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Characterization of semi-continuity in L^p-spaces

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Abstract. Upper and lower semi-continuous functions are important in many areas and play a key role in optimization theory. This paper characterizes the lower and upper semi-continuity of L^p-space functions. We prove that a function $\vartheta : \mathcal{L} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous if and only if each convergent Moore-Smith sequence $\{q_j\}_{j\in\mathbb{N}}$ converging to $q \in \mathcal{L}$ implies that $\int_{\mathcal{L}} \vartheta(q)d\mu \leq \liminf \int_{\mathcal{L}} \vartheta(q_j)d\mu, \forall q \in \mathcal{L}$. We further show that the sum of any two proper lower semi-continuous functions is lower semi-continuous and the product of a lower semi-continuous function by a positive scalar gives a lower semi-continuous function and the case of upper semi-continuous functions follows analogously. Additionally, we prove that for a function in an L^p-space L if $\vartheta(\varphi) = \int_{\mathcal{L}} \varphi d\mu$ such that φ is measurable with respect to a Borel measure μ, then ϑ is upper semi-continuous.

1. Introduction

Lower semi-continuous functions and upper semi-continuous functions play a crucial role in mathematical analysis and are significantly applied in optimization and other fields of science [3]. For this reason they have been intensively studied over time in many mathematical spaces like topological, Hilbert and general Banach spaces. Beer [1], characterized upper semi-continuous functions in compact metric spaces and Hausdorff spaces. In [17] Dini's uniform convergence theorem was extended to characterize sequences of upper semi-continuous functions converging point-wise to a continuous function that converges uniformly. Gool [6] examined lower semi-continuous functions in continuous lattices. The results obtained in Gool's work extended some analytical properties of lsc functions to lsc functions

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whose domain is a continuous lattice[18]. Some results [7] were applied in potential theory to describe solutions of systems of differential equations through generalizing semi-continuity of such functions. Gool's work characterized lsc functions in compact topological spaces but not in L^p-spaces [9].

Correa and Hantoute [4] related the argmin sets of a given function and its semi-continuous convex hull using characterizations involving asymptotic functions. They came up with explicit formulas for Fenchel's sub-differential and the argmin sets of successful Legendre-Fenchel conjugates of functions with real values. The authors thus characterized lsc functions in real locally convex space but not in L^p-spaces. Varagona [16] examined inverse limits with upper semi-continuous bonding functions and decomposability providing sufficient and necessary conditions for the bonding functions to be a decomposable/ indecomposable continuum. Chen, Cho and Yang [2], investigated lower semi-continuous functions in real normed spaces and real reflexive Banach spaces. They introduced the notion of lsc from above. They used this concept to prove that Ekland's and Coristi's theorems hold under semi-continuous functions and convex functions in general Banach spaces ([10], [11] and the references therein).

Mirmostafaee [12] studied usc and lsc functions of multi-valued functions in Baire spaces. His work on characterization of lsc and usc functions in Baire spaces was extended to metrizable spaces [13] and second countable spaces [14] but Mirmostafaee did not consider examining upper and lower semi-continuity in L^p-spaces [15]. Hernandez and Lopez [7] characterized semi-continuous functions in metrizable topological spaces. They showed that a function in a metrizable topological space is lsc if its sublevel set is closed and it is usc if its sublevel set is open. The graphical properties of semi-continuous functions were also studied where it was proved that if the hypograph of a function is closed then the function is usc and similarly if the epigraph of a function is closed then the function is lsc.

However limited research has been conducted in characterizing lower and upper semi-continuity of functions in L^p-spaces. This has been attributed to the challenge posed by intricate nature of p-norm structures and their infinite dimensions. The L^p-spaces are interesting spaces since in these spaces we can measure the changes in lower and upper semi-continuous functions using p-norms.

It is in the interest of this paper to characterize lower semi continuous functions and upper semi-continuous functions in L^p-spaces.

