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Analysis of a class of frictional contact problem for elastic-viscoplastic piezoelectric thermal materials

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Abstract. We consider a quasistatic frictional contact problem with a subdifferential boundary condition for general thermo-electro-elastic-viscoplastic materials. The frictional contact is modeled by a general velocity-dependent dissipation function. We derive a weak formulation of the system and then prove the existence of a unique weak solution to the problem. The proof is based on arguments of evolutionary variational inequalities, parabolic equations, the variational equation, differential equations, and the fixed-point theorem. Finally, we describe a number of concrete contact and friction conditions to which our results apply.

1. Introduction

Due to the importance of contact processes in structural and mechanical systems, significant progress has recently been achieved in modeling and mathematically analysing various processes involved in the contact between deformable bodies. Constitutive laws with internal variables have been employed in numerous publications to model the effect of internal variables on the behavior of real materials such as metals, rocks, polymers, and others, where the rate of deformation depends on these internal variables. Some of the internal state variables considered by many authors include the spatial distribution of dislocations, material work-hardening, absolute temperature, and damage fields. For examples, please refer to [**1**, **14**, **16**, **19**, **18**] for cases involving hardening, temperature, and other internal state variables.

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Various models have been developed to describe the interaction between electric and mechanical fields (see, e.g., [**2, 6, 7, 11**]). Therefore, there is a need to extend the results on models for contact with deformable bodies that include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [**5, 6, 15**], while viscoelastic piezoelectric materials are discussed in [**11, 9**], and elasto-viscoplastic piezoelectric materials have been studied in [**10, 8**].

In this paper, we investigate the mathematical model for the quasistatic process of frictional contact with a subdifferential boundary condition for general thermo-electro-elastic-viscoplastic materials. To do this, we consider a rate-type constitutive equation with two internal variables of the form

$$\begin{aligned}\boldsymbol{\sigma}(t) =&\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}(\varepsilon\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t)\\ &+\int_0^t \mathcal{G}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{E}^*\nabla\varphi(s)), \varepsilon\left(\boldsymbol{u}(s)\right), \theta(s), \boldsymbol{k}(s)\right) ds,\end{aligned} \tag{1}$$

In this context, $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor, and the dot above denotes the derivative with respect to the time variable. $\varepsilon(\boldsymbol{u})$ is the linearized strain tensor, θ represents the absolute temperature, and $\boldsymbol{k}$ is an internal state variable. Here, $\mathcal{A}$ is the viscosity operator, allowed to be nonlinear, $\mathcal{B}$ is the elasticity operator, and $\mathcal{G}$ is a nonlinear constitutive function that describes the visco-plastic behavior of the material. φ is the electric potential, and $\mathcal{E}$ represents the third-order piezoelectric tensor, with $\mathcal{E}^*$is its transposed. It follows from (1) that at each time moment, the stress tensor $\boldsymbol{\sigma}(t)$ is split into three parts $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^{\mathrm{V}}(t)+\boldsymbol{\sigma}^{\mathrm{E}}(t)+\boldsymbol{\sigma}^{\mathrm{R}}(t)$, where $\boldsymbol{\sigma}^V(t) = \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(t)))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}^{\mathrm{E}}(t) = \mathcal{E}^*\nabla\varphi(t)$ represents the electric part of the stress, whereas $\boldsymbol{\sigma}^{\mathrm{R}}(t)$ satisfies a rate-type elastic-viscoplastic relation with absolute temperature and internal state variable

$$\boldsymbol{\sigma}^R(t) = \mathcal{B}(\varepsilon(\boldsymbol{u}(t))) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}^R(s), \varepsilon(\boldsymbol{u}(s)), \theta(s), \boldsymbol{k}(s)\right) ds. \tag{2}$$

When $\mathcal{G} = 0$ in (1), it reduces to the electro-viscoelastic constitutive law given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t).$$

The evolution of the state internal variable field is given by the following differential equation

$$\dot{\boldsymbol{k}} = \Phi\left(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) - \mathcal{E}^*\nabla\varphi, \varepsilon(\boldsymbol{u}), \theta, \boldsymbol{k}\right), \tag{3}$$

Here, Φ is a nonlinear function that also depends on the internal state variable $\boldsymbol{k}$. The following constitutive law is employed for the electric potential

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi), \tag{4}$$

where $\boldsymbol{D}$ is the electric displacement field and $\boldsymbol{B}$ is the electric permittivity tensor.

The differential inclusion used for the evolution of the temperature field is

$$\dot{\theta} - k_0 \Delta\theta = \psi(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}), \varepsilon(\mathbf{u}), \theta, \boldsymbol{k}) + \rho, \tag{5}$$

where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ is a given volume heat source.

Finally, we model the frictional contact with a subdifferential boundary condition of the form

$$\boldsymbol{u} \in U, \quad h(\boldsymbol{v}) - h(\dot{\boldsymbol{u}}) \geq -\boldsymbol{\sigma\nu}(\boldsymbol{v} - \dot{\boldsymbol{u}}), \quad \forall \boldsymbol{v} \in U, \tag{6}$$

In this expression, U represents the set of contact admissible test functions, $\boldsymbol{\sigma\nu}$ denotes the Cauchy stress vector on the contact boundary, and h is a given convex function. The inequality in (6) holds almost everywhere on the contact surface. Examples and detailed explanations of inequality problems in contact mechanics that lead to boundary conditions of this form can be found in the monographs [**17, 20**]. In this condition we can be to the choice of particular forms of the function h, which can be written as the sum of two contact functions, corresponding to the normal and tangential components of the Cauchy stress vector.

The rest of the paper is organized as follows. In Section 2 we present the mechanical problem, some notation, list the assumptions on the problem's data, and we derive the variational formulation of the model. We prove in section 3 the existence and uniqueness of the solution, where it is carried out in several steps and is based on a classical existence and uniqueness result on parabolic inequalities, evolutionary variational equalities, differential equations and fixed point arguments. In Section 4 we describe a number of concrete thermal frictional conditions which may be cast in the abstract form (6) and to which our main results apply.

2. Statement of the Problem

In this section, we present some essential tools for our main results. Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain with a Lipschitz boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b on the other hand, such that $meas\Gamma_1 > 0, meas\Gamma_a > 0$. We denote by $\mathbb{S}^d$ the space of symmetric tensors on $\mathbb{R}^d$. We define the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d,$$
$$\|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}}, \quad \forall \boldsymbol{u} \in \mathbb{R}^d \quad \text{and} \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d.$$

Here and below, the indices i and j run from 1 to d and the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We define the function spaces

$$\begin{aligned}&H = L^2(\Omega)^d = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\}, H_1 = \{\boldsymbol{u} = (u_i) \mid \varepsilon(\boldsymbol{u}) \in \mathcal{H}\} = H^1(\Omega),\\ &\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}.\end{aligned}$$

Here $\varepsilon : H_1 \to \mathcal{H}$ and $\operatorname{Div} : \mathcal{H} \to H$ are the deformation and the divergence operators, respectively, defined by

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}), \quad \operatorname{Div}(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$

$$\begin{aligned}(\mathbf{u}, \boldsymbol{v})_{H_1} &= (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H_1,\\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.\end{aligned}$$

The associated norms on the spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$, are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$, respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\boldsymbol{u} \in H_1$, we also write $\boldsymbol{u}$ for the trace $\gamma\boldsymbol{u}$ of $\boldsymbol{u}$ on Γ and we denote by u_ν and $\boldsymbol{u}_\tau$ the normal and tangential components of $\boldsymbol{u}$ on Γ given by

$$u_\nu = \boldsymbol{u}.\boldsymbol{\nu}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}. \tag{7}$$

