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On some classes of invariant submanifolds of Kenmotsu manifolds

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Abstract. This paper focuses on investigating invariant submanifolds within Kenmotsu manifolds. Specifically, it explores cases where these submanifolds meet the Tachibana conditions related to the parallel second fundamental form, products involving the Riemannian and conformal curvature tensors, and the Ricci curvature tensor along with Riemannian metrics. Under specific conditions, it has been demonstrated that these invariant submanifolds will exhibit the characteristic of being totally geodesic.

1. Introduction

In 1972, Kenmotsu [11] introduced a distinctive category of almost contact metric manifolds characterized by negative curvature. These structures, termed Kenmotsu manifolds, have captured the attention of various researchers. Notable studies on these Kenmotsu manifolds have been conducted by authors including Bagewadi and Venkatesha [1, 2], De and Pathak [6], Ozgur [16]-[17] and many others.

In 1981, Bejancu and Papaghuic introduced the concept of invariant submanifolds, which inherit nearly all the characteristics of ambient manifold. Since then, numerous mathematicians have explored invariant submanifolds in various ambient manifold contexts. Noteworthy studies have been conducted on different ambient manifolds, as evidenced by works such as ([7], [14], [18], [21]-[26], [29]). In contemporary times, this theory has gained substantial importance across diverse fields including image processing, computer design, economic modeling, theoretical physics, and applied mathematics. Furthermore, this theory has found application in the

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field of mechanical engineering. Over the last three decades, American and Chinese researchers [9, 19, 20] have explored this theory to advance the understanding of nonlinear normal modes in vibrating mechanical systems.

The paper is organized as follows: In Section 2, we revisit the concept of Kenmotsu manifolds and review fundamental outcomes related to invariant submanifolds. These foundational understandings serve as a basis for our subsequent investigations. In sections 3, 4, 5, and 6 we have established that invariant submaifolds of Kenmotsu manifolds satisfying $Q(S, \tilde{\nabla}_X \sigma) = 0, Q(S, \tilde{R}\cdot\sigma) = 0, Q(g, \tilde{C}\cdot\sigma) = 0$ and $Q(S, \tilde{C}\cdot\sigma) = 0$ are totally geodesic.

2. Kenmotsu manifolds and their submanifolds

A $(2n+1)$-dimensional differentiable manifold $\tilde{M}$ is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and 1-form η on $\tilde{M}$ such that

$$\phi^2 = -X + \eta(X)\xi, \eta(\xi) = 1 \; and \; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

for any vector fields X, Y on $\tilde{M}$. In specific in an almost contact metric manifold also we have

$$\phi\xi = 0, \eta\cdot\phi = 0. \tag{2}$$

An almost contact metric manifold is said to be an almost Kenmotsu manifold [11] if the following relations hold:

$$(\tilde{\nabla}_X\phi)Y = -g(X, \phi Y) - \eta(Y)\phi X, \tag{3}$$

$$\tilde{\nabla}_X\xi = X - \eta(X)\xi. \tag{4}$$

For a $(2n+1)$-dimensional Riemannian manifold, the concircular curvature tensor $\tilde{C}$ is defined by

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{5}$$

for any vector fields $X, Y, Z \in \chi(\tilde{M})$.

On a Riemannian manifold $\tilde{M}^*$, for a $(0,k)$-type tensor field $T(k \geq 1)$ and a $(0,2)$-type tensor field B, we denote by $Q(B,T)$ a $(0,k+2)$-type tensor field [28] defined as follows

$$\begin{aligned} Q(B,T)(X_1, X_2, \cdots, X_k; X, Y) = -T((X \wedge_B Y)X_1, X_2, \cdots, X_k) \\ -T(X_1, (X \wedge_B Y)X_2, \cdots, X_k) - T(X_1, X_2, \cdots, (X \wedge_B Y)X_k), \end{aligned} \tag{6}$$

where $(X \wedge_B Y)$ is defined by

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y. \tag{7}$$

If $Q(B,T)(X_1, X_2, \cdots, X_k; X, Y) = 0$, then the manifold is said to satisfy Tachibana condition. Let M be a $(2m+1)$-dimensional $(m < n)$ immersed submanifold of a