2. Preliminaries

In this section properties of lower and upper semi-continuity which are used in later discussion are stated and key concepts are defined. We start by defining a special type of Banach space referred to as the L^p-space.

Definition 2.1. [17] Let $(\mathcal{L}, \mathfrak{X}, \mu)$ be a measure space and a number p be given such that $1 \leq p < \infty$. Then an L^p space which consists of measurable functions is defined as $L^p(\mathcal{L}, \mathfrak{X}, \mu) = \{\vartheta : \mathcal{L} \to \mathbb{K} : \vartheta \text{is measurable}, \int_{\mathcal{L}} |\vartheta|^p d\mu < \infty\}$. The L^p-norm of $\vartheta \in L^p(\mathcal{L})$ is given by

$$\|\vartheta\|_p = \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}}.$$

In the proposition below we show that $(\mathcal{L}, \|.\|_p)$ is indeed an L^p-space.

Proposition 2.1. *Let $L^p(\mathcal{L}, \mathfrak{X}, \mu)$ denote an L^p-space on a measure space $(\mathcal{L}, \mathfrak{X}, \mu)$ with $1 \leq p < \infty$. Let $\|.\|_p$ be an L^p norm. Then $(\mathcal{L}, \|.\|_p)$ is an $L^p(\mathcal{L}, \mathfrak{X}, \mu)$-space on $(\mathcal{L}, \mathfrak{X}, \mu)$.*

Proof. We need to show that $\|.\|_p$ is a norm on $L^p(\mathcal{L}, \mathfrak{X}, \mu)$. Let ϑ be a measurable function in $\mathcal{L}$. From the definition of L^p norm, we have, $\|\vartheta\|_p = \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}}$. We proceed to show that $\|\vartheta\|_p$ satisfies the three axioms for a norm below:

(i). $\|\vartheta\|_p \geq 0$ and $\|\vartheta\|_p = 0 \iff \vartheta = 0$. By absoluteness property, $|\vartheta|^p \geq 0$ implying that $\int_{\mathcal{L}} |\vartheta|^p d\mu \geq 0$, hence $\left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} \geq 0$ showing that $\|\vartheta\|_p \geq 0$. Let $\|\vartheta\|_p = 0$, then $\left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} = 0$ implying that $|\vartheta|^p = 0$ hence $\vartheta = 0$. Conversely suppose $|\vartheta|^p = 0$, then $\int_{\mathcal{L}} |\vartheta|^p d\mu = 0$ implying that $\left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} = 0$. Thus, $\|\vartheta\|_p = 0$.

(ii). $\|\kappa\vartheta\|_p = |\kappa|\|\vartheta\|_p, \forall \kappa \in \mathbb{K}$. Now $\|\kappa\vartheta\|_p = \left(\int_{\mathcal{L}} |\kappa\vartheta|^p d\mu\right)^{\frac{1}{p}} = \left(\int_{\mathcal{L}} |\kappa|^p |\vartheta|^p d\mu\right)^{\frac{1}{p}} = |\kappa| \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}}$. Thus, $\|\kappa\vartheta\|_p = |\kappa|\|\vartheta\|_p$.

(iii). $\|\vartheta + \varphi\|_p \leq \|\vartheta\|_p + \|\varphi\|_p, \forall \vartheta, \varphi \in \mathcal{L}$. Let $\vartheta, \varphi \in \mathcal{L}$, then $\|\vartheta + \varphi\|_p = \left(\int_{\mathcal{L}} |\vartheta + \varphi|^p d\mu\right)^{\frac{1}{p}}$. So by Minkowski's inequality we have $\left(\int_{\mathcal{L}} |\vartheta + \varphi|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_{\mathcal{L}} |\vartheta|^p d\mu\right)^{\frac{1}{p}} + \left(\int_{\mathcal{L}} \varphi|^p d\mu\right)^{\frac{1}{p}}, \forall p \geq 1$. This is equivalent to $\|\vartheta + \varphi\|_p \leq \|\vartheta\|_p + \|\varphi\|_p$.

