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Fixed point theorem of integral type mapping in S_b-metric space

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Abstract. The purpose of this paper is to establish the existence and uniqueness of a common coupled Fxed point result in the framework of complete symmetric S_b-metric space. The obtained results generalize and extend some of the well-known results in the Literature.

1. Introduction

Sedghi, Shobe and Aliouche [17] introduced the notion of S-metric space and proved some fixed point theorem by modifying D-metric and G-metric spaces. Sedghi and Van Dung [19] remarked that every S-metric space is topologically equivalent to a metric space. Bakhtin [3] introduced the concept of b-metric space as a generalization of metric space. Czerwik [7] extended the Banach contraction principle is b-metric space. Inspired by the works of Bakhtin [3] and Sedghi, Shobe and Aliouche [17], Souayah and Mlaiki [23] introduced the concept of S_b-metric space.

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existences and uniqueness of a fixed point. His result is called Banach's Fixed point Theorem or the Banach Contraction principle. This theorems provides a technique for solving a variety of problems of applied nature in mathematical science and engineering. Many authors have extended, generalized and improved Banach's Fixed point Theorem in Different ways.

2. Preliminaries

Definition 2.1. Let X be a non-empty set. A function $S : X^3 \to [0, \infty)$ is said to be an S-metrics on X if for each x, y, z, a ∈ X,

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(1) $S(x, y, z) \geq 0$
(2) $S(x, y, z) = 0$ if and only if $x = y = z$
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

A pair (x, S) is called an S-metric space.

Definition 2.2. Let (X, S) be an S-metric space. A map $F : X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that $S(F(x), F(x), F(y)) \leq LS(x, x, y)$.

Let (X, S) be an S-metric space, then we have

Lemma 2.1. [17] *S(x,x,y)=S(y,y,x), for all* $x, y, \in X$.

Lemma 2.2. [17] *The limit of* $\{x_n\}$ *in S-metric space is unique.*

Lemma 2.3. [17] *The convergent sequence* $\{x_n\}$ *in* X *is Cauchy.*

Lemma 2.4. [17] *If the sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$. *Then,* $\lim_{n\to\infty} (x_n, x_n, y_n) = S(x, x, y)$.

Definition 2.3. Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R_+$ is called a b-metric provided that for all $x, y, z \in X$.

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \leq s\,[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space. It is clear that the b-metric space is an extension of usual metric space.

Definition 2.4. [17] Let X be a nonempty set and let $s \geq 1$ be a given number. A function $S_b : X^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, t \in X$, the following conditions hold:

(1) $S_b(x, y, z) = 0$ if and only if $x = y = z$.
(2) $S_b(x, x, y) = S_b(y, y, x)$.
(3) $S_b(x, y, z) \leq s\,[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$.

The pair (X, S_b) is called a S_b-metric space.

Definition 2.5. [17] A sequence $\{x_n\}$ in X converges if and only if there exists z $\in X$ such that $S_b(x_n, x_n, z) \to 0$ as $n \to \infty$. In this case we write $\lim_{n\to\infty} x_n = z$.

Definition 2.6. [17] A sequence $\{x_n\}$ in X is said to be Cauchy sequence if and only if $S_b(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 2.7. [17] The S_b-metric space (X, S_b) is said to be complete if every cauchy sequence is convergent.

Definition 2.8. A S_b-metric space is said to be symmetric if

$$S_b(x, x, y) = S_b(y, y, x)$$

for all $x, y \in$X.

Theorem 2.5. *(Banach's contraction principle) Let (X, d) be a complete metric space, c ∈(0,1) and $f : X \to X$ be a mapping such that for each $x, y \in X$, $d(fx, fy) \leq cd(x, y)$. Then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim n_{\to\infty} f^n(x) = a$.*

In 2002, Branciari [6] analysed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a general contractive condition of integral type.

Theorem 2.6. *(Branciari) Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq c \int_0^{d(x,y)},$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$, then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n(x) = a$.

After the paper of Branciari [6], a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties.