Similarly, normal and tangential components of the stress field $\boldsymbol{\sigma}$ are denoted by

$$\sigma_\nu = \sigma\nu \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{8}$$

and for all $\boldsymbol{\sigma} \in \mathcal{H}_1$ the following Green's formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\boldsymbol{v} da \quad \forall v \in H_1. \tag{9}$$

We suppose that $U \subset H_1, U + \mathcal{D}(\Omega)^d \subset U$, and let $\mathcal{V}$ denote the closed subspace of H_1 defined by

$$\mathcal{V} = \{\boldsymbol{v} \in H_1(\Omega) \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\} \cap U,$$

The set of admissible internal state variables is given by

$$Y = \left\{\boldsymbol{\varpi} = (\varpi_i) \mid \varpi_i \in L^2(\Omega), 1 \leq i \leq m\right\}.$$

Let V denote the closed subspace of $L^2(\Omega)$ given by

$$V = \left\{\zeta \in L^2(\Omega) \mid \varepsilon_{ij}(\zeta) \in L^2(\Omega)\right\} = H^1(\Omega),$$

Since $meas(\Gamma)_1 > 0$, Korn's inequality holds and thus, there exists a positive constant C_k depending only on Ω and Γ_1 such that

$$\|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}} \geq C_k \|\boldsymbol{v}\|_{H^1(\Omega)^d}, \quad \forall \boldsymbol{v} \in V.$$

On $\mathcal{V}$, we consider the inner product and the associated norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}} = (\varepsilon(\boldsymbol{v}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \|\boldsymbol{v}\|_{\mathcal{V}} = \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}, \tag{10}$$

From Korn's inequality, it follows that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_{\mathcal{V}}$ are equivalent norms on $\mathcal{V}$ and therefore $(\mathcal{V}, (,)_{\mathcal{V}})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\tilde{C}_0$, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \|\boldsymbol{v}\|_{\mathcal{V}}, \quad \forall \boldsymbol{v} \in \mathcal{V}. \tag{11}$$

Moreover, we denote by V' the dual of the space V. Identifying $L^2(\Omega)$, with its own dual, we have the inclusions

$$V \subset L^2(\Omega) \subset V'.$$

We use the notation $(.,.)_{V \times V'}$ to represent the duality pairing between V, V'

For the electric displacement field we use the Hilbert space

$$\mathcal{W} = \left\{ \mathbf{D} \in H \mid \operatorname{div} \mathbf{D} \in L^2(\Omega) \right\},$$

endowed with the inner product

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = (\mathbf{D}, \mathbf{E})_H + (\operatorname{div} \mathbf{D}, \operatorname{div} \mathbf{E})_{L^2(\Omega)},$$

and the associated norm $\|.\|_{\mathcal{W}}$, The electric potential field is to be found in

$$W = \left\{ \xi \in H^1(\Omega), \xi = 0 \text{ on } \Gamma_a \right\}.$$

Since meas $(\Gamma_a) > 0$, the Poincaré-Friedrichs inequality holds

$$\|\nabla \zeta\|_H \geq c_F \|\zeta\|_{H^1(\Omega)}, \quad \forall \zeta \in W, \tag{12}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W we use the inner product

$$(\varphi, \xi)_W = (\nabla \varphi, \nabla \xi)_H, \tag{13}$$

and $\|.\|_W$ the associated norm. It follows from (12) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space.

For any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$ and $k > 1$. For $T > 0$ we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\boldsymbol{f}\|_{C(0,T;X)} = \max_{t \in [0,T]} \|\boldsymbol{f}(t)\|_X.$$

$$\|\boldsymbol{f}\|_{C^1(0,T;X)} = \max_{t \in [0,T]} \|\boldsymbol{f}(t)\|_X + \max_{t \in [0,T]} \|\dot{\boldsymbol{f}}(t)\|_X.$$

The physical setting is the following. Let us consider electro- thermo-elastic-viscoplastic bodie occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a smooth boundary Γ, Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there, surface tractions of density f_0 act on $\Gamma_2 \times (0, T)$ and a volume force of density f_2 is applied in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. We admit a possible external heat source applied in $\Omega \times (0, T)$, given by the function ρ. The contact is frictional, the process is quasi-static and use (6) as boundary contact condition.

The classical formulation of the mechanical problem of electro-thermo-elastic-viscoplastic material with internal state variable, frictional, contact may be stated as follows.

Problem P**.** Find a displacement field $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0, T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0, T) \to \mathbb{R}$, an internal state variable field $\boldsymbol{k} : \Omega \times (0, T) \to \mathbb{R}^m$, and a electric displacement field $\boldsymbol{D} : (0, T) \to \mathbb{R}^d$ such that such that

$$\begin{aligned}\boldsymbol{\sigma}(t) = {} & \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}(\varepsilon \boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t) \\ & + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{E}^*\nabla\varphi(s)), \varepsilon\left(\boldsymbol{u}(s)\right), \theta(s), \boldsymbol{k}(s)\right) ds \\ & \qquad \text{in } \Omega \times (0, T),\end{aligned} \tag{14}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi) \quad \text{in } \Omega \times (0, T), \tag{15}$$

$$\dot{\boldsymbol{k}} = \Phi\left(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) - \mathcal{E}^*\nabla\varphi, \varepsilon(\boldsymbol{u}), \theta, \boldsymbol{k}\right) \quad \text{in } \Omega \times (0, T), \tag{16}$$

$$\dot{\theta} - k_0\Delta\theta = \psi(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}), \varepsilon(\mathbf{u}), \theta, \boldsymbol{k}) + \rho \quad \text{in } \Omega \times (0, T), \tag{17}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \qquad \text{in } \Omega \times (0, T), \tag{18}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \qquad \text{in } \Omega \times (0, T), \tag{19}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0, T), \tag{20}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0, T), \tag{21}$$

$$\boldsymbol{u} \in U, \quad h(\boldsymbol{v}) - h(\dot{\boldsymbol{u}}) \geq -\sigma\nu(\boldsymbol{v} - \dot{\boldsymbol{u}}), \quad \forall \boldsymbol{v} \in U \quad \text{on } \Gamma_3 \times (0, T), \tag{22}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0, T), \tag{23}$$

$$\boldsymbol{D}.\nu = q_2 \qquad \text{on } \Gamma_b \times (0, T), \tag{24}$$

$$k_0\frac{\partial\theta}{\partial\nu} + \delta\theta = 0 \qquad \text{on } \Gamma \times (0, T), \tag{25}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{k}(0) = \boldsymbol{k}_0, \quad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{26}$$

First, equations (14)-(16) represent the thermo-electro elastic-viscoplastic constitutive law with internal state variable, were $\mathcal{A}$ is the viscosity operator, allowed to be nonlinear, $\mathcal{B}$ is the elasticity operator and $\mathcal{G}$ is a nonlinear constitutive function describing the viscoplastic behavior of the material and depending on the internal state variable $\boldsymbol{k}$, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathcal{E}^*$ is its transposition. Equation (17) represents the energy conservation. Equations (18) and (19) represent the equilibrium equations for the stress and electric displacement fields. Equations (20)-(21) are the displacement-traction conditions. Condition (22) represents a subdifferential boundary condition on Γ_3 and $h : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ is a measurable convex function.