Kenmotsu manifold $\tilde{M}$. Let $\chi(M)$ be the Lie algebra of vector fields on M and $\chi^{\perp}(M)$ the set of all vector fields perpendicular to M. Let ∇ denotes the covariant differentiation in M determined by the induced metric g. Let σ be the second fundamental form of M. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \ \ X,Y \in \chi(M), \tag{8}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \ \ N \in \chi^{\perp}(M), \tag{9}$$

where $g(A_N X, Y) = g(\sigma(X,Y), N)$ and $\nabla^{\perp}_X N$ denotes the covariant derivative of a cross section N of the normal bundle $T^{\perp}M$ in the direction of X with respect to the connection in $T^{\perp}M$. A submanifold M of a Kenmotsu manifold $\tilde{M}$ is totally geodesic if $\sigma(X,Y) = 0$, for $X, Y \in TM$. The covariant derivative of σ is given by

$$(\tilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{10}$$

for any $X, Y, Z \in TM$. In 1985, Deprez [8] defined the immersion

$$(\tilde{R}(X,Y)\cdot\sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V), \tag{11}$$

for all $X, Y, U, V \in TM$. If $\tilde{R}\cdot\sigma = 0$, then M is said to be semi-parallel. A submanifold M of a Kenmotsu manifold $\tilde{M}$ is said to be invariant if $\phi(TM) \subset TM$. On an invariant submanifold M, it follows that $\xi \in \chi(M)$. In an invariant submanifold of a Kenmotsu manifold $\tilde{M}$, the following relations hold [7]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{12}$$

$$\sigma(X,\xi) = 0, \tag{13}$$

$$(\nabla_X \phi) = -g(X, \phi Y) - \eta(Y)\phi X, \tag{14}$$

$$\sigma(X, \phi Y) = \phi\sigma(X,Y), \tag{15}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{16}$$

$$S(X,\xi) = -2m\eta(X), \tag{17}$$

for any vector field $X, Y \in TM$. So we can state the following:

Theorem 2.1. [13] *An invariant submanifold M of a Kenmotsu manifold $\tilde{M}$ is a Kenmotsu manifold.*

3. Invariant submanifolds of Kenmotsu manifolds satisfying $Q(S, \tilde{\nabla}\sigma) = 0$

In this section we study invariant submanifolds of Kenmotsu manifolds satisfying $Q(S, \tilde{\nabla}\sigma) = 0$. Consider,

$$Q(S, \tilde{\nabla}_X \sigma)(Y, Z; U, V) = 0.$$

By using (6) in the above equation, we have

$$-(\tilde{\nabla}_X \sigma)((U \wedge_S V)Y, Z) - (\tilde{\nabla}_X \sigma)(Y, (U \wedge_S V)Z) = 0.$$

Applying (7) in the above equation, we get

$$\begin{aligned} 0 &= -(\tilde{\nabla}_X\sigma)(S(V,Y)U,Z) + (\tilde{\nabla}_X\sigma)(S(U,Y)V,Z) \\ &\quad -(\tilde{\nabla}_X\sigma)(Y,S(V,Z)U) + (\tilde{\nabla}_X\sigma)(Y,S(U,Z)V). \end{aligned} \tag{18}$$

By using equation (3) in (18), we have

$$\begin{aligned} 0 &= -\nabla^{\perp}_X\sigma(S(V,Y)U,Z) + \sigma(\nabla_X S(V,Y)U,Z) + \sigma(S(V,Y)U,\nabla_X Z) \\ &\quad +\nabla^{\perp}_X\sigma(S(U,Y)V,Z) - \sigma(\nabla_X S(U,Y)V,Z) - \sigma(S(U,Y)V,\nabla_X Z) \\ &\quad -\nabla^{\perp}_X\sigma(U,S(V,Z)U) + \sigma(\nabla_X Y,S(V,Z)U) + \sigma(Y,\nabla_X S(V,Z)U) \\ &\quad +\nabla^{\perp}_X\sigma(Y,S(U,Z)V) - \sigma(\nabla_X Y,S(U,Z)V) - \sigma(Y,\nabla_X S(U,Z)V). \end{aligned}$$

By virtue of (13) and putting $Y = Z = V = \xi$ in the above equation we have

$$0 = 2S(\xi,\xi)\sigma(U,\nabla_X\xi). \tag{19}$$

Using the equation (12) and (17) in (19) we have

$$0 = m\sigma(U,X).$$

Hence the submanifold is totally geodesic. The converse is trivial. Thus, we can state the following:

Theorem 3.1. *An invariant submanifold of a Kenmotsu manifold satisfies $Q(S,\tilde{\nabla}\sigma) = 0$ if and only if it is totally geodesic.*

