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# Boundedness of some operators on variable exponent Fofana's spaces and their preduals

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Abstract. Let $1 \leq p \leq \alpha \leq q \leq \infty$. The Fofana's spaces $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ were introduced in 1988 by Fofana on the basis of Wiener amalgam spaces and their predual spaces $\mathcal{H}(p', q', \alpha')(\mathbb{R}^d)$ have been described by Feichtinger and Feuto in 2019. Recently, in 2023, Yang and Zhou generalized these spaces by replacing the constant exponent $p$ with the variable exponent $p(\cdot)$ and defining so the variable exponent Fofana's spaces $\left(L^{p(\cdot)}, \ell^q\right)^\alpha(\mathbb{R}^d)$ and their preduals $\mathcal{H}(p'(\cdot), q', \alpha')(\mathbb{R}^d)$. The purpose of this paper is to investigate the boundedness of classical operators such as Riesz potentials operators, maximal operators, Calderon-Zygmund operators and some generalized sublinear operators in both $\left(L^{p(\cdot)}, \ell^q\right)^\alpha(\mathbb{R}^d)$ and $\mathcal{H}(p'(\cdot), q', \alpha')(\mathbb{R}^d)$. In order to do this, we prove some properties of these spaces. Our results extend and/or improve those of classical Fofana's spaces and their preduals.

## 1. Introduction

The aim of this paper is to investigate the boundedness of some classical operators on Fofana's spaces with variable exponent and their preduals.

A variable exponent on $\mathbb{R}^d$ is a measurable function $p$ from $\mathbb{R}^d$ to $[1, \infty)$. In order to distinguish between variable and constant exponents, we shall always denote variable exponent by $p(\cdot)$. In the last decades, function spaces with variable exponents have been intensely studied. Examples of such spaces include the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ which first appeared in literature in 1931 with an article written by Orlicz, but the major study of these spaces was initiated by Kovacik and Rakosnik [24] in 1991. Let us notice that, variable exponent Lebesgue spaces have

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many properties in common with classical Lebesgue spaces (see [8, 9, 14] and the references therein). In 2012, Aydin and Gürkanli [4] extended the theory of variable exponent Lebesgue spaces via introducing amalgam spaces with variable exponent $\left(L^{p(\cdot)}, \ell^q\right)(\mathbb{R}^d)$. We refer to [2, 3, 4, 5, 21, 25, 28] and the references therein for a historical background of variable exponent amalgam spaces.

We present now a motivation and the importance for the study of Fofana's spaces. For $1 \leq p, q \leq \infty$, the amalgam space $(L^p, \ell^q)(\mathbb{R}^d)$ is defined as the set of complex values functions $f$ on $\mathbb{R}^d$, which are locally in the classical Lebesgue space $L^p(\mathbb{R}^d)$ and such that the sequence $\{\|f\chi_{I_k}\|_p\}_{k\in\mathbb{Z}^d}$ belongs to $\ell^q(\mathbb{Z}^d)$, where $\|\cdot\|_p$ is the usual Lebesgue norm on $L^p(\mathbb{R}^d)$, $I_k = k+[0,1)^d$ and $\chi_{I_k}$ denotes its characteristic function. Amalgam spaces have been introduced by N. Wiener [31] since 1926, however their systematic study began with the work of F. Holland [22] in 1975. Since then, they have been widely studied (see [20] and the references therein). It is well know that the classical Lebesgue space $L^p(\mathbb{R}^d)$ coincides with the amalgam space $(L^p, \ell^p)(\mathbb{R}^d)$ while it is a proper subspace of $(L^p, \ell^q)(\mathbb{R}^d)$ when $p < q$. There are some useful properties of Lebesgue spaces which are not fulfilled in amalgam spaces. For instance, for $0 < \rho, \alpha < \infty$, the dilation operator $St_\rho^{(\alpha)} : f \mapsto \rho^{-\frac{d}{\alpha}} f(\rho^{-1}\cdot)$ is not isometric in proper amalgam spaces, contrary to Lebesgue spaces. In order to compensate for this shortfall, in 1988, I. Fofana [17] introduced the functions spaces $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ $(1 \leq p \leq \alpha \leq q \leq \infty)$ which consist of functions $f \in (L^p, \ell^q)(\mathbb{R}^d)$ satisfying $\sup\limits_{\rho>0} \left\| \left\{ \left\| \left(St_\rho^{(\alpha)} f\right) \chi_{I_k} \right\|_p \right\}_{k\in\mathbb{Z}^d} \right\|_{\ell^q} < \infty$ and named them "integrable fractional mean function spaces". Nowadays, there is an increasing interest in the study of these spaces and they are called Fofana's spaces by many authors. In 2019, Feichtinger and Feuto [15] introduced the predual spaces $\mathcal{H}(p', q', \alpha')(\mathbb{R}^d)$ of these spaces, where for $1 \leq s \leq \infty$, $s'$ denotes the conjugate exponent of $s$, that is $1/s' = 1 - 1/s$. Let us point out that, Fofana's spaces form a chain of Banach spaces beginning with the classical Lebesgue space $L^\alpha(\mathbb{R}^d) = (L^p, \ell^\alpha)^\alpha(\mathbb{R}^d)$ and ending by the classical Morrey space $\mathcal{M}_p^\alpha(\mathbb{R}^d) = (L^p, \ell^\infty)^\alpha(\mathbb{R}^d)$ (see [18] for more precision). Actually, we recall that the concept of Morrey spaces was introduced in 1938 by C. Morrey [29] in order to study regularity problems arising in calculus of variations. Many results in Fourier analysis, well-known and widely used in Lebesgue, Morrey or amalgam spaces, have been obtained in the framework of Fofana's spaces (see [7, 10, 11, 13, 15, 16, 19] and the references therein).

Recently, in 2023, Yang and Zhou [33] introduced the variable exponent Fofana's spaces $(L^{p(\cdot)}, \ell^q)^\alpha(\mathbb{R}^d)$ and described their predual spaces $\mathcal{H}(p'(\cdot), q', \alpha')(\mathbb{R}^d)$. In fact, they showed some basic properties of these spaces and proved the boundedness of Riesz potentials operators and their commutators on $(L^{p(\cdot)}, \ell^q)^\alpha(\mathbb{R}^d)$.