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \int_0^{\max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}} \phi(t)dt.$$

3. Main Result

Theorem 3.1. *Let (X, S_b) be a complete symmetric S_b-metric space with parameter $s \geq 1$and let the mappings f,g : $X^2 \to$X satisfying*

$$\int_0^{S_b(f(x,y),f(x,y),g(u,v))}\phi(t)dt \tag{1}$$

$$\begin{aligned}
&\leq a_1\int_0^{\frac{S_b(x,x,u)+S_b(y,y,v)}{2}}\phi(t)dt + a_2\int_0^{\frac{S_b(f(x,y),f(x,y),g(u,v))S_b(x,x,u)}{1+S_b(x,x,u)+S_b(y,y,v)}}\phi(t)dt\\
&+ a_3\int_0^{\frac{S_b(f(x,y),f(x,y),g(u,v))S_b(y,y,v)}{1+S_b(x,x,u)+S_b(y,y,v)}}\phi(t)dt + a_4\int_0^{\frac{S_b(x,x,f(x,y))S_b(x,x,u)}{1+S_b(x,x,u)+S_b(y,y,v)}}\phi(t)dt\\
&+ a_5\int_0^{\frac{S_b(x,x,f(x,y))S_b(y,y,v)}{1+S_b(x,x,u)+S_b(y,y,v)}}\phi(t)dt + a_6\int_0^{\frac{S_b(u,u,g(u,v))S_b(x,x,u)}{1+S_b(x,x,u)+S_b(y,y,v)}}\phi(t)dt\\
&+ a_7\int_0^{\frac{S_b(u,u,g(u,v))S_b(y,y,v)}{1+S_b(x,x,u)+S_b(y,y,v)}}\phi(t)dt\\
&+ a_8\int_0^{\max\{S_b(f(x,y),f(x,y),g(u,v)),S_b(u,u,g(u,v))\}}\phi(t)dt + a_9\int_0^{\min\{S_b(x,x,u),S_b(y,y,v)\}}\phi(t)dt
\end{aligned} \tag{2}$$

where $a_i \geq 0$ (i=1,...,9) and $\sum_{i=1}^{9} a_i < 1$, $s < \frac{1-a_2-a_3-a_6-a_7-a_8}{a_1+a_4+a_5+a_9}$, $\phi : [0,+\infty) \to [0,+\infty)$ is a non-negative Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$, and for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$. Then f and g have a unique common coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be an arbitrary points in X. We can construct a sequence $\{x_k\}$ and $\{y_k\}$ in X such that $x_{2k+1} = f(x_{2k}, y_{2k}), y_{2k+1} = f(y_{2k}, x_{2k})$, $x_{2k+2} = g(x_{2k+1}, y_{2k+1})$, and $y_{2k+2} = g(y_{2k+1}, x_{2k+1})$ for $k = 0, 1, \ldots$. Then

$$\begin{aligned}
\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt &= \int_0^{S_b(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))}\phi(t)dt\\
&\leq a_1\int_0^{\frac{S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}{2}}\phi(t)dt\\
&+ a_2\int_0^{\frac{S_b(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S_b(x_{2k},x_{2k},x_{2k+1})}{1+S_b(x_{2k},x_{2k},u)+S_b(y_{2k},y_{2k},y_{2k+1})}}\phi(t)dt\\
&+ a_3\int_0^{\frac{S_b(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S_b(y_{2k},y_{2k},y_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}}\phi(t)dt\\
&+ a_4\int_0^{\frac{S_b(x_{2k},x_{2k},f(x_{2k},y_{2k}))S_b(x_{2k},x_{2k},x_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}}\phi(t)dt\\
&+ a_5\int_0^{\frac{S_b(x_{2k},x_{2k},f(x_{2k},y_{2k}))S_b(y_{2k},y_{2k},y_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}}\phi(t)dt
\end{aligned}$$