4. Invariant submanifolds of Kenmotsu manifold satisfying $Q(S,\tilde{R}\cdot\sigma) = 0$

This section deals with the study of invariant submanifolds of Kenmotsu manifolds satisfying $Q(S,\tilde{R}\cdot\sigma) = 0$. Consider

$$Q(S,\tilde{R}(X,Y)\cdot\sigma)(Z,W;U,V) = 0.$$

By using (6) in the above equation, we have

$$-(\tilde{R}(X,Y)\cdot\sigma)((U\wedge_S V)Z,W) - (\tilde{R}(X,Y)\cdot\sigma)(Z,(U\wedge_S V)W) = 0. \tag{20}$$

Applying (7) in equation (20), one can get

$$\begin{aligned} 0 &= -S(V,Z)(\tilde{R}(X,Y)\cdot\sigma)(U,W) + S(U,Z)(\tilde{R}(X,Y)\cdot\sigma)(V,W) \\ &\quad -S(V,W)(\tilde{R}(X,Y)\cdot\sigma)(Z,U) + S(U,W)(\tilde{R}(X,Y)\cdot\sigma)(Z,V). \end{aligned} \tag{21}$$

Plugging equation (11) in (21), we have

$$\begin{aligned} 0 &= -S(V,Z)[R^{\perp}(X,Y)\sigma(U,W) + \sigma(R(X,Y)U,W) + \sigma(U,R(X,Y)W) \\ &\quad +S(U,Z)[R^{\perp}(X,Y)\sigma(V,W) - \sigma(R(X,Y)V,W) - \sigma(V,R(X,Y)W) \\ &\quad -S(V,W)[R^{\perp}(X,Y)\sigma(Z,U) + \sigma(R(X,Y)Z,U) + \sigma(Z,R(X,Y)U) \\ &\quad +S(U,W)[R^{\perp}(X,Y)\sigma(Z,V) - \sigma(R(X,Y)Z,V) - \sigma(Z,R(X,Y)V). \end{aligned}$$

By virtue of (13) and putting $Y = Z = W = V = \xi$, in the above equation we have

$$0 = S(\xi,\xi)\sigma(U, R(X,\xi)\xi). \tag{22}$$

By using equation (16) and (17) in (22), we have

$$0 = m\sigma(U, X).$$

Hence, the submanifold is totally geodesic. Conversely, let M be totally geodesic invariant submanifold, then from (21) we get $Q(S, \tilde{R}\cdot\sigma) = 0$. Hence we can state the following:

Theorem 4.1. *An invariant submanifold of a Kenmotsu manifold satisfies* $Q(S, \tilde{R}\cdot\sigma) = 0$ *if and only if it is totally geodesic.*

5. Invariant submanifolds of Kenmotsu manifold satisfying $Q(g, \tilde{C}\cdot\sigma) = 0$

In this section, we study of invariant submanifolds of Kenmotsu manifolds satisfying $Q(g, \tilde{C}\cdot\sigma) = 0$. Consider,

$$Q(g, \tilde{C}(X,Y)\cdot\sigma)(Z,W;U,V) = 0.$$

By using (6) in the above equation, we have

$$-(\tilde{C}(X,Y)\cdot\sigma)((U\wedge_g V)Z, W) - (\tilde{C}(X,Y)\cdot\sigma)(Z, (U\wedge_g V)W) = 0. \tag{23}$$

Applying (7) in equation (23), one can get

$$\begin{aligned} 0 \;=\; & -g(V,Z)(\tilde{C}(X,Y)\cdot\sigma)(U,W) + g(U,Z)(\tilde{C}(X,Y)\cdot\sigma)(V,W) \\ & -g(V,W)(\tilde{C}(X,Y)\cdot\sigma)(Z,U) + g(U,W)(\tilde{C}(X,Y)\cdot\sigma)(Z,V). \end{aligned} \tag{24}$$

Plugging equation (11) in (24), we have

$$\begin{aligned} 0 \;=\; & -g(V,Z)[C^{\perp}(X,Y)\sigma(U,W) + \sigma(C(X,Y)U,W) + \sigma(U,C(X,Y)W) \\ & +g(U,Z)[C^{\perp}(X,Y)\sigma(V,W) - \sigma(C(X,Y)V,W) - \sigma(V,C(X,Y)W) \\ & -g(V,W)[C^{\perp}(X,Y)\sigma(Z,U) + \sigma(C(X,Y)Z,U) + \sigma(Z,C(X,Y)U) \\ & +g(U,W)[C^{\perp}(X,Y)\sigma(Z,V) - \sigma(C(X,Y)Z,V) - \sigma(Z,C(X,Y)V). \end{aligned}$$

