, J. Math. Anal., **10**(2) (2019), 58-68.

[9] V.A. Khan, H.F. Yasmeen, H. Altaf, and Q.M. Danish Lohani, *Intuitionistic fuzzy $\mathfrak{I}$-convergent sequence spaces defined by compact operator*, Cogent Math., **3** (2016), 1267904.

[10] A.N. Mangayarkkarasi, M. Jeyaraman, and V. Jeyanthi, *On stability of a cubic functional equation in neutrosophic normed spaces*, Adv. Appl. Math. Sci., **21**(4) (2022), 1975-1988.

[11] M. Mursaleen, S.A. Mohiuddine, H. Osama, and H. Edely, *On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces*, Comput. Math. Appl. **59** (2010), 603-611.

[12] R. Saadati and J.H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos Solitons Fractals, **27** (2006), 331–344.

[13] T. Salat, B.C. Tripathy and M. Ziman, *On $\mathfrak{I}$-convergence field*, Ital. J. Pure Appl. Math., **17** (2005), 45-54.

[14] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 1998.

[15] F. Smarandache, *Neutrosophic set a generalization of the intuitionistic fuzzy sets*, J. Defense Resources Manag., **1**(1) (2010), 107-116.

[16] B.C.I. Tripathy and B. Hazarika, *Some $\mathfrak{I}$-Convergent sequence spaces defined by Orliczfunction*, Acta Math. Appl. Sinica, **27**(1) (2011), 149-154.

[17] L.A. Zadeh, *Fuzzy sets*, Inf. Control, **8** (1965), 338-353.

P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India

Email address: jeya.math@gmail.com

Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India

Email address: iiswariya1234@gmail.com,

Received : *September* 2023
Accepted : *October* 2023