(23) and (24) represent the electric boundary conditions. The relation (25) represents a Fourier boundary condition for the temperature on Γ_2. Finally, (26) is the initial condition.

In the study of the problem P, we consider the following assumptions

The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\text{There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b)\text{There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{27}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{B}} > 0 \quad \text{such that} \\ \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|, \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{28}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \quad \text{such that} \\ \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{k}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{k}_2)\| \leq \\ L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|), \\ \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d, \text{ and } \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{R}^m, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ and } \boldsymbol{k} \in \mathbb{R}^m, \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{k}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{29}$$

The function $\psi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\psi} > 0 \quad \text{such that} \\ \|\psi(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{k}_1) - \psi(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{k}_2)\| \leq \\ L_{\psi}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|), \\ \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d, \text{ and } \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{R}^m, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ and } \boldsymbol{k} \in \mathbb{R}^m, \boldsymbol{x} \mapsto \psi(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{k}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \psi(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \in L^2(\Omega)^m. \end{cases} \tag{30}$$

The function $\Phi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\Phi} > 0 \quad \text{such that} \\ \|\Phi(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{k}_1) - \Phi(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{k}_2)\| \leq \\ L_{\Phi}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|), \\ \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d, \text{ and } \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{R}^m, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ and } \boldsymbol{k} \in \mathbb{R}^m, \boldsymbol{x} \mapsto \Phi(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{k}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \Phi(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \in L^2(\Omega)^m. \end{cases} \tag{31}$$

Electric permittivity operator $\boldsymbol{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \boldsymbol{B}(\boldsymbol{x}, E) = (b_{ij}(\boldsymbol{x})E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \quad b_{ij} = b_{ji} \in L^\infty(\Omega), 1 \leq i, j \leq d. \\ (c) \text{ There exists a constant } m_B > 0 \text{ such that} \\ \qquad \boldsymbol{B}(E.E) \geq m_B\|E\|^2, \forall E = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{cases} \tag{32}$$

The piezoelectric operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{E} = (e_{ijk}), e_{ijk} \in L^\infty(\Omega), 1 \le i, j, k \le d. \\ (b) \quad \mathcal{E}(\mathbf{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{\tau} \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases} \tag{33}$$

The piezoelectric tensor $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}$

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega).$$

The forces, tractions and the volume heat source have the regularity

$$\boldsymbol{f}_0 \in C(0, T; H), \quad \boldsymbol{f}_2 \in C\left(0, T; L^2(\Gamma_2)^d\right). \tag{34}$$

$$q_0 \in C\left(0, T; L^2(\Omega)\right), \quad q_2 \in C\left(0, T; L^2(\Gamma_b)\right), \quad \rho \in C(0, T; L^2(\Omega)). \tag{35}$$

The energy coefficient k_0 and the functions δ satisfy

$$k_0 > 0, \quad \delta > 0. \tag{36}$$

The initial data satisfy

$$\boldsymbol{u}_0 \in \mathcal{V}, \quad \theta_0 \in V, \quad \boldsymbol{k}_0 \in Y. \tag{37}$$

We introduce the following bilinear form $a : V \times V \to \mathbb{R}$

$$a(\zeta, \varphi) = k_0 \int_\Omega \nabla\zeta.\nabla\xi dx + \int_\Gamma \zeta.\xi d\gamma. \tag{38}$$

Next we define the functional $j : \mathcal{V} \to (-\infty, +\infty]$ by

$$j(\boldsymbol{v}) = \begin{cases} \displaystyle\int_{\Gamma_3} h(\boldsymbol{v}) \mathrm{d}a & \text{if } h(\boldsymbol{v}) \in L^1(\Gamma_3), \\ +\infty & \text{otherwise}. \end{cases} \tag{39}$$

and we suppose that

$$j \text{ is a convex lower semicontinuous function on } \mathcal{V} \text{ such that } j \not\equiv +\infty. \tag{40}$$

Next, we use the Riesz representation theorem to define $\mathbf{f} : [0, T] \to \mathcal{V}$

$$(\mathbf{f}(t), \boldsymbol{v})_{\mathcal{V}} = \int_\Omega \mathbf{f}_0(t) \cdot \boldsymbol{v} dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \boldsymbol{v} da, \tag{41}$$

for all $\boldsymbol{v} \in \mathcal{V}$, $t \in [0, T]$. Then conditions (34) and (41) imply

$$\mathbf{f}(t) \in C(0, T; \mathcal{V}). \tag{42}$$

and we denote by $q : [0, T] \to W$ the function defined by

$$(q(t), \zeta)_W = \int_\Omega q_0(t)\zeta dx - \int_{\Gamma_b} q_2(t)\zeta da. \tag{43}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (14)-(26).

Problem PV**.** Find a displacement field $\boldsymbol{u}:(0,T)\to\mathcal{V}$, a stress field $\boldsymbol{\sigma}:(0,T)\to\mathcal{H}$, an electric potential $\varphi:(0,T)\to W$, a temperature field $\theta:(0,T)\to V$, and an internal state variable field $\boldsymbol{k}:(0,T)\to Y$, and a electric displacement field $\boldsymbol{D}:(0,T)\to\mathcal{W}_1$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t)=&\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t))+\mathcal{B}(\varepsilon\boldsymbol{u}(t))+\mathcal{E}^*\nabla\varphi(t)\\&+\int_0^t\mathcal{G}\left(\boldsymbol{\sigma}(s)-\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)-\mathcal{E}^*\nabla\varphi(t)),\varepsilon\left(\boldsymbol{u}(s),\theta(s),\boldsymbol{k}(s)\right)\right)ds,\end{aligned}\tag{44}$$

$$\dot{\boldsymbol{k}}(t)=\Phi\left(\boldsymbol{\sigma}(t)-\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t))-\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\theta(t),\boldsymbol{k}(t)\right)\tag{45}$$

$$\mathbf{D}=\mathcal{E}\varepsilon(\mathbf{u})-\boldsymbol{B}\nabla(\varphi),\tag{46}$$

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}+j(\boldsymbol{v})-j(\dot{\boldsymbol{u}}(t))\\&\qquad\geq(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_{\mathcal{V}},\quad\forall\boldsymbol{v}\in\mathcal{V},\quad a.e.t\in(0,T).\end{aligned}\tag{47}$$

$$\left(\mathcal{E}\varepsilon\left(\boldsymbol{u}(t)\right)+\boldsymbol{B}\left(E\left(\varphi(t)\right)\right),\nabla\phi\right)_H=(-q(t),\phi)_W,\forall\phi\in W.\tag{48}$$

$$\begin{aligned}(\dot{\theta}(t),\mathbf{v})_{V'\times V}+a_0(\theta(t),\mathbf{v})=(\psi(\sigma(t),\varepsilon(\dot{\mathbf{u}}(t)),\theta(t)),\mathbf{v})_{V'\times V}\\+(\rho(t),\mathbf{v})_{L^2(\Omega)},\quad\forall\mathbf{v}\in V,\ \text{a.e. } t\in(0,T).\end{aligned}\tag{49}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad\theta(0)=\theta_0,\quad\boldsymbol{k}(0)=\boldsymbol{k}_0.\tag{50}$$

Our main existence and uniqueness result for Problem PV is in the following section.