$$
\begin{aligned}
&+ a_6 \int_0^{\frac{S_b(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S_b(x_{2k},x_{2k},x_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_7 \int_0^{\frac{S_b(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S_b(y_{2k},y_{2k},y_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_8 \int_0^{max\{S_b(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1})),S_b(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))\}} \phi(t)dt \\
&+ a_9 \int_0^{min\{S_b(x_{2k},x_{2k},x_{2k+1}),S_b(y_{2k},y_{2k},y_{2k+1})\}} \phi(t)dt \\
=& a_1 \int_0^{\frac{S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}{2}} \phi(t)dt + a_2 \int_0^{\frac{S_b(x_{2k+1},x_{2k+1},x_{2k+2})S_b(x_{2k},x_{2k},x_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_3 \int_0^{\frac{S_b(x_{2k+1},x_{2k+1},x_{2k+2})S_b(y_{2k},y_{2k},y_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_4 \int_0^{\frac{S_b(x_{2k},x_{2k},x_{2k+1})S_b(x_{2k},x_{2k},x_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_5 \int_0^{\frac{S_b(x_{2k},x_{2k},x_{2k+1})S_b(y_{2k},y_{2k},y_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_6 \int_0^{\frac{S_b(x_{2k+1},x_{2k+1},x_{2k+2})S_b(x_{2k},x_{2k},x_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_7 \int_0^{\frac{S_b(x_{2k+1},x_{2k+1},x_{2k+2})S_b(y_{2k},y_{2k},y_{2k+1})}{1+S_b(x_{2k},x_{2k},x_{2k+1})+S_b(y_{2k},y_{2k},y_{2k+1})}} \phi(t)dt \\
&+ a_8 \int_0^{max\{S_b(x_{2k+1},x_{2k+1},x_{2k+2}),S_b(x_{2k+1},x_{2k+1},x_{2k+2})\}} \phi(t)dt \\
&+ a_9 \int_0^{min\{S_b(x_{2k},x_{2k},x_{2k+1}),S_b(y_{2k},y_{2k},y_{2k+1})\}} \phi(t)dt
\end{aligned}
$$

So,

$$
\begin{aligned}
\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})} \phi(t)dt \leq& \left(\frac{a_1}{2} + a_4 + a_5\right) \int_0^{S_b(x_{2k},x_{2k},x_{2k+1})} \phi(t)dt \\
&+ \frac{a_1}{2} \int_0^{S_b(y_{2k},y_{2k},y_{2k+1})} \phi(t)dt \\
&+ (a_2 + a_3 + a_6 + a_7 + a_8) \int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})} \phi(t)dt \\
&+ a_9 \int_0^{min\{S_b(x_{2k},x_{2k},x_{2k+1}),S_b(y_{2k},y_{2k},y_{2k+1})\}} \phi(t)dt \qquad (3)
\end{aligned}
$$

Case 1. If $\min\{S_b(x_{2k},x_{2k},x_{2k+1}),S_b(y_{2k},y_{2k},y_{2k+1})\}=S_b(x_{2k},x_{2k},x_{2k+1})$. From equation (2), we get

$$(1-a_2-a_3-a_6-a_7-a_8)\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt$$

$$\leq\left(\frac{a_1}{2}+a_4+a_5+a_9\right)\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt+\frac{a_1}{2}\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt$$

and therefore,

$$\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt\leq\frac{\frac{a_1}{2}+a_4+a_5+a_9}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt+\frac{\frac{a_1}{2}}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt. \tag{4}$$

Proceeding similarly one can prove that

$$\int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt\leq\frac{\frac{a_1}{2}+a_4+a_5+a_9}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt+\frac{\frac{a_1}{2}}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt. \tag{5}$$

Adding equation (3) and equation (4) we have

$$\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt+\int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt$$

$$\leq\frac{a_1+a_4+a_5+a_9}{1-a_2-a_3-a_6-a_7-a_8}\left[\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt+\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt\right].$$

Therefore,

$$\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt+\int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt$$

$$\leq h\left[\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt+\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt\right],$$

where h= $\frac{a_1+a_4+a_5+a_9}{1-a_2-a_3-a_6-a_7-a_8}<1$.