The main purpose of this paper is to establish the boundedness of classical operators such as Riesz potentials operators, maximal operators, Calderón-Zygmund operators and also some generalized sublinear operators on both $(L^{p(\cdot)}, \ell^q)^\alpha(\mathbb{R}^d)$ and

$\mathcal{H}(p'(\cdot), q', \alpha')(\mathbb{R}^d)$. In order to do this, we first prove some complementary properties of these spaces. We note that some of our results extend and/or improve those of the classical Fofana's spaces $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ and their preduals $\mathcal{H}(p', q', \alpha')(\mathbb{R}^d)$, while others are new results.

The paper contains seven sections. In Section 2 we summarize basic results on functions spaces with variable exponent. Section 3 is devoted to prove some complementary results on $(L^{p(\cdot)}, \ell^q)^\alpha(\mathbb{R}^d)$ and $\mathcal{H}(p'(\cdot), q', \alpha')(\mathbb{R}^d)$. Sections 4, 5, 6 and 7 deal with the boundedness of Riesz potential operators, maximal operators, Calderón-Zygmund operators and some generalized sublinear operators, respectively.

Throughout the remainder of this paper we shall use the following notations.

Let $p(\cdot)$ be a variable exponent on $\mathbb{R}^d$.

• The conjuguate exponent $p'(\cdot)$ of $p(\cdot)$ is defined by the formula $\frac{1}{p'(\cdot)} = 1 - \frac{1}{p(\cdot)}$ with the convention $\frac{1}{\infty} = 0$.

• $\mathcal{P}(\mathbb{R}^d)$ denotes the set of all variable exponents $p(\cdot)$ on $\mathbb{R}^d$ such that $1 < p_- \leq p(\cdot) \leq p_+ < \infty$, where

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} p(x) \ \ , \ \ p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} p(x).$$

• $\mathcal{P}^{\log}(\mathbb{R}^d)$ is the set of all elements $p(\cdot)$ of $\mathcal{P}(\mathbb{R}^d)$ satisfying the following two conditions :

$(i)$ $p(\cdot)$ is locally log-Hölder continuous : there exists a constant $C_0$ such that :

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x-y|)} \ , \ \ x, y \in \mathbb{R}^d \ , \ \ |x-y| \leq \frac{1}{2} \tag{1}$$

$(ii)$ $p(\cdot)$ is log-Hölder continuous at infinity : there exists a constant $C_\infty$ such that :

$$|p(x) - p(y)| \leq \frac{C_\infty}{\log(e+|x|)} \ , \ \ x, y \in \mathbb{R}^d \ , \ \ |y| > |x|. \tag{2}$$

The letter $C$ is used for non-negative constants independent of the relevant variables that may change from one occurrence to another. We propose the following abbreviation $A \lesssim B$ for the inequalities $A \leq CB$, where $C$ is a positive constant independent of the main parameters. $|E|$ and $\chi_E$ stand for the Lebesgue measure and the characteristic function, respectively, of any subset $E$ of $\mathbb{R}^d$.

## 2. Basic facts about variable exponent spaces

We recall some fundamental definitions and properties of some functions spaces with variable exponent.

**2.1. Variable exponent Lebesgue spaces.** Let $p(\cdot)$ be a variable exponent on $\mathbb{R}^d$. The variable exponent Lebesgue spaces $L^{p(\cdot)} := L^{p(\cdot)}(\mathbb{R}^d)$ consist of all measurable functions $f$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^d} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

The space $L^{p(\cdot)}$ becomes a Banach space when equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^d} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

If $p(\cdot) = p$ is a constant function then $\|\cdot\|_{p(\cdot)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$ and so $L^{p(\cdot)}$ is equal to $L^p$.

The spaces $L^{p(\cdot)}_{\mathrm{loc}} := L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^d)$ are defined by

$$L^{p(\cdot)}_{\mathrm{loc}} = \left\{f : f\chi_K \in L^{p(\cdot)},\ \text{for all compact subset } K \text{ of } \mathbb{R}^d\right\}.$$

Variable exponent Lebesgue spaces have many properties in common with the classical Lebesgue spaces. For instance, we have the following results.

**Proposition 2.1.** *([8]) Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $E$ a subset of $\mathbb{R}^d$ such that $|E| < \infty$.*
*1) If $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$ then $fg \in L^1$ and*

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq (1 + 1/p_- - 1/p_+)\, \|f\|_{p(\cdot)}\, \|g\|_{p'(\cdot)}. \tag{3}$$

*2) If $p(\cdot) \leq q(\cdot)$ then $L^{q(\cdot)}(E) \subseteq L^{p(\cdot)}(E)$ and*

$$\|f\|_{p(\cdot)} \leq (1 + |E|)\, \|f\|_{q(\cdot)}, \qquad f \in L^{q(\cdot)}(E). \tag{4}$$

**Proposition 2.2.** [23] *Assume that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$ and $B$ is a ball in $\mathbb{R}^d$. Then there exists a constant $C > 0$ such that*

$$\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \leq C|B|. \tag{5}$$

**2.2. Variable exponent amalgam spaces.** Let $p(\cdot)$ be a variable exponent on $\mathbb{R}^d$, $f \in L^{p(\cdot)}_{\mathrm{loc}}$ and $1 \leq q \leq \infty$. we set

$$\|f\|_{p(\cdot),q} = \left\|\left\{\|f\chi_{Q_k}\|_{p(\cdot)}\right\}_{k\in\mathbb{Z}^d}\right\|_{\ell^q} = \begin{cases} \left(\displaystyle\sum_{k\in\mathbb{Z}^d} \|f\chi_{Q_k}\|^q_{p(\cdot)}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \displaystyle\sup_{k\in\mathbb{Z}^d} \|f\chi_{Q_k}\|_{p(\cdot)} & \text{if } q = \infty, \end{cases} \tag{6}$$

where

$$Q_k = \prod_{j=1}^{d} [k_j, k_j + 1)\ , \quad k = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d.$$

**Definition 2.1.** Let $p(\cdot)$ be a variable exponent on $\mathbb{R}^d$ and $1 \leq q \leq \infty$. The variable exponent amalgam spaces $\left(L^{p(\cdot)}, \ell^q\right)$ are defined by

$$(L^{p(\cdot)}, \ell^q) := (L^{p(\cdot)}, \ell^q)(\mathbb{R}^d) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}} \ / \ \|f\|_{p(\cdot),q} < \infty \right\}.$$

The following results are well known.