3. Main Results

Theorem 3.1. *Assume that (27)-(40) hold, Then there exists a unique solution* $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\theta,\boldsymbol{k},\boldsymbol{D})$ *to problem* PV. *Moreover, the solution has the regularity*

$$\boldsymbol{u}(t)\in C^1(0,T;\mathcal{V}),\tag{51}$$

$$\varphi\in C(0,T;W),\tag{52}$$

$$\boldsymbol{\sigma}\in C(0,T;\mathcal{H}),\tag{53}$$

$$\theta\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;V).\tag{54}$$

$$\boldsymbol{k}\in C^1\left(0,T;Y\right).\tag{55}$$

$$\boldsymbol{D}\in C(0,T;\mathcal{W}).\tag{56}$$

The functions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, θ, $\boldsymbol{k}$, and $\boldsymbol{D}$ which satisfy (44)-(50)are called a weak solution of the contact problem P. We conclude that, under the assumptions (27)-(40), the mechanical problem (14)-(26) has a unique weak solution satisfying (51)-(56).

The proof of Theorem 3.1, is carried out is several steps and is based on a classical existence and uniqueness result on evolutionary variational inequalities, differentiel equations and fixed point argument.

We denote by C a constant whose value may change from line to line when no confusing can arise.

Let $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in C(0,T;\mathcal{H}\times Y)$ be given, in the first step, we consider the following variational problem.

Problem $\mathcal{P}^1_\eta$. Find a displacement field $\mathbf{u}_\eta : (0,T) \to \mathcal{V}$, such that

$$(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}_\eta(t)), \varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\boldsymbol{u}_\eta(t)), \varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)))_{\mathcal{H}} + \left(\boldsymbol{\eta}^1(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)\right)_{\mathcal{H}} \\ + j\left(\boldsymbol{v}\right) - j\left(\dot{\boldsymbol{u}}_\eta(t)\right) \geq \left(\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)\right)_{\mathcal{V}}, \forall \boldsymbol{v} \in \mathcal{V}, \text{ a.e. } t \in (0,T), \tag{57}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{58}$$

We have the following result for $\mathcal{P}^1_\eta$

Lemma 3.2. *There exists a unique solution to Problem $\mathcal{P}^1_\eta$ with the regularity (51).*

Proof. Let us introduce operators $A : \mathcal{V} \to \mathcal{V}$ and $B : \mathcal{V} \to \mathcal{V}$

$$(\mathcal{A}\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}} = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}. \tag{59}$$

$$(\mathcal{B}\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}} = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}. \tag{60}$$

Therefore, (57) can be rewritten as follows

$$(A\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\mathcal{V}} + (B\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\mathcal{V}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geq (\mathbf{f}_\eta(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\mathcal{V}}, \tag{61}$$

where

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}^1(t), \quad a.e. t \in [0,T].$$

It follows from (10), (59), (60) and hypothesis (27), (28) that there exist three positive constants $m_A = m_{\mathcal{A}}$, $L_A = L_{\mathcal{A}}$ and $L_B = L_{\mathcal{B}}$, such that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\mathcal{V}} \geq m_A \|\boldsymbol{u} - \boldsymbol{v}\|^2_{\mathcal{V}}$$
$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{\mathcal{V}} \leq L_A \|\boldsymbol{u} - \boldsymbol{v}\|_{\mathcal{V}}$$
$$\|B\boldsymbol{u} - B\boldsymbol{v}\|_{\mathcal{V}} \leq L_B \|\boldsymbol{u} - \boldsymbol{v}\|_{\mathcal{V}}.$$

Moreover, by using (37), (40), (42) and classical arguments of functional analysis concerning Evolutionary variational inequalities [**3, 21**] we can easily prove the existence and uniqueness of $\boldsymbol{u}_\eta$ satisfying (51). □

In the second step we use the displacement field obtained in Lemma 3.2, to construct the following variational problem for the an electrical potential.

Problem $\mathcal{P}^2_\eta$. Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$\left(\mathcal{E}\varepsilon\left(\boldsymbol{u}_\eta(t)\right) + \mathbf{B}\left(E\left(\varphi(t)_\eta\right)\right), \nabla\phi\right)_H = (-q(t), \phi)_W, \forall\phi \in W. \tag{62}$$

We have the following result for problem $\mathcal{P}^2_\eta$

Lemma 3.3. *Problem (62) has unique solution φ_η which satisfies the regularity (52).*

Moreover, if φ_η represents the solution to Problem $\mathcal{P}^2_\eta$ for η_i, $i = 1, 2$, then there exists $C > 0$ such that

$$\|\varphi_1(t) - \varphi_2(t)\|_W \leq C\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V, \quad \forall t \in (0,T). \tag{63}$$

Proof. we consider the form $G : W \times W \to \mathbb{R}$

$$G(\varphi, \phi) = (\mathbf{B}\nabla\varphi, \nabla\phi)_H \quad \forall\varphi, \phi \in W, \tag{64}$$

we use (12), (13), (32) and (64) to show that the form G is bilinear continuous, symmetric and coercive on W, moreover using (43) and the Riesz representation theorem we may define an element $\xi_\eta : [0,T] \to W$ such that

$$(\xi_\eta(t), \phi)_W = (q(t), \phi)_W + \left(\mathcal{E}\varepsilon\left(\boldsymbol{u}_\eta(t)\right), \nabla\phi\right)_H \quad \forall\phi \in W, t \in (0,T),$$

we apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$G\left(\varphi_\eta(t), \phi\right) = (\xi_\eta(t), \phi)_W, \quad \forall\phi \in W. \tag{65}$$

It follows from (65) that φ_η is a solution of the equation (62). Let $\varphi_{\eta_i} = \varphi_i$, and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1, 2$. We use (62) to obtain

$$\|\varphi_1(t) - \varphi_2(t)\|_W \leq C\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathcal{V}}, \quad \forall t \in (0,T).$$

Now since $\boldsymbol{u}_\eta \in C^1(0,T;\mathcal{V})$, it implies that $\varphi_\eta \in C(0,T;W)$.This completes the proof. □

For $\lambda \in C(0,T;V')$, we consider the following variational problem.

Problem $\mathcal{P}_\lambda$. Find the temperature field $\theta_\lambda : (0,T) \to \mathbb{R}$,

$$\left(\dot{\theta}_\lambda(t), \mathbf{v}\right)_{V'\times V} + a_0\left(\theta_\lambda(t), \mathbf{v}\right) = (\lambda(t) + \rho(t), \mathbf{v})_{V'\times V} \quad \forall\mathbf{v} \in V, \text{ a.e. } t \in (0,T), \tag{66}$$

$$\theta_\lambda(0) = \theta_0, \text{ in } \Omega. \tag{67}$$

Lemma 3.4. *There exists a unique solution θ_λ to the auxiliary problem $\mathcal{P}_\lambda$ satisfying (55).*

Proof. By an application of the Poincaré-Friedrichs inequality, we can find a constant $\delta' > 0$ such that

$$\int_\Omega \|\nabla\zeta\|^2 dx + \frac{\delta}{k_0}\int_\Gamma \|\zeta\|^2 d\gamma \geq \delta' \int_\Omega \|\zeta\|^2 dx, \quad \forall \zeta \in V.$$

Thus, we obtain

$$\mathfrak{a}_0(\zeta,\zeta) \geq c_1 \|\xi\|_V^2, \quad \forall \zeta \in V,$$

where $c_1 = k_0 min(1,\delta')/2$, which implies that a_0 is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (66) has a unique solution θ_λ satisfying (54). □

Now, define $\boldsymbol{k}_\eta \in C^1(0,T;Y)$ by

$$\boldsymbol{k}_\eta(t) = \boldsymbol{k}_0 + \int_0^t \boldsymbol{\eta}^2(s)ds. \tag{68}$$

In the fourth step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 3.2 and $\boldsymbol{k}_\eta$ defined in (68) to consider the following Cauchy problem for the stress field.