Case 2. If $\min\{S_b(x_{2k},x_{2k},x_{2k+1}),S_b(y_{2k},y_{2k},y_{2k+1})\}=S_b(y_{2k},y_{2k},y_{2k+1})$. From equation (2), we get

$$(1-a_2-a_3-a_6-a_7-a_8)\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt$$

$$\leq\left(\frac{a_1}{2}+a_4+a_5\right)\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt+\left(\frac{a_1}{2}+a_9\right)\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt$$

and therefore,

$$\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt \leq \frac{\frac{a_1}{2}+a_4+a_5}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt + \frac{\frac{a_1}{2}+a_9}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt. \tag{6}$$

Proceeding similarly one can prove that

$$\int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt \leq \frac{\frac{a_1}{2}+a_4+a_5}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt + \frac{\frac{a_1}{2}+a_9}{1-a_2-a_3-a_6-a_7-a_8}\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt. \tag{7}$$

Adding equation (5) and equation (6) we have

$$\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt + \int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt$$
$$\leq \frac{a_1+a_4+a_5+a_9}{1-a_2-a_3-a_6-a_7-a_8}\left[\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt + \int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt\right].$$

Therefore,

$$\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt + \int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt$$
$$\leq h\left[\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt + \int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt\right]$$

where h= $\frac{a_1+a_4+a_5+a_9}{1-a_2-a_3-a_6-a_7-a_8} < 1$. Also, we can show that

$$\int_0^{S_b(x_{2k+2},x_{2k+2},x_{2k+3})}\phi(t)dt + \int_0^{S_b(y_{2k+2},y_{2k+2},y_{2k+3})}\phi(t)dt$$
$$\leq h\left[\int_0^{S_b(x_{2k+1},x_{2k+1},x_{2k+2})}\phi(t)dt + \int_0^{S_b(y_{2k+1},y_{2k+1},y_{2k+2})}\phi(t)dt\right]$$
$$\leq h^2\left[\int_0^{S_b(x_{2k},x_{2k},x_{2k+1})}\phi(t)dt + \int_0^{S_b(y_{2k},y_{2k},y_{2k+1})}\phi(t)dt\right].$$

Continuing this way, we have

$$
\begin{aligned}
&\int_0^{S_b(x_n,x_n,x_{n+1})}\phi(t)dt+\int_0^{S_b(y_n,y_n,y_{n+1})}\phi(t)dt\\
\leq& h\left[\int_0^{S_b(x_{n-1},x_{n-1},x_n)}\phi(t)dt+\int_0^{S_b(y_{n-1},y_{n-1},y_n)}\phi(t)dt\right]\\
\leq& h^2\left[\int_0^{S_b(x_{n-2},x_{n-2},x_{n-1})}\phi(t)dt+\int_0^{S_b(y_{n-2},y_{n-2},y_{n-1})}\phi(t)dt\right]\\
&\vdots\\
\leq& h^n\left[\int_0^{S_b(x_0,x_0,x_1)}\phi(t)dt+\int_0^{S_b(y_0,y_0,y_1)}\phi(t)dt\right].
\end{aligned}
$$