Since all axioms for a norm are satisfied we conclude that $\|.\|_p$ is a norm on $L^p(\mathcal{L}, \mathfrak{X}, \mu)$ and therefore the ordered pair $(\mathcal{L}, \|.\|_p)$ is an $L^p(\mathcal{L}, \mathfrak{X}, \mu)$-space on $(\mathcal{L}, \mathfrak{X}, \mu)$.

□

We now proceed to define the notion of semi-continuity of functions and show its implication in continuity of functions in normed spaces.

Definition 2.2. [5] Let L be a nonempty normed space. A function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ is said to be semi-continuous if it is either lower semi-continuous or upper semi-continuous.

Definition 2.3. [14] Let $\mathcal{L}$ be a normed linear space. A function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ is *lsc* if, given a sequence $\{q_n\} \in \mathcal{L}$, $\vartheta(q) \leq \liminf_{n\to\infty} \vartheta(q_n)$ as n approaches infinity and q_n approaches $q \in \mathcal{L}$.

Definition 2.4. [18] Let $\mathcal{L}$ be a normed linear space. A function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ is *usc* if, for any sequence $\{q_n\} \in \mathcal{L}$, $\vartheta(q) \geq \limsup_{n\to\infty} \vartheta(q_n)$ as n approaches infinity and q_n approaches $q \in \mathcal{L}$.

Definition 2.5. [3] A function that is both lower semi-continuous and upper semi-continuous is said to be a continuous function.

The following results represent some properties of semi-continuous functions in an L^p-space.

Proposition 2.2. *Let ϑ be a function from a normed space $\mathcal{Q}$ to the extended line $\overline{\mathbb{R}}$. If the function ϑ is convex then the following are equivalent:*

(i). *ϑ is weakly-lsc.*
(ii). *ϑ is weakly sequentially-lsc.*
(iii). *ϑ is sequentially-lsc.*
(iv). *ϑ is lsc.*

Theorem 2.3. *Let $\{\vartheta_j\}_{j=1}^{n}$ be a sequence of continuous functions from an L^p-space to the extended real line $\overline{\mathbb{R}}$ such that $\vartheta_1 q \leq \vartheta_2 q ... \leq \vartheta_n q \leq ... \; \forall q \in \mathcal{L}$. Then $\{\vartheta_j\}_{j=1}^{n}$ converges uniformly to a function $\vartheta \in \mathcal{L}$.*

Proposition 2.4. *For a function $\vartheta : \mathcal{L} \to \overline{\mathbb{R}}$ the following statements are equivalent:*

(i). *ϑ is lower semi-continuous.*
(ii). *$\overline{epi(\vartheta)} \in \mathcal{L}$;*

3. Main Results

Now we give the main results of this study. We begin with the following proposition in which lower semi-continuity is characterized using convex conjugates and bi-conjugates. We show that if a conjugate is convex then it is weak*-lower semi-continuous ($w^* - lsc$) and a convex bi-conjugate is weak lower semi-continuous.

Proposition 3.1. *Suppose ϑ is any function in an L^p-space $\mathcal{L}$. Then ϑ^*, ϑ^{**} are convex functions, and*

(i). *ϑ^* is $w^* - lsc$.*
(ii). *ϑ^{**} is $w - lsc$.*
(iii). *$\vartheta^{**} \leq \vartheta$*

Furthermore, if ϑ_1, ϑ_2 are convex functions satisfying $\vartheta_1 \leq \vartheta_2$, then $\vartheta_1^* \geq \vartheta_2^*$.