Problem $\mathcal{P}_{\eta,\lambda}$. Find the stress field $\boldsymbol{\sigma}_{\eta,\lambda} : (0,T) \to \mathbb{S}_n$ which is a solution of the problem

$$\boldsymbol{\sigma}_{\eta,\lambda}(t) = \mathcal{B}\left(\varepsilon\left(\boldsymbol{u}_\eta(t)\right)\right) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_{\eta,\lambda}(s), \varepsilon\left(\boldsymbol{u}_\eta(s), \theta_\lambda(s), \boldsymbol{k}_\eta(s)\right)\right) ds, \quad \forall t \in [0,T]. \tag{69}$$

Lemma 3.5. *There exists a unique solution of Problem $\mathcal{P}_{\eta,\lambda}$ and it satisfies (53). Moreover, if $\boldsymbol{u}_{\eta_i}, \theta_{\eta_i}$ and $\sigma_{\eta_i,\lambda_i}$ represent the solutions of problems $\mathcal{P}_\eta^1, \mathcal{P}_\eta^2, \mathcal{P}_\lambda$ and $\mathcal{P}_{\eta,\lambda}$, respectively, for $i = 1,2$, then there exists $C > 0$ such that*

$$\begin{aligned} &\|\boldsymbol{\sigma}_{\eta_1,\lambda_1}(t) - \boldsymbol{\sigma}_{\eta_2,\lambda_2}(t)\|_{\mathcal{H}}^2 \leq C \left(\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_{\mathcal{V}}^2\right. \\ &\left.+ \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|_{\mathcal{V}}^2 + \|\theta_{\lambda_1}(t) - \theta_{\lambda_2}(t)\|_V^2 + \|\boldsymbol{k}_{\eta_1}(t) - \boldsymbol{k}_{\eta_2}(t)\|_Y^2\, ds\right). \end{aligned} \tag{70}$$

Proof. Let $\mathcal{T}_{\eta,\lambda} : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ be the operator given by

$$\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}(t) = \mathcal{B}\left(\varepsilon\left(\boldsymbol{u}_\eta(t)\right)\right) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}(s), \varepsilon\left(\boldsymbol{u}_\eta(s), \theta_\lambda(s), \boldsymbol{k}_\eta(s)\right)\right) ds, \quad \forall t \in [0,T]. \tag{71}$$

For $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$, we use (29) and (71) to obtain for all $t \in [0,T]$

$$\|\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_1) - \mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_1)\|_{\mathcal{H}}^2 \leq L_{\mathcal{G}}^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds.$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \|\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_1) - \mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_1)\|_{\mathcal{H}}^2 dt_1 \leq L_{\mathcal{G}}^2 T \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds dt_1.$$

Therefore,

$$\|\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_2)-\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_2)\|^2_{\mathcal{H}}\le L^4_{\mathcal{G}}T^2\int_0^{t_2}\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,dsdt_1.$$

For $t_1,t_2,...,t_p\in(0,T)$, we generalize the procedure above by recurrence on p. We obtain the inequality

$$\begin{aligned}&\|\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_p)-\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_p)\|^2_{\mathcal{H}}\\&\qquad\le L^{2p}_{\mathcal{G}}T^p\int_0^{tp}\cdots\int_0^{t_2}\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,dsdt_1\dots dt_{p-1}.\end{aligned}$$

Which implies

$$\|\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_1(t_p)-\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_2(t_p)\|^2_{\mathcal{H}}\le\frac{L^{2p}_{\mathcal{G}}T^{p+1}}{p!}\int_0^T\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\|\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_1-\mathcal{T}_{\eta,\lambda}\boldsymbol{\sigma}_2\|^2_{C(0,T;\mathcal{H})}\le\frac{L^{2p}_{\mathcal{G}}T^{p+2}}{p!}\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|^2_{C(0,T;\mathcal{H})}.$$

It follows from this inequality that for large p enough, the operator $\mathcal{T}^{(p)}_{\eta,\lambda}$ is a contraction on the Banach space $C(0,T;\mathcal{H})$, and therefore there exists a unique element $\boldsymbol{\sigma}_{\eta,\lambda}\in C(0,T;\mathcal{H})$ such that $\mathcal{T}^{(p)}_{\eta,\lambda}\boldsymbol{\sigma}_{\eta,\lambda}=\boldsymbol{\sigma}_{\eta,\lambda}$. Moreover, $\boldsymbol{\sigma}_{\eta,\lambda}$ is the unique solution of Problem $\mathcal{P}_{\eta,\lambda}$, and using (69), the regularity of $\boldsymbol{u}_\eta$, the regularity of θ_λ, and the properties of the operators $\mathcal{B}$, $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_{\eta,\lambda}\in C(0,T;\mathcal{H})$. Consider now $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in C(0;T,\mathcal{H}\times Y)$, $\lambda_1,\lambda_2\in C(0;T,V')$ and for $i=1,2$ denote $\boldsymbol{u}_{\boldsymbol{\eta_i}}=\boldsymbol{u_i},\boldsymbol{\sigma}_{\boldsymbol{\eta_i},\boldsymbol{\lambda_i}}=\boldsymbol{\sigma_i},\boldsymbol{k}_{\boldsymbol{\eta_i}}=\boldsymbol{k_i}$ and $\theta_{\lambda_i}=\theta_i$. We have

$$\boldsymbol{\sigma}_i(t)=\mathcal{B}(\varepsilon(\boldsymbol{u}_i(t))+\int_0^t\mathcal{G}\left(\boldsymbol{\sigma}_i(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_i(s)),\theta_i(s),\boldsymbol{k}_i\right)ds,\qquad\text{a.e. }t\in(0,T),$$

and using the properties (28), (29) of $\mathcal{B}$, $\mathcal{G}$ we find

$$\begin{aligned}&\|\sigma_1(t)-\sigma_2(t)\|^2_{\mathcal{H}}\\&\le C\Bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\mathcal{V}}+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathcal{H}}\,ds+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\mathcal{V}}\,ds\\&\quad+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s)\|^2_Y\,ds\Bigg),\quad\forall t\in[0,T].\end{aligned}\tag{72}$$

We use Gronwall argument in the previous inequality to deduce (70), which concludes the proof of Lemma 3.5. □

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$ the operator $\mathcal{E}$, the function S for $t\in(0,T)$, we consider the element

$$\Lambda(\boldsymbol{\eta},\lambda)(t)=\left(\Lambda^1(\boldsymbol{\eta},\lambda)(t),\Lambda^2(\boldsymbol{\eta},\lambda)(t),\Lambda^3(\boldsymbol{\eta},\lambda)(t)\right)\in\mathcal{H}\times Y\times V',\tag{73}$$

defined by

$$\begin{aligned}(\Lambda^1(\boldsymbol{\eta},\lambda)(t),\boldsymbol{v})_{\mathcal{H}\times\mathcal{V}} &= (\mathcal{E}^*\nabla\varphi_\eta(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\\ &+\left(\int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_{\eta,\lambda}(s),\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\eta(s)\right),\theta_\lambda(s),\boldsymbol{k}_{\boldsymbol{\eta}}(s)\right)ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}},\forall\boldsymbol{v}\in\mathcal{V},\end{aligned} \tag{74}$$

$$\Lambda^2\left(\boldsymbol{\eta},\lambda\right)(t)=\Phi\left(\boldsymbol{\sigma}_{\eta,\lambda},\varepsilon\left(\boldsymbol{u}_\eta(t)\right),\alpha_\lambda(t),\theta_\lambda(t),\boldsymbol{k}_{\boldsymbol{\eta}}(t)\right). \tag{75}$$

$$\Lambda^3\left(\boldsymbol{\eta},\lambda\right)(t)=\psi\left(\boldsymbol{\sigma}_{\eta,\lambda},\varepsilon\left(\boldsymbol{u}_\eta(t)\right),\alpha_\lambda(t),\theta_\lambda(t),\boldsymbol{k}_{\boldsymbol{\eta}}(t)\right). \tag{76}$$