**Proposition 2.3.** *([5, 28, 33]) Let $p(\cdot)$ be a variable exponent on $\mathbb{R}^d$ and $1 \leq q \leq \infty$.*
*1) $\left(L^{p(\cdot)}, \ell^q\right)$ is a linear subspace of $L^{p(\cdot)}_{\mathrm{loc}}$ and, when endowed with $\|.\|_{p(\cdot),q}$, a Banach space.*
*2) If $f \in \left(L^{p(\cdot)}, \ell^q\right)$ and $g \in \left(L^{p'(\cdot)}, \ell^{q'}\right)$ then $fg \in L^1$ and*

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, dx \lesssim \|f\|_{p(\cdot),q} \, \|g\|_{p'(\cdot),q'}. \tag{7}$$

*3) If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $1 \leq q < \infty$ then the topological dual of $\left(L^{p(\cdot)}, \ell^q\right)$ is $\left(L^{p'(\cdot)}, \ell^{q'}\right)$.*
*4) If $1 \leq p(\cdot), q < \infty$ then the set $\mathcal{C}_{\mathrm{c}} := \mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$ of all continuous and compactly supported functions on $\mathbb{R}^d$ is a dense subspace of $\left(L^{p(\cdot)}, \ell^q\right)$.*

**2.3. Variable exponent Fofana's spaces and their preduals.** Let $0 < \alpha < \infty$ and $0 < \rho < \infty$. For any measurable function $f$, the dilation operator $St^{(\alpha)}_\rho$ is defined by

$$St^{(\alpha)}_\rho f = \rho^{-\frac{d}{\alpha}} f\left(\rho^{-1}.\right). \tag{8}$$

It is easy to see that $\left\{St^{(\alpha)}_\rho \ : \ 0 < \rho < \infty\right\}$ is a commutative group of operators on $L^{p(\cdot)}_{\mathrm{loc}}$, isomorphic to the multiplicative group $(0, \infty)$; that is

$$\begin{cases} \text{for any real number } \rho > 0,\ St^{(\alpha)}_\rho \text{ maps } L^{p(\cdot)}_{\mathrm{loc}} \text{ into itself} \\ St^{(\alpha)}_1 f = f, \quad f \in L^{p(\cdot)}_{\mathrm{loc}} \\ St^{(\alpha)}_{\rho_1} \circ St^{(\alpha)}_{\rho_2} = St^{(\alpha)}_{\rho_1\rho_2} = St^{(\alpha)}_{\rho_2} \circ St^{(\alpha)}_{\rho_1}, \quad 0 < \rho_1, \rho_2 < \infty. \end{cases}$$

**Definition 2.2.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $1 \leq \alpha, q \leq \infty$. The variable exponent Fofana's spaces $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ are defined by

$$\left(L^{p(\cdot)}, \ell^q\right)^\alpha := \left(L^{p(\cdot)}, \ell^q\right)^\alpha(\mathbb{R}^d) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}} \ / \ \|f\|_{p(\cdot),q,\alpha} < \infty \right\}$$

where

$$||f||_{p(\cdot),q,\alpha} = \sup_{\rho > 0} \left\| St^{(\alpha)}_\rho f \right\|_{p(\cdot),q}. \tag{9}$$

These spaces have been introduced by Yang and Zhou [33] initially by means of "continuous" norms $\widetilde{||\cdot||}_{p(\cdot),q,\alpha}$ (equivalent to $||\cdot||_{p(\cdot),q,\alpha}$) defined as follows

$$\widetilde{||f||}_{p(\cdot),q,\alpha} = \sup_{r>0} \left\| |B(\cdot,r)|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \left\| f\chi_{B(\cdot,r)} \right\|_{p(\cdot)} \right\|_q , \tag{10}$$

where $B(\cdot,r)$ denotes a ball in $\mathbb{R}^d$ with radius $r$. Let us recall some basic properties of these spaces.

**Proposition 2.4.** [33] *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $1 \le \alpha, q \le \infty$.*
*1) $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ are linear subspaces of $L^{p(\cdot)}_{\text{loc}}$ and, when endowed with $\|.\|_{p(\cdot),q,\alpha}$ , Banach spaces.*
*2) $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ are non trivial if and only if $p(\cdot) \le \alpha \le q$.*
*3) If $p(\cdot) \le \alpha \le q$ then $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ are continuously included in $\left(L^{p(\cdot)}, \ell^q\right)$.*
*4) If $p(\cdot) = p$ then $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ is just the classical Fofana's space $(L^p, \ell^q)^\alpha$.*

**Definition 2.3.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p(\cdot) \le \alpha \le q \le \infty$.
1) A sequence $\{(c_n, \rho_n, f_n)\}_{n\ge1}$ of elements of $\mathbb{C} \times (0,\infty) \times (L^{p'(\cdot)}, \ell^{q'})$ is called a $\mathfrak{h}$-decomposition of an element $f$ of $L^{p(\cdot)}_{\text{loc}}$ if

$$\begin{cases} \|f_n\|_{p'(\cdot),q'} \le 1 , \quad n \ge 1 \\ \\ \sum_{n\ge1} |c_n| < \infty \\ f = \sum_{n\ge1} c_n St^{(\alpha')}_{\rho_n} f_n \quad \text{in the sense of } L^{p(\cdot)}_{\text{loc}} . \end{cases}$$