PROOF. Let the convex function $\varrho \in \mathcal{L}^* : \varrho \to \langle q, \varrho \rangle, \forall q \in \text{dom}(\vartheta)$ be *weak** continuous function. Then, $\varrho \to \langle q, \varrho \rangle$ is $w^* - lsc$ for all finite or infinite function $\vartheta(q)$. Define ϑ^* as $\vartheta^* = \sup\{q : q \in lsc(\mathcal{L})\}$. Since the supremum of any collection of $lsc(\mathcal{L})$ is a lsc function, then ϑ^* supremum is a convex and $weak^* - lsc$ function. Now, the function $q \mapsto \langle q, \varrho \rangle$ is convex and weakly lsc. Define ϑ^{**} as $\vartheta^{**} = \sup\{q : q \mapsto \langle q, \varrho \rangle\} \forall \varrho \in \mathcal{L}^*$. Then ϑ^{**} is convex and weakly lsc. For each $q \in \text{dom}(\vartheta)$, $\langle q, \varrho \rangle - \vartheta^*(\varrho) \leq \vartheta(q)$. Hence $\forall q \in \text{dom}(\vartheta), \vartheta^{**}(q) = \sup\{(\langle q, \varrho \rangle - \vartheta^*(\varrho))\} \leq \vartheta(q)$ this implies that $\vartheta^{**} \leq \vartheta$. Let $\vartheta_1 \leq \vartheta_2$, then, $\vartheta_2^*(\varrho) = \sup\{(\langle q, \varrho \rangle - \vartheta_2(q))\} \leq \sup\{(\langle q, \varrho \rangle - \vartheta_1(q))\} = \vartheta_1^*(\varrho)$ and this clearly shows that $\vartheta_1^* \geq \vartheta_2^*$. □

The next result presents a characterization of lower semi-continuous functions in terms of Moore-Smith sequences.

Lemma 3.2. *Let $\mathcal{L}$ be an L^p-space and $lsc(\mathcal{L})$ denote a collection of all lsc functions. Let ϑ be a measurable function and $\{(q_j)\}_{j \in \mathbb{N}} \in dom(\vartheta)$ be a Moore-Smith sequence. Then $\vartheta \in lsc(\mathcal{L})$, if and only if, $\int_{\mathcal{L}} \vartheta(q) \leq \liminf \int_{\mathcal{L}} \vartheta(q_j), \forall q \in dom(\vartheta)$ whenever $q_j \to q$.*

PROOF. Let $\vartheta \in lsc(\mathcal{L})$. Assume $\{(q_j)_{j \in \mathbb{N}}\}$ is a Moore-Smith sequence converging to $q \in dom(\vartheta)$. If $r < \vartheta(q), \forall r \in \overline{\mathbb{R}}$, then $\vartheta^{-1}(r, \infty)$ is open since $\vartheta \in lsc(\mathcal{L})$. Now, $\forall q \in \vartheta^{-1}(r, \infty)$ and $q_j \to q$, $\exists j_r \leq j$ satisfying $q_j \in \vartheta^{-1}(r, \infty)$. So given $j \geq j_r$ we have $\vartheta(q_j) > r$ implying that $\liminf \vartheta(q_j) \geq r$. Now $\forall r < \vartheta(q)$ we obtain $\liminf \vartheta(q_j) \geq \vartheta(q)$ (by Banach-Steinhaus Theorem). Hence, equivalently $\vartheta(q) \leq \liminf \vartheta(q_j)$. Given that ϑ is measurable over $\mathcal{L}$ and $q_j \to q$, then by Fatou's Lemma we have $\int_{\mathcal{L}} \vartheta(q) \leq \liminf \int_{\mathcal{L}} \vartheta(q_j)$.

Conversely, for each Moore-Smith sequence $q_j \to q$ let $\int_{\mathcal{L}} \vartheta(q) \leq \liminf \int_{\mathcal{L}} \vartheta(q_j)$. Then $\vartheta(q) \leq \liminf \vartheta(q_j)$. Let $\Omega = \vartheta^{-1}(-\infty, r], \forall r \in \overline{\mathbb{R}}$. If $q \in \overline{\Omega}$ and given $q_j \to q$, then $\{q_j\} \in \Omega$. Since $\Omega = \vartheta^{-1}(-\infty, r]$, we deduce that $\vartheta(q_j) \leq r$ for each j and so $\vartheta(q) \leq r$. Hence $q \in \Omega$, implying that Ω is closed and so the complement of Ω ($\Omega^c = \vartheta^{-1}(r, \infty)$) is open, showing that ϑ is lower semi-continuous. □

The following result considers proper functions that are bounded below. It affirms that the sum of any two lower semi-continuous functions whose values do not take to $-\infty$ is lower semi-continuous and so is the product of these functions with a positive scalar.