Here, for every $(\boldsymbol{\eta},\lambda)\in C(0,T;\mathcal{H}\times Y\times V')$, $\boldsymbol{u}_\eta,\varphi_\eta,\theta_\lambda$ and $\boldsymbol{k}_{\boldsymbol{\eta}}$ represent the displacement field, the electric potential field, the adhesion field and the stress field obtained in Lemmas 3.2, 3.3, 3.4, respectively, and $\boldsymbol{k}_{\boldsymbol{\eta}}$ is the internal state variable given by (68). We have the following result.

Lemma 3.6. *The mapping Λ has a fixed point $(\boldsymbol{\eta}^*,\lambda^*)\in C(0,T;\mathcal{H}\times Y\times V')$, such that $\Lambda\left(\boldsymbol{\eta}^*,\lambda^*\right)=\left(\boldsymbol{\eta}^*,\lambda^*\right)$.*

Proof. Let $t\in(0,T)$ and $\left(\boldsymbol{\eta}_1,\lambda_1\right),\left(\boldsymbol{\eta}_2,\lambda_2\right)\in C\left(0,T;\mathcal{H}\times Y\times V'\right)$. We use the notation that $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i}=\dot{\boldsymbol{u}}_i$, $\theta_{\lambda_i}=\theta_i$, $\varphi_{\eta_i}=\varphi_i$, $\boldsymbol{k}_{\boldsymbol{\eta_i}}=\boldsymbol{k_i}$ and $\boldsymbol{\sigma}_{\eta_i,\lambda_i}=\boldsymbol{\sigma}_i$ for $i=1,2$. Using (10)),(30),(31), and (33) to find

$$\begin{aligned}&\left\|\Lambda\left(\boldsymbol{\eta}_1,\lambda_1\right)(t)-\Lambda\left(\boldsymbol{\eta}_2,\lambda_2\right)(t)\right\|^2_{\mathcal{H}\times Y\times V'}\\ \leq& C(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\int_0^t(\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\mathcal{V}}\\ &+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}+\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s)\|^2_Y\,ds)\\ &+\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\mathcal{V}}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\\ &+\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s)\|^2_Y),\end{aligned} \tag{77}$$

we use estimates (63), (70) to obtain

$$\begin{aligned}&\left\|\Lambda\left(\boldsymbol{\eta}_1,\lambda_1\right)(t)-\Lambda\left(\boldsymbol{\eta}_2,\lambda_2\right)(t)\right\|^2_{\mathcal{H}\times Y\times V'}\\ \leq& C(\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\mathcal{V}}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}+\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s)\|^2_Y\\ &+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\mathcal{V}}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}+\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s)\|^2_Y\,ds.\end{aligned} \tag{78}$$

Using (57), we derive the relation

$$\begin{aligned}(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}_1)-\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}_2),\varepsilon(\dot{\boldsymbol{u}}_1)-\varepsilon(\dot{\boldsymbol{u}}_2)_{\mathcal{H}}+(\mathcal{B}\varepsilon(\boldsymbol{u}_1)-\mathcal{B}\varepsilon(\boldsymbol{u}_2),\varepsilon(\dot{\boldsymbol{u}}_1)-\varepsilon(\dot{\boldsymbol{u}}_2)_{\mathcal{H}}\\ \leq(\boldsymbol{\eta}^1_1-\boldsymbol{\eta}^1_2,\varepsilon(\dot{\boldsymbol{u}}_1)-\varepsilon(\dot{\boldsymbol{u}}_2))_{\mathcal{H}},\end{aligned} \tag{79}$$

then we use assumptions (27) and (28) to derive

$$\|\dot{\boldsymbol{u}}_1-\dot{\boldsymbol{u}}_2\|_{\mathcal{V}}\leq C\left(\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\mathcal{V}}+\left\|\boldsymbol{\eta}^1_1-\boldsymbol{\eta}^1_2\right\|_{\mathcal{H}}\right). \tag{80}$$

Since $\boldsymbol{u}_i\left(t\right) = \int\limits_0^t \dot{\boldsymbol{u}}_i\left(s\right) ds + \boldsymbol{u}_0, \forall t \in (0,T)$, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathcal{V}}^2 \leq \int_0^T \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_{\mathcal{V}}^2 \, ds. \tag{81}$$

Combining (80) and (81), and using the Gronwall's inequality, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathcal{V}} \leq C \int_0^t \left\|\boldsymbol{\eta}_1^1 - \boldsymbol{\eta}_2^1\right\|_{\mathcal{H}} ds, \quad t \in (0,T). \tag{82}$$

On the other hand, if we take the substitution $\lambda = \lambda_1, \lambda = \lambda_2$ in (66) and subtracting the two obtained equations, we deduce by choosing $v = \theta_{\lambda_1} - \theta_{\lambda_2}$ as test function

$$\begin{aligned}\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 + C_1 \int_0^t \|\theta_1(t) - \theta_2(t)\|_V^2 \\ \leq \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{V'} \|\theta_1(s) - \theta_2(s)\|_V \, ds, \quad \forall t \in (0,T),\end{aligned}$$

employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\theta_1(s) - \theta_2(s)\|_V^2 \, ds \\ \leq C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{V'}^2 \, ds, \quad \forall t \in (0,T),\end{aligned}$$

we use the inclusion $L^2(\Omega) \subset V$, we get

$$\begin{aligned}\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 \, ds \\ \leq C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{V'}^2 \, ds, \quad \forall t \in (0,T),\end{aligned}$$

from this inequality, combined with Gronwall's inequality, we deduce that

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leq C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{V'}^2 \, ds. \tag{83}$$

Furthermore, from (68) we have

$$\|\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)\|_Y^2 \leq C \int_0^t \left\|\boldsymbol{\eta}_1^2(s) - \boldsymbol{\eta}_2^2(s)\right\|_Y^2 ds. \tag{84}$$

Form the previous inequality and estimates (83), (82) and (77) it follows now that

$$\begin{aligned}\|\Lambda\left(\boldsymbol{\eta}_1, \lambda_1\right)(t) - \Lambda\left(\boldsymbol{\eta}_2, \lambda_2\right)(t)\|_{\mathcal{H}\times Y\times V'}^2 \\ \leq C \int_0^t \|\left(\boldsymbol{\eta}_1, \lambda_1\right)(s) - \left(\boldsymbol{\eta}_2, \lambda_2\right)(s)\|_{\mathcal{H}\times Y\times V'}^2 \, ds.\end{aligned} \tag{85}$$