If $S_b(x_n,x_n,x_{n+1})+S_b(y_n,y_n,y_{n+1})=S_{bn}$, then

$$S_{bn}\leq hS_{bn-1}\leq h^2S_{bn-2}\leq ...\leq h^nS_{b0}.$$

So for $m>n$,

$$
\begin{aligned}
&S_b(x_n,x_n,x_m)+S_b(y_n,y_n,y_m)\leq s\left[2S_b(x_n,x_n,x_{n+1})+S_b(x_{n+1},x_{n+1},x_m)\right]\\
&\quad+2\left[S_b(y_n,y_n,y_{n+1})+S_b(y_{n+1},y_{n+1},y_m)\right]\\
=&2s\left[S_b(x_n,x_n,x_{n+1})+S_b(y_n,y_n,y_{n+1})\right]+s\left[S_b(x_{n+1},x_{n+1},x_m)+S_b(y_{n+1},y_{n+1},y_m)\right]\\
\leq&2s\left[S_b(x_n,x_n,x_{n+1})+S_b(y_n,y_n,y_{n+1})\right]\\
&\quad+2s^2\left[S_b(x_{n+1},x_{n+1},x_{n+2})+S_b(y_{n+1},y_{n+1},y_{n+2})\right]\\
&\quad+s^2\left[S_b(x_{n+2},x_{n+2},x_m)+S_b(y_{n+2},y_{n+2},y_m)\right]\\
\leq&2s\left[S_b(x_n,x_n,x_{n+1})+S_b(y_n,y_n,y_{n+1})\right]\\
&\quad+2s^2\left[S_b(x_{n+1},x_{n+1},x_{n+2})+S_b(y_{n+1},y_{n+1},y_{n+2})\right]\\
&\quad+2s^3\left[S_b(x_{n+2},x_{n+2},x_{n+3})+S_b(y_{n+2},y_{n+2},y_{n+3})\right]+\cdots+\\
&\quad+2s^{m-n-1}\left[S_b(x_{m-2},x_{m-2},x_{m-1})+S_b(y_{m-2},y_{m-2},y_{m-1})\right]\\
&\quad+2s^{m-n}\left[S_b(x_{m-1},x_{m-1},x_m)+S_b(y_{m-1},y_{m-1},y_m)\right]\\
\leq&2\left(sh^n+s^2h^{n+1}+s^3h^{n+2}+---+s^{m-n}h^{m-1}\right)S_{b0}\\
<&2sh^n\left[1+sh+(sh)^2+...\right]S_{b0}\\
=&\frac{2sh^n}{1-sh}S_{b0}
\end{aligned}
$$

Therefore, we have

$$\int_0^{S_b(x_n,x_n,x_m)+S_b(y_n,y_n,y_m)}\phi(t)dt\to 0\quad as\quad n\to\infty. \tag{8}$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose that they are not. Then there exits an $\epsilon > 0$ and subsequence $\{x_{m(p)}\}$ and $\{x_{n(p)}\}$ such that $m(p) < n(p) < m(p+1)$ with

$$S(y_{n(p)}, y_{n(p)}, y_{m(p)}) \geq \epsilon \quad , S(y_{n(p)-1}, y_{n(p)-1}, y_{m(p)}) < \epsilon \tag{9}$$

Now,

$$\begin{aligned} S(y_{m(p)-1}, y_{m(p-1)}, y_{n(p-1)}) &\leq S(y_{m(p)-1}, y_{m(p-1)}, y_{m(p)}) + S(y_{m(p)-1}, y_{m(p-1)}, y_{m(p)}) \\ &\quad + S(y_{n(p)-1}, y_{n(p-1)}, y_{m(p)}) \\ &< 2S(y_{m(p)-1}, y_{m(p-1)}, y_{m(p)}) + \epsilon. \end{aligned} \tag{10}$$

From equations (7) and (9), we get

$$\lim_{p\to\infty} \int_0^{S(y_{m(p)-1}, y_{m(p-1)}, y_{n(p-1)})} \phi(t)dt \leq \int_0^{\epsilon} \phi(t)dt. \tag{11}$$

Using equations (7), (9) and (10), we get

$$\begin{aligned} \int_0^{\epsilon} \phi(t)dt &\leq \int_0^{S(y_{n(p)}, y_{n(p)}, y_{m(p)})} \phi(t)dt \\ &\leq k\int_0^{S(y_{n(p)-1}, y_{n(p)-1}, y_{m(p)-1})} \phi(t)dt \\ &\leq k\int_0^{\epsilon} \phi(t)dt. \end{aligned}$$

Which is contradiction. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. As X is a complete S_b-metric space, so there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now, we will prove that $x = f(x, y)$ and $y = f(y, x)$. On the contrary suppose that $x \neq f(x, y)$ or $y \neq f(x, y)$. Then $S_b(x, x, f(x, y)) = l_1 > 0$ or $S_b(y, y, f(y, x)) =$