Theorem 3.3. *Let $\mathcal{L}$ be an L^p space. Suppose two functions ϑ and ϱ are lower semi-continuous in $\mathcal{L}$ such that $\vartheta > -\infty, \varrho > -\infty$, then their sum $\vartheta + \varrho$ is lower semi-continuous in $\mathcal{L}$ and furthermore $\forall \lambda > 0$, $\lambda\vartheta$ is lower semi-continuous in $\mathcal{L}$).*

PROOF. Let $\{(q_j)_{j\in\mathbb{N}}\} \in \mathcal{Q}$ be a Moore-Smith sequence converging to $q \in \mathcal{Q}$, then by Lemma 3.2 we have,

$$\begin{aligned}\int_{\mathcal{L}}(\varrho+\vartheta)(q) &\leq \liminf\int_{\mathcal{L}}\varrho(q_j)+\liminf\int_{\mathcal{L}}\vartheta(q_j)\\ &\leq \liminf\int_{\mathcal{L}}(\varrho(q_j)+\vartheta(q_j))\\ &= \liminf\int_{\mathcal{L}}(\varrho+\vartheta)(q_j)\end{aligned}$$

Thus $\int_{\mathcal{L}}(\varrho+\vartheta)(q) \leq \liminf\int_{\mathcal{L}}(\varrho+\vartheta)(q_j)$ showing that $\varrho+\vartheta$ is lower semi-continuous. It also follows that,

$$\begin{aligned}\int_{\mathcal{L}}(\lambda\vartheta)(q) &= \int_{\mathcal{L}}\lambda\vartheta(q)\\ &\leq \lambda\liminf\int_{\mathcal{L}}\vartheta(q_j)\\ &= \liminf\lambda\int_{\mathcal{L}}\vartheta(q_j)\\ &= \liminf\int_{\mathcal{L}}(\lambda\vartheta)(q_j)\end{aligned}$$

Hence, $\lambda\vartheta$ is lower semi-continuous. □

Theorem 3.4. *For an L^p space L, if the sequence of lsc ϑ_n is finite $\forall\theta : L \to \overline{\mathbb{R}}$, and if θ_n converges uniformly in L to ϑ, then ϑ is lower semi-continuous.*

PROOF. Given $\xi > 0$, an integer $N \leq n$ exists such that $|\vartheta_n(q) - \vartheta(q)| < \xi$, $\forall q \in L$. Define η by $\eta = \sup|\vartheta_n(q) - \vartheta(q)| : q \in L$. Now, $\forall\xi > 0$, $\eta \leq \xi$ whenever $n \geq N$. If $N \leq n$ and given a Moore-Smith sequence $\vartheta(q_j)_{j\in N}$ converging in L, then

$$\begin{aligned}\vartheta(q) &\leq \vartheta_n(q)\\ &= \eta+\liminf\vartheta_n(q_j)\\ &= 2\eta+\liminf\vartheta(q_j)\\ &= 2\xi+\liminf\vartheta(q_j)\end{aligned}$$

This holds for all $\xi > 0$. Thus, we get $\vartheta(q) \leq \liminf\theta(q_j)$. It therefore follows that ϑ is lower semi-continuous.