Let is introduce the following notations

$$\begin{cases} I_1 = \int_0^t \|(\boldsymbol{\eta}_1, \lambda_1)(s) - (\boldsymbol{\eta}_2, \lambda_2)(s)\|_{\mathcal{H}\times Y\times V'}\, ds, \\ \vdots \\ I_k = \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_1} \|(\boldsymbol{\eta}_1, \lambda_1)(r) - (\boldsymbol{\eta}_2, \lambda_2)(r)\|_{\mathcal{H}\times Y\times V'}, \end{cases}$$

and by induction, by denoting by Λ^m the m power of the operator Λ, we obtain

$$\begin{aligned} &\|\Lambda^m(\boldsymbol{\eta}_1, \lambda_1)(t) - \Lambda^m(\boldsymbol{\eta}_2, \lambda_2)(t)\|_{\mathcal{H}\times Y\times V'} \\ &\quad \le C^m \left(\sum_{k=1}^m C_m^k I^{m-k} \|(\boldsymbol{\eta}_1, \lambda_1)(t) - (\boldsymbol{\eta}_2, \lambda_2)(t)\|_{\mathcal{H}\times Y\times V'} \right), \end{aligned}$$

for all $t \in (0, T)$ and $m \in \mathbb{N}$,

$$\begin{aligned} I^{m-k}\|((\boldsymbol{\eta}_1, \lambda_1) - (\boldsymbol{\eta}_2, \lambda_2))\|_{\mathcal{H}\times Y\times V'} &= \int_{(m-k)\text{ fois}} . \int \|(\boldsymbol{\eta}_1, \lambda_1) - (\boldsymbol{\eta}_2, \lambda_2)\|_{\mathcal{H}\times Y\times V'} \\ &\le \int_0^s \int \cdots \int_{(m-k)\text{ fois}} \|(\boldsymbol{\eta}_1, \lambda_1) - (\boldsymbol{\eta}_2, \lambda_2)\|_{C(0,T;\mathcal{H}\times Y\times V')} \\ &\le \frac{t^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \lambda_1) - (\boldsymbol{\eta}_2, \lambda_2)\|_{C(0,T;\mathcal{H}\times Y\times V')} \\ &\le \frac{T^{m-k}}{k!} \|(\boldsymbol{\eta}_1, \lambda_1) - (\boldsymbol{\eta}_2, \lambda_2)\|_{C(0,T;\mathcal{H}\times Y\times V')}, \end{aligned}$$

$$\begin{aligned} &\|\Lambda^m(\boldsymbol{\eta}_1, \lambda_1)(t) - \Lambda^m(\boldsymbol{\eta}_2, \lambda_2)(t)\|_{C(0,T;\mathcal{H}\times Y\times V')} \\ &\quad \le C^m \left(\sum_{k=1}^m C_m^k \frac{T^{m-k}}{k!} \|(\eta_1, \lambda_1)(t) - (\eta_2, \lambda_2)(t)\|_{C(0,T;\mathcal{H}\times Y\times V')} \right) \\ &\quad \le \frac{(CT)^m}{m!} \|(\boldsymbol{\eta}_1, \lambda_1)(t) - (\boldsymbol{\eta}_2, \lambda_2)(t)\|^2_{C(0,T;\mathcal{H}\times Y\times V')}, \end{aligned}$$

Thus implies that for m large enough, the operator Λ^m of Λ is a contraction on Banach space $C(0, T; \mathcal{H} \times Y \times V')$. So Λ^m has a unique fixed point $(\boldsymbol{\eta}^*, \lambda^*) \in C(0, T; \mathcal{H} \times Y \times V')$, and therefore $(\boldsymbol{\eta}^*, \lambda^*)$ is a unique fixed point of Λ.

□

Now we have every thing that is required to prove theorem 3.1.

Existence. Let $(\boldsymbol{\eta}^*, \lambda^*) \in C(0, T; \mathcal{H} \times Y \times V')$ be the fixed point of Λ and

$$\boldsymbol{u} = \boldsymbol{u}_{\eta^*}, \quad \boldsymbol{k} = \boldsymbol{k}_{\eta^*}, \quad \varphi_{\eta^*} = \varphi, \quad \theta = \theta_{\lambda^*} \tag{86}$$

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{E}^*\nabla\varphi(t) + \boldsymbol{\sigma}_{\eta^*\lambda^*}, \tag{87}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi). \tag{88}$$

We prove that $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{k}, \varphi, \theta, \boldsymbol{D})$ satisfies (44)-(50) and (51)-(56). Indeed, we write (69) for $\boldsymbol{\eta}^* = \boldsymbol{\eta}, \quad \lambda^* = \lambda$ and use (86)-(87) to obtain that (44) is satisfied. Now we consider (57) for $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ and use the first equality in (86) to find

$$\begin{gathered}(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}_\eta(t)), \varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)))_{\mathcal{H}} \\ + \left(\boldsymbol{\eta}^{1*}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)\right)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}_\eta(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_{\mathcal{V}} \\ \forall \boldsymbol{v} \in V, t \in [0, T].\end{gathered} \tag{89}$$

The equalities $\Lambda^1(\boldsymbol{\eta}^*, \lambda^*) = \boldsymbol{\eta}^{1*}$, $\Lambda^2(\boldsymbol{\eta}^*, \lambda^*) = \boldsymbol{\eta}^{2*}$, and $\Lambda^3(\boldsymbol{\eta}^*, \lambda^*) = \lambda^*$. combined with (74)-(76), (86) and (87) show that for all $\boldsymbol{v} \in \mathcal{V}$,

$$\begin{aligned}(\boldsymbol{\eta}^{\mathbf{1}^*}(t), \boldsymbol{v})_{\mathcal{H}\times\mathcal{V}} &= (\mathcal{B}(\varepsilon\boldsymbol{u}(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \\ &+ \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) - \mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(s)), \theta(s), \boldsymbol{k}(s))ds, \varepsilon(\boldsymbol{v})\right)_{\mathcal{H}},\end{aligned} \tag{90}$$

$$\boldsymbol{\eta}^{\mathbf{2}^*}(t) = \Phi\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(t)), \theta(t), \boldsymbol{k}\right). \tag{91}$$

$$\lambda^*(t) = \psi\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s) - \mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(t)), \theta(t), \boldsymbol{k}\right). \tag{92}$$

From (92) and (68) we see that (45) is satisfied. We substitute (90) in (89)) and use (44) to see that (47) is satisfied.

We write now (62) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (86) to find (48). Next, (50), The regularities (51), (52), and (55) follow from Lemmas 3.2, 3.3, and the relation (68). We write (66) for $\lambda = \lambda^*$ and use (86) and (92) to find that (49) is satisfied, and the regularity (54) follows from lemma 3.4. The regularity $\boldsymbol{\sigma} \in C(0, T; \mathcal{H})$ follows from Lemmas 3.5.

Let now $t_1, t_2 \in [0, T]$, from (12), (32), (33) and (88), we conclude that there exists a positive constant $C > 0$ verifying

$$\|\boldsymbol{D}(t_1) - \boldsymbol{D}(t_2)\|_H \leq C\left(\|\varphi(t_1) - \varphi(t_2)\|_W + \|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_{\mathcal{V}}\right).$$

The regularity of $\boldsymbol{u}$ and φ given by (51) and (52) implies

$$\boldsymbol{D} \in C(0, T; H). \tag{93}$$

We choose $\phi \in D(\Omega)^d$ in (48) and using (43) we find

$$\operatorname{div} \boldsymbol{D}_*(t) = q_0(t), \quad \forall t \in [0, T]. \tag{94}$$

Property (56) follows from (35),(93) and (94) which concludes the existence part the theorem.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ.