By virtue of (13) and putting $Y = Z = W = V = \xi$, in the above equation we have

$$0 = g(\xi,\xi)\sigma(U, C(X,\xi)\xi). \tag{25}$$

By using equation (5) and (16) in (25), we have

$$0 = \left[\frac{\tilde{r}}{2m(2m+1)}\right]\sigma(U,X).$$

This implies $\sigma(U,X)=0$ provided $\tilde{r}\neq 0$. Hence M is totally geodesic provided $\tilde{r}\neq 0$. Conversely, let M be totally geodesic with $\tilde{r}\neq 0$, then from (24) we get $Q(g,\tilde{C}\cdot\sigma)=0$. Hence, we can state the following:

Theorem 5.1. *An invariant submanifold of a Kenmotsu manifold with $\tilde{r}\neq 0$ satisfies $Q(g,\tilde{C}\cdot\sigma)=0$ if and only if it is totally geodesic.*

6. Invariant submanifolds of Kenmotsu manifold satisfying $Q(S,\tilde{C}\cdot\sigma)=0$

This section deals with the study of invariant submanifolds of Kenmotsu manifolds satisfying $Q(S,\tilde{C}\cdot\sigma)=0$. Consider,

$$Q(S,\tilde{C}(X,Y)\cdot\sigma)(Z,W;U,V)=0.$$

By using (6) in the above equation, we have

$$-(\tilde{C}(X,Y)\cdot\sigma)((U\wedge_S V)Z,W)-(\tilde{C}(X,Y)\cdot\sigma)(Z,(U\wedge_S V)W)=0. \tag{26}$$

Applying (7) in equation (26), one can get

$$\begin{aligned} 0 \;=\;& -S(V,Z)(\tilde{C}(X,Y)\cdot\sigma)(U,W)+S(U,Z)(\tilde{C}(X,Y)\cdot\sigma)(V,W)\\ & -S(V,W)(\tilde{C}(X,Y)\cdot\sigma)(Z,U)+S(U,W)(\tilde{C}(X,Y)\cdot\sigma)(Z,V). \end{aligned}\tag{27}$$

Plugging equation (11) in (27), we have

$$\begin{aligned} 0 \;=\;& -S(V,Z)[C^{\perp}(X,Y)\sigma(U,W)+\sigma(C(X,Y)U,W)+\sigma(U,C(X,Y)W)\\ & +S(U,Z)[C^{\perp}(X,Y)\sigma(V,W)-\sigma(C(X,Y)V,W)-\sigma(V,C(X,Y)W)\\ & -S(V,W)[C^{\perp}(X,Y)\sigma(Z,U)+\sigma(C(X,Y)Z,U)+\sigma(Z,C(X,Y)U)\\ & +S(U,W)[C^{\perp}(X,Y)\sigma(Z,V)-\sigma(C(X,Y)Z,V)-\sigma(Z,C(X,Y)V). \end{aligned}$$

By virtue of (13) and putting $Y=Z=W=V=\xi$, in the above equation we have

$$0=S(\xi,\xi)\sigma(U,C(X,\xi)\xi,U). \tag{28}$$

By using equation (5) and (17) in (28), we have

$$0=\left[\frac{\tilde{r}}{(2m+1)}\right]\sigma(U,X)=0.$$

This implies $\sigma(U,X)=0$ provided $\tilde{r}\neq 0$. Hence M is totally geodesic provided $\tilde{r}\neq 0$. Conversely, let M be totally geodesic with $\tilde{r}\neq 0$, then from (27) we get $Q(S,\tilde{C}\cdot\sigma)=0$. Hence we can state the following:

Theorem 6.1. *An invariant submanifold of a Kenmotsu manifold with $\tilde{r}\neq 0$ satisfies $Q(S,\tilde{C}\cdot\sigma)=0$ if and only if it is totally geodesic.*

7. Conclusion

The study leads to the conclusion that the Tachibana conditions, encompassing the parallel second fundamental form of the ambient manifold, along with products involving the Riemannian curvature tensor and the second fundamental form, as well as the conformal curvature tensor and the second fundamental form, in relation to the Ricci curvature tensor and Riemannian metric of invariant submanifolds within Kenmotsu manifolds, are interrelated and equivalent due to the property of total geodesicity.

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References

[1] C. S. Bagewadi and Venkatesha, *Some curvature tensor on a Kenmotsu manifold,* Tensor, **68** (2007), 140-147.

[2] C. S. Bagewadi and Venkatesha, *On pseudo projective ϕ-recurrent Kenmotsu manifolds,* Soochow J. Math., **32** (2006), 1-7.

[3] A. Bejancu and N. Papaghuic, *Semi-invariant submanifolds of a Sasakian manifold,* An Sti. Univ. AL I CUZA Iasi, **27** (1981), 163-170.