□

Corollary 3.5. *Let $\mathcal{L}$ be L^p-space. Denote a collection of continuous functions taking $\mathcal{L} \to [0,1]$ by $C(\vartheta)$. If a function ϑ is lower semi-continuous, then each $\vartheta > 0$, is given by $\vartheta = sup\{\varphi : \varphi is\ continuous, \forall\varphi \leq \vartheta\}$*

Proof. Suppose $\varphi \in C(\vartheta) : \varphi \leq \vartheta$. Since $C(\vartheta)$ is nonempty let $\beta = \vartheta^{-1}(-\infty, \vartheta(q)-\xi], \forall q \in \mathcal{L}, \xi > 0$. Now, $\mathcal{L} \setminus \beta = \beta^c = \vartheta^{-1}(\vartheta(q)-\xi, \infty)$ is open, so β is closed. Therefore, $q \notin \beta$. Hence, $\exists \rho : \mathcal{L} \to [0,1]$ a continuous function satisfying $\rho(L) = 0$ and $\rho(q) = 1$. Assuming $\vartheta(q) - \xi \leq 0$ then, given $\rho \geq 0$ and $\vartheta \geq 0$, $(\vartheta(q) - \xi)\varphi \leq \varphi$. If we let $\vartheta(q) - \xi > 0$ and $w \in \beta$, then $(\vartheta(q) - \xi)\varphi(w) \leq \varphi(w)$ when $w \notin \beta$ and $(\vartheta(q) - \xi)\varphi(w) = 0$ when $w \in \beta$. Thus $(\vartheta(q) - \xi)\varphi \leq \vartheta$ and since $(\vartheta(q) - \varphi)\varphi$ is continuous, then $(\vartheta(q) - \varphi)\varphi \in C(\vartheta)$. Thus, we obtain

$$M(C(\vartheta))(q)) \geq (\vartheta(q)-\xi)\varphi(q) = \vartheta(q)-\xi where M(C(\vartheta) = sup\{\varphi : \varphi \text{is continuous}, \forall \varphi \leq \vartheta\}.$$

Since ϑ was arbitrary we therefore have $M(C(\vartheta))(q) \geq \vartheta(q)$ and since q was also arbitrary we have $M(C(\vartheta)) \leq \vartheta$. Therefore we have $\vartheta = M(C(\vartheta))$ showing that $\vartheta = sup\{\varphi : \varphi \text{is continuous}, \forall \varphi \leq \vartheta\}$

□

Proposition 3.6. *Let ϑ be a function from an L^p-space $\mathcal{L}$ to the extended real line $\overline{\mathbb{R}}$. Let B be a compact subset of $\overline{\mathbb{R}}$. If ϑ is a lower semi-continuous function, then $\vartheta(q) \geq \vartheta(\bar{q})$ for all $q \in B$.*

Proof. By way of contradiction, suppose that ϑ has no lower bound. Then, $\exists q_r \in B, \forall r \in \mathbb{N}$ satisfying $\vartheta(q_r) < -r$. Compactness of B implies that a subsequence q_{r_k} of q_r exists which converges to $q_0 \in B$. Lower semi-continuity of ϑ means that ϑ is lower semi-continuous at every point $q_0 \in B$ in the convergent sequence q_r tending to q. Thus, by Lemma 3.2, $\liminf_{k\to\infty} \vartheta(q_{r_k}) \geq \vartheta(q_r)$. This shows a contradiction since $\liminf_{k\to\infty} \vartheta(q_{r_k}) = -\infty$. Hence, ϑ is bounded below.

Suppose $F = \inf \vartheta(q) : q \in B$. Then, $F \in \mathbb{R}$ because $\vartheta(q)$ is not empty and is bounded below.

Let the sequence $b_r \in B$ be such that $\vartheta(b_r)$ converges to F. Then, since B is compact, there is b_{r_k} a subsequence of b_r with limit $\bar{q} \in B$. Now, $F = \lim_{k\to\infty} \vartheta(b_{r_k}) = \liminf_{k\to\infty} \vartheta(b_{r_k}) \geq \vartheta(\bar{q}) \geq F$. This shows that $F = \vartheta(\bar{q})$. Therefore, $\vartheta(q) \geq \vartheta(\bar{q})$, $\forall q \in B$. □

In the next theorem we have characterized a *lsc* function in terms of its epigraph.