2) The space $\mathcal{H}(p'(\cdot), q', \alpha') = \mathcal{H}(p'(\cdot), q', \alpha')(\mathbb{R}^d)$ is defined as the set of all elements of $L^{p(\cdot)}_{\text{loc}}$ whose set of $\mathfrak{h}$-decompositions is nonvoid; in other words

$$\mathcal{H}(p'(\cdot), q', \alpha') = \{f \in L^{p(\cdot)}_{\text{loc}} \ : \ \|f\|_{\mathcal{H}(p'(\cdot),q',\alpha')} < \infty\}$$

with

$$\|f\|_{\mathcal{H}(p'(\cdot),q',\alpha')} = \inf \left\{ \sum_{n\ge1} |c_n| \ : \ f = \sum_{n\ge1} c_n St^{(\alpha')}_{\rho_n} f_n \right\}, \tag{11}$$

where the infimum is taken over all $\mathfrak{h}$-decompositions of $f$ with the convention $\inf \varnothing = \infty$.

The following results have been obtained in [33] by Yang and Zhou.

**Proposition 2.5.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p(\cdot) \le \alpha \le q \le \infty$.*
*1) $\left(L^{p'(\cdot)}, \ell^{q'}\right)$ is a dense subspace of $\mathcal{H}(p'(\cdot), q', \alpha')$.*
*2) If $f \in \left(L^{p(\cdot)}, \ell^q\right)^\alpha$ and $g \in \mathcal{H}(p'(\cdot), q', \alpha')$ then $fg \in L^1$ and*

$$\left| \int_{\mathbb{R}^d} f(x)g(x)\,dx \right| \le \|f\|_{p(\cdot),q,\alpha} \, \|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}. \tag{12}$$

*3) A predual of* $\left(L^{p(\cdot)},\ell^q\right)^\alpha$ *is the space* $\mathcal{H}(p'(\cdot),q',\alpha')$ *in the following sense :*

*The operator* $T : f \mapsto T_f$*, defined by*

$$T_f(g) = \int_{\mathbb{R}^d} f(x)g(x)\,dx\ ,\quad f\in\left(L^{p(\cdot)},\ell^q\right)^\alpha\ ,\ g\in\mathcal{H}(p'(\cdot),q',\alpha'),\tag{13}$$

*is an isometric isomorphism of* $\left(L^{p(\cdot)},\ell^q\right)^\alpha$ *into the topological dual* $\mathcal{H}(p'(\cdot),q',\alpha')^*$ *of* $\mathcal{H}(p'(\cdot),q',\alpha')$*.*

## 3. Complementary properties on $\left(L^{p(\cdot)},\ell^q\right)^\alpha$ and $\mathcal{H}\left(p'(\cdot),q',\alpha'\right)$

In this section we show some properties of the spaces $\left(L^{p(\cdot)},\ell^q\right)^\alpha$ and $\mathcal{H}(p'(\cdot),q',\alpha')$. Let us start with the following remark.

**Remark 3.1.** Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ and $p(\cdot)\leq\alpha\leq q\leq\infty$. It readily follows from Point 3) of Proposition 2.5 that

$$\|f\|_{p(\cdot),q,\alpha} = \sup\left\{\int_{\mathbb{R}^d}|f(x)g(x)|\,dx\ ,\ g\in\mathcal{H}(p'(\cdot),q',\alpha'),\ \|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\leq 1\right\}\tag{14}$$

and therefore the Hahn-Banach theorem leads to

$$\|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} = \sup\left\{\int_{\mathbb{R}^d}|g(x)f(x)|\,dx\ ,\ f\in\left(L^{p(\cdot)},\ell^q\right)^\alpha,\ \|f\|_{p(\cdot),q,\alpha}\leq 1\right\}.\tag{15}$$

It is well known that the variable exponent Lebesgue spaces $L^{p(\cdot)}$ are solid spaces; that is : if $g$ is a measurable function and $f\in L^{p(\cdot)}$ such that $|g|\leq|f|$ almost everywhere then $g$ belongs to $L^{p(\cdot)}$ and $\|g\|_{p(\cdot)}\leq\|f\|_{p(\cdot)}$. An analogous result holds true for variable exponents amalgam spaces (see [4, Proposition 2.2]). The following proposition shows that this result extends to the setting of both variable exponent Fofana's spaces $\left(L^{p(\cdot)},\ell^q\right)^\alpha$ and their preduals $\mathcal{H}(p'(\cdot),q',\alpha')$.

**Proposition 3.1.** *Let* $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ *and* $p(\cdot)\leq\alpha\leq q\leq\infty$*. Then* $\left(L^{p(\cdot)},\ell^q\right)^\alpha$ *and* $\mathcal{H}(p'(\cdot),q',\alpha')$ *are solid spaces.*

Proof. Let $f$ and $g$ be two elements of $L^{p(\cdot)}_{\rm loc}$ such that $|g|\leq|f|$ almost everywhere.

1) Let us suppose that $f\in\left(L^{p(\cdot)},\ell^q\right)^\alpha$. It is easy to see that, for any $\rho>0$, $\left|St_\rho^{(\alpha)}g\right|\leq\left|St_\rho^{(\alpha)}f\right|$. Since $\left(L^{p(\cdot)},\ell^q\right)$ is a solid space, we have

$$\left\|St_\rho^{(\alpha)}g\right\|_{p(\cdot),q}\leq\left\|St_\rho^{(\alpha)}f\right\|_{p(\cdot),q}\ ,\qquad 0<\rho<\infty$$

and therefore, taking the supremum over all $\rho>0$, we get

$$\|g\|_{p(\cdot),q,\alpha}\leq\|f\|_{p(\cdot),q,\alpha}<\infty.$$

Hence $g$ belongs to $\left(L^{p(\cdot)},\ell^q\right)^\alpha$.

2) Let us suppose that $f \in \mathcal{H}(p'(\cdot), q', \alpha')$ and $\{(c_n, \rho_n, f_n)\}_{n\geq 1}$ is a $\mathfrak{h}$-decomposition of $f$; that is : $f = \sum_{n\geq 1} c_n St_{\rho_n}^{(\alpha')} f_n$ with $\|f_n\|_{p'(\cdot),q'} \leq 1$ for any integer $n \geq 1$.