[4] D. E. Blair, *The theory of quasi-Sasakian structure,* J. Differ. Geom., **1** (1967), 331-345.

[5] D. E. Blair, *Contact Manifolds in Riemannian Geometry,* Lecture Notes in Mathematics, No 509. Springer, 1976.

[6] U. C. De and G. Pathak, *On 3-dimensional Kenmotsu manifolds,* Ind. J. Pure Appl. Math., **35** (2004), 159-165.

[7] U. C. De and P. Majhi, *On invariant submanifolds of Kenmotsu manifolds,* J. Geometry, **106** (2015), 109-122.

[8] J. Deprez, *Semi-parallel surfaces in the Euclidean space,* J. Geometry, **25** (1985), 192-200.

[9] Z. Guojing and W. Jianguo, *Invariant sub-manifolds and modes of nonlinear autonomous systems,* Appl. Math. Mech., **19**(7) (1998), 687-693.

[10] S. K. Hui and J. Roy, *Invariant and anti-invariant submanifolds of special quasi-Sasakian manifolds,* J. Geometry, **109** (2018), 1-16.

[11] K. Kenmotsu, *A class of almost contact Riemannian manifolds,* Tohoku Math. J., **24** (1972), 93-103.

[12] J.H. Kwon and B.H. Kim, *A new class of almost contact Riemannian manifolds,* Commun. Korean Math. Soc., **8** (1993), 455-465.

[13] V. Mangione, *Totally geodesic submanifolds of a Kenmotsu space form,* Math. Rep., **57**(4) (2005), 315-324.

[14] D. Nirmala, M. S. Siddesha, and C. S. Bagewadi, *On invariant submanifolds of Lorentzian β-Kenmotsu manifold,* Asian J. Math. Comput. Res., **19**(4) (2017), 203-213

[15] Z. Olszak, *On three-dimensional conformally flat quasi-sasakian manifolds,* Period. Math. Hung., **33** (1996), 105-113.

[16] C. Ozgur, *On weakly symmetric Kenmotsu manifolds,* Differ. Geom. Dyn. Syt., **8** (2006), 204-209.

[17] C. Ozgur, *On generalized recurrent Kenmotsu manifolds,* World Appl. Sci. J. **2** (2007), 29-33.
[18] A. A. Shaikh, Y. Matsuyama, and S. K. Hui, *On invariant submanifolds of* $(LCS)_n$*-manifolds,* J. Egypt. Math. Soc., **24**(2) (2016), 263-269.
[19] S.W. Shaw and C. Pierre, *Normal modes for nonlinear vibratory systems,* J. Sound Vib., **164**(1) (1993), 85-124.
[20] S. W. Shaw and C. Pierre, *Normal modes of vibration for nonlinear continuous systems,* J. Sound Vib., **169**(3) (1994), 319-347.
[21] M. S. Siddesha and C. S. Bagewadi, *Submanifold of a* (k,μ)*-Contact manifold,* CUBO A Mat. J., **18**(1) (2016), 59-68.
[22] M. S. Siddesha and C. S. Bagewadi, *Totally geodesic submanifolds of* (k,μ)*-contact manifolds,* ISOR J. Math., **12** (2016), 1-6.
[23] M. S. Siddesha and C. S. Bagewadi, *On some classes of invariant submanifolds* (k,μ)*-contact manifold,* J. of Inf. Mat. Sci., **9**(1) (2017), 13-26.
[24] M. S. Siddesha and C. S. Bagewadi, *Invariant submanifolds of* (k,μ)*-contact manifolds admitting quarter symmetric metric connection,* Int. J. Math. Trends Tech., **34**(2) (2016), 48-53.
[25] M.S. Siddesha and C.S. Bagewadi, *Submanifolds of a conformal* (k,μ)*-contact manifold,* BSG Proceedings, 28 (2021), 88-97.
[26] P. Somashekhara, M. S. Siddesha, C. S. Bagewadi, and M.M. Praveena, *On invariant submanifolds of SQ-Sasakian manifolds,* J. Pharm. Negative Results, **14**(3) (2023), 1151-1158.
[27] S. Sular and C. Ozgur, *On some submanifolds of Kenmotsu manifolds,* Chaos Solitons Fractals, **42**(4) (2009), 1990-1995.
[28] L. Verstraelen, *Comments on pseudosymmetry in the sense of Ryszard Deszcz In: Geometry and Topology of submanifolds,* World Sci. Pub. River Edge, **6** (1994), 199-209.
[29] A. Yildiz and C. Murathan, *Invariant submanifolds of Sasakian space forms,* J. Geometry, **95** (2009), 135-150.

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