Theorem 3.7. *For an L^p-space L, epi(ϑ) $\subseteq L \times \mathbb{R}$ is closed if and only if ϑ is lower semi-continuous.*

Proof. Let ϑ be lower semi-continuous and $\forall q \in L$, $\forall k \in \mathbb{R}$ assume $(q_i, k_i) \in \text{epi}(\vartheta)$ converges to $(q, k) \in L \times \mathbb{R}$. Then $q_i \to q$ and $k_i \to k$. By Proposition 2.1, we have $\vartheta(q) \leq \liminf \vartheta(q_i) \leq \liminf k_i \leq \lim k_i = k$. Thus, $\vartheta(q) \leq k$ implies that $(q, k) \in \text{epi}(\vartheta)$. Then, the set $(L \times k) \cap \text{epi}(\vartheta) = (q, k) : k \geq \vartheta(q) \subseteq L \times \mathbb{R}$ is closed. This implies that $\vartheta^{-1}(-\infty, k)$ as a subset of L is closed, but $\vartheta^{-1}(k, \infty), \forall q \in L$, is open. Since this holds true for each $k \in \mathbb{R}$, thus ϑ is lower semi-continuous. □

The next result shows that weakly lower semi-continuity implies lower semi-continuity.

Corollary 3.8. *Suppose $\vartheta : L \to \overline{\mathbb{R}}$ is a convex function in an L^p-space L. Then, ϑ is lower semi-continuous if ϑ is weakly lower semi-continuous.*

PROOF. It is a fact that $\mathbb{R}$ is locally convex, so the product $L \times \mathbb{R}$ is also locally convex. Therefore, $L_\vartheta \times \mathbb{R}$ forms a weak topology in L. Hence, weak closure in $L \times \mathbb{R}$ implies closure in $L_\vartheta \times \mathbb{R}$. A convex subset of $L \times \mathbb{R}$ is always closed in $L_\vartheta \times \mathbb{R}$ as well as the whole domain of $L \times \mathbb{R}$. Thus, epi(ϑ) is closed, implying that $\vartheta \subseteq L_\vartheta \times \mathbb{R}$ is weakly semi-continuous. Since epi(ϑ) $\subseteq L \times \mathbb{R}$ is closed, then ϑ is lower semi-continuous. □

The next theorem proves that if a function of a measurable function ϑ is equal to the Lebesgue integral of φ with respect to the Borel measure μ, then the function is upper semi-continuous.

Theorem 3.9. *Let L be an L^p-space. Let $\vartheta \in L$ be a function, and suppose the function φ is Lebesgue integrable over L with respect to the Borel measure μ. If $\vartheta(\varphi) = \int_L \varphi, d\mu$ such that φ is measurable, then ϑ is an upper semi-continuous function.*

PROOF. Assume φ belongs to a set of continuous upper semi-continuous functions, and the upper semi-continuous sequence φ_n converges to φ. Suppose the upper semi-continuous sequence g_j decreases to $g_j = \varphi^+_{1/j}$. Then $\lim_{j\to\infty} \vartheta(g_j) = \vartheta(\varphi)$. Since φ is bounded above and $\mu(L) < \infty$, then $\vartheta(\varphi)$ is finite. Now, for each $\xi > 0$, suppose j satisfies $\vartheta(g_j) < \vartheta(\varphi) + \xi$. Then $|\varphi_n - \varphi| \leq 1/j$ since $\varphi_n \to \varphi$. Therefore, $\varphi_n \leq g_j$, implying that $\vartheta(\varphi_n) \leq \vartheta(g_j) \leq \vartheta(\varphi) + \xi$. □

The corollaries below follow from Theorem 3.9. We show that upper semi-continuous property of functions is preserved under uniform continuity of any sequence of upper semi-continuous functions.