We set, for any integer $n \geq 1$,

$$g_n(x) = \begin{cases} 0 & \text{if } f(\rho_n x) = 0 \\ \\ \frac{g(\rho_n x)}{f(\rho_n x)} f_n(x) & \text{otherwise.} \end{cases}$$

By hypothesis, $|g| \leq |f|$. This implies that, for any integer $n \geq 1$, $|g_n| \leq |f_n|$ and therefore, since $\left(L^{p'(\cdot)}, \ell^{q'}\right)$ is a solid space, we have

$$\|g_n\|_{p'(\cdot),q'} \leq \|f_n\|_{p'(\cdot),q'} \leq 1\ , \quad n \geq 1. \qquad (*)$$

We also have

$$\sum_{n\geq 1} c_n St_{\rho_n}^{(\alpha')} g_n = g. \qquad (**)$$

Let us prove $(**)$. Suppose that $x$ is an element of $\mathbb{R}^d$.

• $1^{st}$ case : $f(x) = 0$.

We have $0 \leq |g(x)| \leq |f(x)| = 0$ and so $g(x) = 0$. Furthermore, for any $n \geq 1$,

$$f(x) = 0 \Longrightarrow f\left(\rho_n(\rho_n^{-1}x)\right) = 0 \Longrightarrow g_n(\rho_n^{-1}x) = 0.$$

Therefore

$$\sum_{n\geq 1} c_n St_{\rho_n}^{(\alpha')} g_n(x) = \sum_{n\geq 1} c_n\, \rho_n^{-\frac{d}{\alpha'}} g_n(\rho_n^{-1}x) = 0.$$

Hence

$$\sum_{n\geq 1} c_n St_{\rho_n}^{(\alpha')} g_n(x) = 0 = g(x).$$

• $2^{nd}$ case : $f(x) \neq 0$. We have

$$\begin{aligned}\sum_{n\geq 1} c_n \left[St_{\rho_n}^{(\alpha')} g_n\right](x) &= \sum_{n\geq 1} c_n \left[St_{\rho_n}^{(\alpha')}\left(\frac{g(\rho_n\,\cdot)}{f(\rho_n\,\cdot)}\, f_n\right)\right](x)\\ &= \sum_{n\geq 1} c_n\ \rho_n^{-\frac{d}{\alpha'}}\ \frac{g(\rho_n\rho_n^{-1}x)}{f(\rho_n\,\rho_n^{-1}x)}\, f_n(\rho_n^{-1}x)\\ &= \sum_{n\geq 1} c_n\ \rho_n^{-\frac{d}{\alpha'}}\ \frac{g(x)}{f(x)}\, f_n(\rho_n^{-1}x)\\ &= \frac{g(x)}{f(x)} \sum_{n\geq 1} c_n\ \rho_n^{-\frac{d}{\alpha'}}\ f_n(\rho_n^{-1}x)\\ &= \frac{g(x)}{f(x)} \sum_{n\geq 1} c_n\ St_{\rho_n}^{(\alpha')} f_n(x)\\ &= \frac{g(x)}{f(x)}\, f(x)\\ &= g(x).\end{aligned}$$

Hence $(**)$ holds. Thus, $(*)$ and $(**)$ imply that $\{(c_n, \rho_n, g_n)\}_{n\geq 1}$ is a $\mathfrak{h}$-decomposition of $g$ and so

$$\|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq \sum_{n\geq 1} |c_n|.$$

Taking the infimum with respect to all $\mathfrak{h}$-decompositions of $f$, we get

$$\|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq \inf\left\{\sum_{n\geq 1} |c_n|\right\} = \|f\|_{\mathcal{H}(p'(\cdot),q',\alpha')} < \infty.$$

Hence $g$ belongs to $\mathcal{H}(p'(\cdot), q', \alpha')$. □

From Proposition 3.1 and [6, Theorem 1.7] we deduce what follows.

**Corollary 3.2.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p(\cdot) \leq \alpha \leq q \leq \infty$. Then we have*

$$\|\,|f|\,\|_{p(\cdot),q,\alpha} = \|f\|_{p(\cdot),q,\alpha}\ , \qquad f \in \left(L^{p(\cdot)}, \ell^q\right)^\alpha \tag{16}$$

*and*

$$\|\,|g|\,\|_{\mathcal{H}(p'(\cdot),q',\alpha')} = \|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\ , \qquad g \in \mathcal{H}(p'(\cdot), q', \alpha'). \tag{17}$$

Let us show the following embedding results.

**Proposition 3.3.** *Let $p(\cdot),\ p_1(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $1 \leq \alpha, q, q_1 \leq \infty$.*
*1) If $p_1(\cdot) \leq p(\cdot) \leq \alpha \leq q \leq \infty$ then*

$$\|f\|_{p_1(\cdot),q,\alpha} \leq 2\,\|f\|_{p(\cdot),q,\alpha}\ , \qquad f \in L^{p(\cdot)}_{\rm loc} \tag{18}$$

*and therefore $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ is continuously embedded in $\left(L^{p_1(\cdot)}, \ell^q\right)^\alpha$.*

*2) If $p(\cdot) \leq \alpha \leq q \leq q_1 \leq \infty$ then*

$$\|f\|_{p(\cdot),q_1,\alpha} \leq \|f\|_{p(\cdot),q,\alpha}\ , \qquad f \in L^{p(\cdot)}_{\mathrm{loc}} \tag{19}$$

*and therefore $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ is continuously embedded in $\left(L^{p(\cdot)}, \ell^{q_1}\right)^\alpha$.*

Proof. Let $f$ be an element of $L^{p(\cdot)}_{\mathrm{loc}}$.