$l_2 > 0$. Using inequality (1),

$$\begin{aligned}
l_1 =& S_b(x,x,f(x,y)) \\
\leq& s\left[2S_b(x,x,x_{n+1}) + S_b(x_{n+1},x_{n+1},f(x,y))\right] \\
=& s\left[2S_b(x,x,x_{n+1}) + S_b(f(x_n,y_n),f(x_n,y_n),f(x,y))\right] \\
\leq& 2S_b(x,x,x_{n+1}) + s\left[a_1\int_0^{\frac{S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}{2}}\phi(t)dt\right] \\
&+ s\left[a_2\int_0^{\frac{S_b(f(x_n,y_n),f(x_n,y_n),g(x,y))S_b(x_n,x_n,x)}{1+S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}}\phi(t)dt\right] \\
&+ s\left[a_3\int_0^{\frac{S_b(f(x_n,y_n),f(x_n,y_n),g(x,y))S_b(y_n,y_n,y)}{1+S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}}\phi(t)dt\right] \\
&+ s\left[a_4\int_0^{\frac{S_b(x_n,x_n,f(x_n,y_n))S_b(x_n,x_n,x)}{1+S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}}\phi(t)dt\right] + s\left[a_5\int_0^{\frac{S_b(x_n,x_n,f(x_n,y_n))S_b(y_n,y_n,y)}{1+S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}}\phi(t)dt\right] \\
&+ s\left[a_6\int_0^{\frac{S_b(x,x,g(x,y))S_b(x_n,x_n,x)}{1+S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}}\phi(t)dt\right] + s\left[a_7\int_0^{\frac{S_b(x,x,g(x,y))S_b(y_n,y_n,y)}{1+S_b(x_n,x_n,x)+S_b(y_n,y_n,y)}}\phi(t)dt\right] \\
&+ s\left[a_8\int_0^{max\{S_b(f(x_n,y_n),f(x_n,y_n),g(x,y)),S_b(x,x,g(x,y))\}}\phi(t)dt\right] \\
&+ s\left[a_9\int_0^{min\{S_b(x_n,x_n,x),S_b(y_n,y_n,y)\}}\phi(t)dt\right]
\end{aligned}$$

Since x_n and y_n are convergent to x and y, by taking limit as $n \to \infty$, we get $l_1 \leq 0$, which is a contradiction. So, $S_b(x,x,f(x,y)) = 0$ which gives $x = f(x,y)$.

Similarly, we can prove that $y = f(y,x)$. Also, we can prove that $x = g(x,y)$ and $y = g(y,x)$. Hence, (x,y) is a common coupled fixed point of f and g. In order to prove the uniqueness of the coupled fixed point, if possible let (p,q) be the second common coupled fixed point of f and g. Then by using inequality (1), we have

$$\begin{aligned}
\int_0^{S_b(x,x,p)}\phi(t)dt =& \int_0^{S_b(f(x,y),f(x,y),g(p,q))}\phi(t)dt \\
\leq& a_1\int_0^{\frac{S_b(x,x,p)+S_b(y,y,q)}{2}}\phi(t)dt + a_2\int_0^{\frac{S_b(f(x,y),f(x,y),g(p,q))S_b(x,x,p)}{1+S_b(x,x,p)+S_b(y,y,q)}}\phi(t)dt \\
&+ a_3\int_0^{\frac{S_b(f(x,y),f(x,y),g(p,q))S_b(y,y,q)}{1+S_b(x,x,p)+S_b(y,y,q)}}\phi(t)dt + a_4\int_0^{\frac{S_b(x,x,f(x,y))S_b(x,x,p)}{1+S_b(x,x,p)+S_b(y,y,q)}}\phi(t)dt \\
&+ a_5\int_0^{\frac{S_b(x,x,f(x,y))S_b(y,y,q)}{1+S_b(x,x,p)+S_b(y,y,q)}}\phi(t)dt + a_6\int_0^{\frac{S_b(p,p,g(p,q))S_b(x,x,p)}{1+S_b(x,x,p)+S_b(y,y,q)}}\phi(t)dt
\end{aligned}$$

$$+ \quad a_7 \int_0^{\frac{S_b(p,p,g(p,q))S_b(y,y,q)}{1+S_b(x,x,p)+S_b(y,y,q)}} \phi(t)dt + a_8 \int_0^{max\{S_b(f(x,y),f(x,y),g(p,q)),S_b(p,p,g(p,q))\}} \phi(t)dt$$
$$+ \quad a_9 \int_0^{min\{S_b(x,x,p),S_b(y,y,q)\}} \phi(t)dt.$$