Corollary 3.10. *Let $\vartheta \in L$ be an integral induced by a Borel measure μ. Suppose $\varphi_n \in usc(L)$ converges to a measurable function $\varphi \in usc(L)$. Define $\vartheta_r(\varphi_n) = \vartheta[(\varphi_n)1/r^+]$ such that $\vartheta_r : \varphi_n : n \in \mathbb{Z}^+ \to \mathbb{R}$, $\forall r \in \mathbb{Z}^+$. Then, $\lim n \to \infty\vartheta(\varphi_n) = \vartheta(\varphi)$ if and only if φ_r converges uniformly to ϑ on φ_n.*

PROOF. Let $F = \varphi \cup \varphi_n$. Since $\lim_{n\to\infty} \vartheta(\varphi_n) = \vartheta(\varphi)$, ϑ is continuous on F and φ is its unique limit point. Suppose $\vartheta_r(\varphi) = \vartheta[(\varphi)1/r^+]$. Then, ϑ is upper semi-continuous and forms a decreasing sequence. Therefore, for all $a \in F$, $\vartheta_1(a) \geq \vartheta_2(a) \geq \cdots \geq \vartheta(a)$, hence $\lim r \to \infty\vartheta_r(a) = \vartheta(a)$. So, by Dini's theorem, it follows that ϑ_r is uniformly convergent on φ_n.

On the converse, let $\lim_{r\to\infty} \vartheta_r(a) \neq \vartheta(a)$. Since ϑ is upper semi-continuous at φ, we know that $|\varphi_n - \varphi| \leq 1/n$ and $\vartheta(\varphi_n) < \vartheta(\varphi) - \xi$, $\forall \xi > 0$. Now, $(\varphi_n)1/n^+ > \varphi$ implies that $\vartheta_n(\varphi_n) \geq \vartheta(\varphi) > \vartheta(\varphi_n) + \xi$. Thus, ϑ_r cannot converge uniformly to ϑ on φ_n. Hence, $\lim r \to \infty\vartheta_r(a) = \vartheta(a)$. □

Corollary 3.11. *Let ϑ be an upper semi-continuous function. Then, for any positive integer δ, the function ϑ_δ^+ is upper semi-continuous and bounded in an L^p-space L.*

PROOF. To show that ϑ_δ^+ is bounded below, assume ϑ_δ^+ is not bounded below. Then, there exists $(q_n, h_n) \in \text{hypo}(\vartheta_\delta^+)^c$ such that $h_n < -n$ for all n. The distance between (q_n, h_n) and each point of $\text{hypo}(\vartheta) \geq \delta$. We assume that $q_n \to q$. Thus, (q_n, ϑ_n) is arbitrarily close to the half line (q, ϑ): $\vartheta(q) \geq q$, as $n \to \infty$. Hence, $(q, \vartheta) : q \in \vartheta(q)$ is a subset of the hypo graph of ϑ, which is a contradiction to our assumption. Therefore, ϑ_δ^+ is bounded below.

To show that ϑ_δ^+ is upper semi-continuous, we apply Theorem 2.7. Since ϑ_δ^+ is closed and bounded, we have $\beta_\delta \cdot \text{hypo}(\vartheta) = \text{hypo}(\vartheta_\delta^+)$ because parallel bodies of closed sets are closed. Therefore, for every $\delta > 0$, ϑ_δ^+ is upper semi-continuous. □

4. Conclusion

We have characterized lower semi-continuity in L^p-spaces using Moore-Smith sequences and shown that a function $\vartheta : L \to \overline{\mathbb{R}}$ is lower semi-continuous if, for any Moore-Smith sequence $q_{j\,j\in\mathbb{N}}$ in the domain of ϑ that converges to an L^p-space function ϑ, implies that if $q \in \text{dom}(\vartheta)$, then $\vartheta(q)$ is less than or equal to the limit inferior of $\vartheta(q_j)$. We have further proved that ϑ, defined as $\vartheta(\varphi) = \int_L \varphi, d\mu$ for a continuous function φ, is an upper semi-continuous function.

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