1) Assume that $p_1(\cdot) \leq p(\cdot) \leq \alpha \leq q \leq \infty$. For any $\rho > 0$ and $k \in \mathbb{Z}^d$, (4) implies that

$$\left\|\left(St^{(\alpha)}_\rho f\right) \chi_{Q_k}\right\|_{p_1(\cdot)} \leq 2 \left\|\left(St^{(\alpha)}_\rho f\right) \chi_{Q_k}\right\|_{p(\cdot)}.$$

Therefore, taking the $l^q$-norm, we get

$$\left\|St^{(\alpha)}_\rho f\right\|_{p_1(\cdot),q} \leq 2 \left\|St^{(\alpha)}_\rho f\right\|_{p(\cdot),q}$$

and so, by taking the supremum over all $\rho > 0$, we obtain

$$\|f\|_{p_1(\cdot),q,\alpha} \leq 2 \|f\|_{p(\cdot),q,\alpha}.$$

2) Assume that $p(\cdot) \leq \alpha \leq q \leq q_1 \leq \infty$.

- $1^{st}$ case : $q = q_1 = \infty$. The result is obvious.
- $2^{nd}$ case : $q < \infty$ and $q_1 = \infty$. For any $\rho > 0$, we have

$$\left\|\left\{\left\|\left(St^{(\alpha)}_\rho f\right) \chi_{Q_k}\right\|_{p(\cdot)}\right\}_{k\in\mathbb{Z}^d}\right\|_{\ell^\infty} \leq \left\|\left\{\left\|\left(St^{(\alpha)}_\rho f\right) \chi_{Q_k}\right\|_{p(\cdot)}\right\}_{k\in\mathbb{Z}^d}\right\|_{\ell^q}$$

and so

$$\left\|St^{(\alpha)}_\rho f\right\|_{p(\cdot),\infty} \leq \left\|St^{(\alpha)}_\rho f\right\|_{p(\cdot),q}.$$

Therefore, by taking the supremum over all $\rho > 0$, we obtain

$$\|f\|_{p(\cdot),\infty,\alpha} \leq \|f\|_{p(\cdot),q,\alpha}.$$

- $3^{rd}$ case : $q_1 < \infty$. Since $1 \leq q \leq q_1$, for any $\rho > 0$, we have

$$\left(\sum_{k\in\mathbb{Z}^d} \left\|\left(St^{(\alpha)}_\rho f\right) \chi_{Q_k}\right\|^{q_1}_{p(\cdot)}\right)^{\frac{1}{q_1}} \leq \left(\sum_{k\in\mathbb{Z}^d} \left\|\left(St^{(\alpha)}_\rho f\right) \chi_{Q_k}\right\|^{q}_{p(\cdot)}\right)^{\frac{1}{q}}$$

and so

$$\left\|St^{(\alpha)}_\rho f\right\|_{p(\cdot),q_1} \leq \left\|St^{(\alpha)}_\rho f\right\|_{p(\cdot),q}.$$

Therefore, by taking the supremum over all $\rho > 0$, we obtain

$$\|f\|_{p(\cdot),q_1,\alpha} \leq \|f\|_{p(\cdot),q,\alpha}.$$

□

**Proposition 3.4.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p(\cdot) \leq \alpha \leq q \leq \infty$.*
*1) The spaces $\mathcal{H}(p'(\cdot), q', \alpha')$ are isometrically invariant under the family $\left\{St^{(\alpha')}_\rho\right\}_{\rho>0}$ :*

$$\left\|St^{(\alpha')}_\rho g\right\|_{\mathcal{H}(p'(\cdot),q',\alpha')} = \|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\ , \quad 0 < \rho < \infty\ , \quad g \in \mathcal{H}(p'(\cdot), q', \alpha'). \tag{20}$$

*2) The set $\mathcal{C}_{\mathrm{c}}^{\infty} := \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ of all infinitely differentiable and compactly supported functions on $\mathbb{R}^d$ is a dense subspace of $\mathcal{H}(p'(\cdot), q', \alpha')$.*

Proof. 1) Let $g \in \mathcal{H}(p'(\cdot), q', \alpha')$ and $\{(c_n, \rho_n, g_n)\}_{n\geq 1}$ any $\mathfrak{h}$-decomposition of $g$. For $0 < \rho < \infty$, it is clear that

$$St_{\rho}^{(\alpha')}g = \sum_{n\geq 1} c_n St_{\rho\rho_n}^{(\alpha')} g_n$$

and so that $\{(c_n, \rho\rho_n, g_n)\}_{n\geq 1}$ is a $\mathfrak{h}$-decomposition of $St_{\rho}^{(\alpha')}g$. Consequently, we obtain

$$\left\| St_{\rho}^{(\alpha')}g \right\|_{\mathcal{H}(p'(\cdot),q',\alpha')} = \|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}.$$

2) Let $g$ be an element of $\mathcal{H}(p'(\cdot), q', \alpha')$. By Point 4) of Proposition 2.3 and Point 1) of Proposition 2.5, $g$ can be approximated by some function $\varphi \in \mathcal{C}_{\mathrm{c}}$. Moreover, there exists some element $\phi$ of $\mathcal{C}_{\mathrm{c}}^{\infty}$ such that $\phi * \varphi$ belongs to $\mathcal{C}_{\mathrm{c}}^{\infty}$ and converges to $\varphi$ in $\left(L^{p'(\cdot)}, \ell^{q'}\right)$, and therefore in $\mathcal{H}(p'(\cdot), q', \alpha')$, where $\phi * \varphi$ denotes the convolution product of $\phi$ and $\varphi$. Therefore the following inequality

$$\|\phi * \varphi - g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq \|\phi * \varphi - \varphi\|_{\mathcal{H}(p'(\cdot),q',\alpha')} + \|\varphi - g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}$$

shows that $\phi * \varphi$ converges to $g$. This ends the proof. □

## 4. Riesz potentials operators

The Riesz potentials operator $I_\gamma$ $(0 < \gamma < d)$ is defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\gamma}} dy$$

when this integral makes sense. The boundedness of this operator in variable exponent Fofana's spaces has been studied by Yang and Zhou. Notice that, with very slight modification of the hypotheses, their result (see [33, Theorem 4.1]) remains valid. Actually the following result holds true.