Accordingly,

$$\int_0^{S_b(x,x,p)} \phi(t)dt \leq a_1 \int_0^{\frac{S_b(x,x,p)+S_b(y,y,q)}{2}} \phi(t)dt + a_2 \int_0^{S_b(x,x,p)} \phi(t)dt + a_3 \int_0^{S_b(x,x,p)} \phi(t)dt + a_8 \int_0^{S_b(x,x,p)} \phi(t)dt + a_9 \int_0^{min\{S_b(x,x,p),S_b(y,y,q)\}} \phi(t)dt. \tag{12}$$

Case 1. If $\min\{S_b(x,x,p), S_b(y,y,q)\} = S_b(x,x,p)$. From equation (11),

$$\int_0^{S_b(x,x,p)} \phi(t)dt \leq \left(\frac{a_1}{2} + a_2 + a_3 + a_8 + a_9\right) \int_0^{S_b(x,x,p)} \phi(t)dt + \frac{a_1}{2} \int_0^{S_b(y,y,q)} \phi(t)dt$$

which implies that

$$\int_0^{S_b(x,x,p)} \phi(t)dt \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3 - 2a_8 - 2a_9} \int_0^{S_b(y,y,q)} \phi(t)dt.$$

Case 2. If $\min\{S_b(x,x,p), S_b(y,y,q)\} = S_b(y,y,q)$. From equation (11),

$$\int_0^{S_b(x,x,p)} \phi(t)dt \leq \left(\frac{a_1}{2} + a_2 + a_3 + a_8\right) \int_0^{S_b(x,x,p)} \phi(t)dt + \left(\frac{a_1}{2} + a_9\right) \int_0^{S_b(y,y,q)} \phi(t)dt$$

which implies

$$\int_0^{S_b(x,x,p)} \phi(t)dt \leq \frac{a_1 + 2a_9}{2 - a_1 - 2a_2 - 2a_3 - 2a_8} \int_0^{S_b(y,y,q)} \phi(t)dt.$$

From both the cases, finally, we get

$$\int_0^{S_b(x,x,p)} \phi(t)dt \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3 - 2a_8 - 2a_9} \int_0^{S_b(y,y,q)} \phi(t)dt. \tag{13}$$

Similarly,

$$\int_0^{S_b(y,y,q)} \phi(t)dt \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3 - 2a_8 - 2a_9} \int_0^{S_b(x,x,p)} \phi(t)dt. \tag{14}$$

Adding equation (12) and equation (13), we have

$$\int_0^{S_b(x,x,p)+S_b(y,y,q)} \phi(t)dt \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3 - 2a_8 - 2a_9} \int_0^{S_b(x,x,p)+S_b(y,y,q)} \phi(t)dt$$

and we get

$$\frac{2\,(1 - a_1 - a_2 - a_3 - a_8 - a_9)}{2 - a_1 - 2a_2 - 2a_3 - 2a_8 - 2a_9} \int_0^{[S_b(x,x,p)+S_b(y,y,q)]} \phi(t)dt \leq 0.$$

Since $\sum_{i=1}^{9} a_i < 1$, and $\frac{2(1-a_1-a_2-a_3-a_8-a_9)}{2-a_1-2a_2-2a_3-2a_8-2a_9} > 0$, we have that $S_b(x,x,p) + S_b(y,y,q) = 0$, which implies that $x = p$ and $y = q$, i.e., $(x,y) = (p,q)$. Thus f and g have unique coupled common fixed point. This completes the proof.

□

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