4. Examples of subdifferential conditions with friction

In this section, we report some examples of contact laws that condition (22), which were presented in a previous paper (see reference [4]). Also note that the relevant boundary value problem for each example has a unique weak solution

Example 4.1. Bilateral contact with Tresca's friction law. In this case, the boundary conditions on the contact surface are derived as follows

$$\begin{cases} u_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \leq g, \\ |\boldsymbol{\sigma}_\tau| < g \Longrightarrow \dot{\boldsymbol{u}}_\tau = \mathbf{0}, \qquad \text{on } \Gamma_3 \times (0,T) \\ |\boldsymbol{\sigma}_\tau| = g \Longrightarrow \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \lambda \geq 0 \end{cases} \tag{95}$$

Here λ represents the friction bound, i.e. the magnitude of the limiting friction at which slip occurs. The contact is assumed to be bilateral, i.e. there is no loss of contact during the process.

Let $\mathcal{V}$ denote the closed subspace of H_1 defined by

$$\mathcal{V} = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1, \quad u_\nu = 0 \text{ on } \Gamma_3\}.$$

(22) holds with the choice $h(\boldsymbol{v}) = g|\boldsymbol{v}_\tau|$. Where $g \in L^\infty(\mathbb{R})$, $g \geq 0$.

Example 4.2. Bilateral contact with elastic-viscoplastic friction condition. We consider problems with the boundary conditions

$$u_\nu = 0, \quad \boldsymbol{\sigma}_\tau = -g\,|\dot{\boldsymbol{u}}_\tau|^{p-1}\,\dot{\boldsymbol{u}}_\tau \qquad \text{on } \Gamma_3 \times (0,T) \tag{96}$$

where $g > 0$ is the coefficient of friction and $0 < p \leq 1$. In this case we consider

$$\mathcal{V} = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1, \quad u_\nu = 0 \text{ on } \Gamma_3\}.$$

and

$$h(\boldsymbol{v}) = \frac{g}{p+1}\,|\boldsymbol{v}_\tau|^{p+1}, \quad g \in L^\infty(\mathbb{R}).$$

Example 4.3. elastic-viscoplastic contact with Tresca's friction law. Here the model of the contact reads as follows

$$\begin{cases} -\sigma_\nu = k\,|\dot{u}_\nu|^{q-1}\,\dot{u}_\nu, \quad |\boldsymbol{\sigma}_\tau| \leq g, \\ |\boldsymbol{\sigma}_\tau| < g \Longrightarrow \dot{\boldsymbol{u}}_\tau = 0, \qquad \text{on } \Gamma_3 \times (0,T) \\ |\boldsymbol{\sigma}_\tau| = g \Longrightarrow \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \quad \lambda \geq 0 \end{cases} \tag{97}$$

where $g, k \geq 0$ and $0 < q \leq 1$. We choose $U = H_1$, $\mathcal{V} = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\}$ and

$$h(\boldsymbol{v}) = \frac{k}{q+1}\,|v_\nu|^{q+1} + g\,|\boldsymbol{v}_\tau|, \quad g, k \in L^\infty(\mathbb{R}).$$

Example 4.4. Elastic-viscoplastic contact with friction. We have the following boundary condition

$$-\sigma_\nu = k\,|\dot{u}_\nu|^{q-1}\,\dot{u}_\nu, \quad \boldsymbol{\sigma}_\tau = -g\,|\dot{\boldsymbol{u}}_\tau|^{p-1}\,\dot{\boldsymbol{u}}_\tau \quad \text{on } \Gamma_3 \times (0,T), \tag{98}$$

We take $k, g > 0$, $0 < p, q \leq 1$, $U = H_1, \mathcal{V} = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}$ and

$$h(\boldsymbol{v}) = \frac{k}{q+1}\,|v_\nu|^{q+1} + \frac{g}{p+1}\,|\boldsymbol{v}_\tau|^{p+1}, \quad g, k \in L^\infty(\mathbb{R}).$$

Since the assumptions (40) is satisfied for each example, we may apply Theorem 3.1, and we deduce that there is a unique weak solution to each problem.

References

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Application to Free Boundary Problems*, Wiley-Interscience, Chichester-New York, 1984.

[2] S. Boutechebak, *A dynamic problem of frictionless contact for elastic-thermoviscoplastic materials with damage*, Int. J. Pure Appl. Math., **86** (2013), 173–197.

[3] H. Brézis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier **18** (1968), 115–175.

[4] O. Chau, D. Motreanu, and M. Sofonea, *Quasistatic frictional problems for elastic and viscoelastic materials*, Appl. Math., **47**(4) (2002), 341-360.

[5] H. L. Dai and X. Wang, *Thermo-electro-elastic transient responses in piezoelectric hollow structures*, Int. J. Sol. Struct., **42** (2005), 1151–1171.

[6] T. Haje Ammar, S. Drabla, and B. Benabderrahmane, *Analysis and approximation of frictionless contact problems between two piezoelectric bodies with adhesion*, Georgian Math. J., **44** (2014), 1-15.

[7] T. Haje Ammar, A. Saidi, and A. Azeb Ahmed, *Dynamic contact problem with adhesion and damage between thermo-electro-elasto-viscoplastic bodies*, C. R. Mecani., **345** (2017), 329–336.

[8] A. Hamidat and A. Aissaoui, *A quasistatic frictional contact problem with normal damped response for thermo-electro-elastic-viscoelastic bodies*, Adv. Math.: Sci. J. **10**(12) (2021).

[9] A. Hamidat and A. Aissaoui, *A quasi-static contact problem with friction in electro viscoelasticity with long-term memory body with damage and thermal effects*, Int. J. Nonlinear Anal. Appl., **13**(2) (2022), 205-220.

[10] W. Han, M. Sofonea, and K. Kazmi, *Analysis and numerical solution of a frictionless contact problem for electro-elastic-visco-plastic materials*, Compu. Methods. Appl. Mech. Engrg., **196** (2007), 3915–3926.

[11] Z. Lerguet, M. Shillor, and M. Sofonea, *A frictional contact problem for an electroviscoelastic body*, Electron. J. Differ. Equat., **2007** (170) (2007), 1–16.

[12] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires*, Dunod, 1969.

[13] J. L. Lions and E. Magénes, *Problemes aux limites non homogenes et applications*, vol. 1 et 2, Dunod, Paris, 1968.

[14] F. Messelmi, B. Merouani, and M. Meflah, *Nonlinear thermoelasticity problem*, Anal. Univer. Oradea, Fasc. Math. Tome **15** (2008), 207–217.

[15] R. D. Mindlin, *Polarisation gradient in elastic dielectrics*, Int. J. Solids Struct., **4**(6) (1968), 637–642.

[16] J. Necas and J. Kratochvil, *On existence of the solution boundary value problems for elastic-inelastic solids*, Comment. Math. Univ. Carolinea, 14 (1973), 755–760.

[17] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Basel, 1985.

[18] M. Sofonea, *Functional Methods in Thermo-ElastoVisco-Plasticity*, Ph. D. Thesis, Univ of Bucharest, 1988. [in Romanian]

[19] M. Sofonea, *Quasistatic processes for elastic-viscoplastic materials with internal state variables*, Ann. Sci. Univer. Clermont. Math., **94**(25) (1989), 47-60.

[20] M. Sofonea and M. Shillor, *Variational analysis of quasistatic viscoplastic contact problems with friction*, Comm. Appl. Anal., **5** (2001), 135–151.

[21] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, Cambridge University Press, 2012.

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