**Theorem 4.1.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$, $p(\cdot) \leq \alpha < q \leq \infty$, $0 < \gamma < d\left(\frac{1}{\alpha} - \frac{1}{q}\right)$, $\frac{1}{p_*(\cdot)} = \frac{1}{p(\cdot)} - \frac{\gamma}{d}$ and $1 \leq \beta \leq \infty$. Then the Riesz potentials operator $I_\gamma$ is bounded from $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ to $\left(L^{p_*(\cdot)}, \ell^q\right)^\beta$ if and only if $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{\gamma}{d}$.*

As a consequence of Theorem 4.1 we have the following result.

**Theorem 4.2.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$, $p(\cdot) \leq \alpha < q \leq \infty$, $0 < \gamma < d\left(\frac{1}{\alpha} - \frac{1}{q}\right)$, $\frac{1}{p_*(\cdot)} = \frac{1}{p(\cdot)} - \frac{\gamma}{d}$ and $1 \leq \beta \leq \infty$. Then the Riesz potentials operator $I_\gamma$ is bounded from $\mathcal{H}(p'_*(\cdot), q', \beta')$ to $\mathcal{H}(p'(\cdot), q', \alpha')$ if and only if $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{\gamma}{d}$.*

Proof. 1) Let us assume that $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{\gamma}{d}$ and $g \in \mathcal{H}(p'_*(\cdot), q', \beta')$. A direct calculation, using Fubini's theorem, shows that for any $f \in \left(L^{p(\cdot)}, \ell^q\right)^\alpha$, we have

$$\int_{\mathbb{R}^d} |I_\gamma g(x) f(x)|\, dx \leq \int_{\mathbb{R}^d} |g(x)|\, I_\gamma(|f|)(x) dx.$$

Furthermore, by Theorem 4.1, there exists a constant $C$ such that

$$\|I_\gamma f\|_{p_*(\cdot),q,\beta} \leq C \|f\|_{p(\cdot),q,\alpha} \ , \qquad f \in \left(L^{p(\cdot)}, \ell^q\right)^\alpha.$$

Therefore, using Point 2) of Proposition 2.5 and Corollary 3.2, we obtain

$$\int_{\mathbb{R}^d} |I_\gamma g(x) f(x)|\, dx \leq C\ \|g\|_{\mathcal{H}(p'_*(\cdot),q',\beta')}\ \|f\|_{p(\cdot),q,\alpha}$$

and so

$$\sup\left\{\int_{\mathbb{R}^d} |I_\gamma g(x) f(x)|\, dx\ ,\ f \in \left(L^{p(\cdot)}, \ell^q\right)^\alpha,\ \|f\|_{p(\cdot),q,\alpha} \leq 1\right\} \leq C\ \|g\|_{\mathcal{H}(p'_*(\cdot),q',\beta')}.$$

This implies, by Remark 3.1, that

$$\|I_\gamma g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq C\ \|g\|_{\mathcal{H}(p'_*(\cdot),q',\beta')}.$$

This shows that $I_\gamma$ is bounded from $\mathcal{H}(p'_*(\cdot), q', \beta')$ to $\mathcal{H}(p'(\cdot), q', \alpha')$.

2) Let us assume that $I_\gamma$ is bounded from $\mathcal{H}(p'_*(\cdot), q', \beta')$ to $\mathcal{H}(p'(\cdot), q', \alpha')$.

Arguing as in point 1), we get that $I_\gamma$ is bounded from $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ to $\left(L^{p_*(\cdot)}, \ell^q\right)^\beta$ and therefore by Theorem 4.1 we obtain $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{\gamma}{d}$. This completes the proof. □

## 5. Maximal operators

The fractional maximal operators $\mathcal{M}_\gamma$ $(0 \leq \gamma < d)$ are defined by

$$\mathcal{M}_\gamma f(x) = \sup_{r>0} |Q(x,r)|^{\frac{\gamma}{d}-1} \int_{Q(x,r)} |f(y)| dy\ , \quad x \in \mathbb{R}^d\ ,\ f \in L^1_{\mathrm{loc}},$$

where

$$Q(x,r) = \prod_{j=1}^{d} \left[x_j - \frac{r}{2}, x_j + \frac{r}{2}\right), \qquad x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d\ ,\ 0 < r < \infty.$$

When $\gamma = 0$, we have the Hardy-Littlewood maximal operator $\mathcal{M}_0$ which is one of the most important classical operators in Harmonic Analysis because its controls various other important operators. Whereas, if $0 < \gamma < d$ then the following pointwise control holds :

$$\mathcal{M}_\gamma f(x) \leq C_d\ I_\gamma(|f|)(x)\ , \qquad x \in \mathbb{R}^d\ ,\ f \in L^1_{\mathrm{loc}}, \tag{21}$$

where $C_d$ is a real constant depending only on $d$. Since $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ and $\mathcal{H}(p'(\cdot), q', \alpha')$ are solid spaces (see Proposition 3.1), it readily follows from Theorem 4.1, Theorem 4.2 and inequality (21), the following results.

**Corollary 5.1.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$, $p(\cdot) \leq \alpha < q \leq \infty$, $0 < \gamma < d\left(\frac{1}{\alpha} - \frac{1}{q}\right)$, $\frac{1}{p_*(\cdot)} = \frac{1}{p(\cdot)} - \frac{\gamma}{d}$ and $\frac{1}{\alpha_*} = \frac{1}{\alpha} - \frac{\gamma}{d}$. Then the fractional maximal operator $\mathcal{M}_\gamma$ is bounded from :*
*1) $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ to $\left(L^{p_*(\cdot)}, \ell^q\right)^{\alpha_*}$*
*2) $\mathcal{H}(p'_*(\cdot), q', \alpha'_*)$ to $\mathcal{H}(p'(\cdot), q', \alpha')$.*

The boundedness of $\mathcal{M}_0$ on variable exponent amalgam spaces was investigated by many authors. The one dimensional case of the proposition below is contained in [**21**, **28**].

**Proposition 5.2.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$ and $1 < q < \infty$. Then the Hardy-Littlewood maximal operator $\mathcal{M}_0$ is bounded on $\left(L^{p(\cdot)}, \ell^q\right)$.*

We shall use the fact that the Hardy-Littlewood maximal operator commutes with the dilation operators. A direct calculation shows this result. However, for the reader's convenience we give its detailed proof.

**Lemma 5.3.** *Let $p(\cdot)$ be a variable exponent on $\mathbb{R}^d$, $0 \leq \gamma < d$ and $0 < \rho, \alpha < \infty$. Then for any element $f$ of $L^{p(\cdot)}_{\text{loc}}$ we have*

$$\mathcal{M}_\gamma\left(St^{(\alpha)}_\rho f\right) = \rho^\gamma\ St^{(\alpha)}_\rho\left(\mathcal{M}_\gamma f\right). \tag{22}$$

*In particular, we have*

$$\mathcal{M}_0\left(St^{(\alpha)}_\rho f\right) = St^{(\alpha)}_\rho\left(\mathcal{M}_0 f\right). \tag{23}$$

Proof. Let $f \in L^{p(\cdot)}_{\text{loc}}$, $x \in \mathbb{R}^d$ and $Q(x,r)$ be a cube of $\mathbb{R}^d$. We have

$$\begin{aligned}\mathcal{M}_\gamma\left(St^{(\alpha)}_\rho f\right)(x) &= \mathcal{M}_\gamma\left(\rho^{-\frac{d}{\alpha}} f(\rho^{-1}.)\right)(x) = \rho^{-\frac{d}{\alpha}}\mathcal{M}_\gamma\left(f(\rho^{-1}.)\right)(x)\\ &= \rho^{-\frac{d}{\alpha}}\sup_{r>0}\left\{|Q(x,r)|^{\frac{\gamma}{d}-1}\int_{Q(x,r)}\left|f(\rho^{-1}y)\right|dy\right\}\\ &= \rho^{-\frac{d}{\alpha}}\sup_{r>0}\left\{|Q(x,r)|^{\frac{\gamma}{d}-1}\int_{Q(x,r)}\left|f(\rho^{-1}y)\right|\rho^d\ \rho^{-d}dy\right\}.\end{aligned}$$

By setting $z = \rho^{-1}y$, we have $dz = \rho^{-d}dy$ and

$$y \in Q(x,r) \Longrightarrow z \in Q\left(\rho^{-1}x,\ \rho^{-1}r\right).$$

Therefore, we get

$$\mathcal{M}_\gamma\left(St^{(\alpha)}_\rho f\right)(x) = \rho^{-\frac{d}{\alpha}}\sup_{r>0}\left\{\rho^d\ |Q(x,r)|^{\frac{\gamma}{d}-1}\int_{Q(\rho^{-1}x,\ \rho^{-1}r)}|f(z)|\ dz\right\}.$$

Furthermore, we have

$$\begin{aligned}\rho^d\ |Q(x,r)|^{\frac{\gamma}{d}-1} &= \rho^d\ r^{d\left(\frac{\gamma}{d}-1\right)} = \rho^d\ \rho^{d\left(\frac{\gamma}{d}-1\right)}\ (\rho^{-1}r)^{d\left(\frac{\gamma}{d}-1\right)} = \rho^\gamma(\rho^{-1}r)^{d\left(\frac{\gamma}{d}-1\right)}\\ &= \rho^\gamma\left|Q\left(\rho^{-1}x,\ \rho^{-1}r\right)\right|^{\frac{\gamma}{d}-1}.\end{aligned}$$

Therefore

$$\mathcal{M}_\gamma\left(St_\rho^{(\alpha)}f\right)(x)=\rho^{-\frac{d}{\alpha}}\sup_{r>0}\left\{\rho^\gamma\left|Q\left(\rho^{-1}x,\ \rho^{-1}r\right)\right|^{\frac{\gamma}{d}-1}\int_{Q(\rho^{-1}x,\ \rho^{-1}r)}|f(z)|\ dz\right\}.$$

We set $t=\rho^{-1}r$ and so we obtain

$$\begin{aligned}\mathcal{M}_\gamma\left(St_\rho^{(\alpha)}f\right)(x)&=\rho^\gamma\ \rho^{-\frac{d}{\alpha}}\sup_{t>0}\left\{\left|Q\left(\rho^{-1}x,\ t\right)\right|^{\frac{\gamma}{d}-1}\int_{Q(\rho^{-1}x,\ t)}|f(z)|\ dz\right\}\\&=\rho^\gamma\ \rho^{-\frac{d}{\alpha}}\mathcal{M}_\gamma f(\rho^{-1}x)\\&=\rho^\gamma\ St_\rho^{(\alpha)}\left(\mathcal{M}_\gamma f\right)(x).\end{aligned}$$

Should this equality be true for any $x\in\mathbb{R}^d$, we actually have

$$\mathcal{M}_\gamma\left(St_\rho^{(\alpha)}f\right)=\rho^\gamma\ St_\rho^{(\alpha)}\left(\mathcal{M}_\gamma f\right).$$

As a particular case, by taking $\gamma=0$, we obtain

$$\mathcal{M}_0\left(St_\rho^{(\alpha)}f\right)=St_\rho^{(\alpha)}\left(\mathcal{M}_0 f\right).$$

□

We can now prove the boundedness of the Hardy-Littlewood maximal operator on variable exponent Fofana's spaces.

**Theorem 5.4.** *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^d)$ and $p(\cdot)\leq\alpha\leq q<\infty$. Then the Hardy-Littlewood maximal operator $\mathcal{M}_0$ is bounded on $\left(L^{p(\cdot)},\ell^q\right)^\alpha$.*

Proof. Let $f\in\left(L^{p(\cdot)},\ell^q\right)^\alpha$ and $\rho>0$. Using Lemma 5.3, we get

$$\left\|St_\rho^{(\alpha)}\left(\mathcal{M}_0 f\right)\right\|_{p(\cdot),q}=\ \left\|\mathcal{M}_0\left(St_\rho^{(\alpha)}f\right)\right\|_{p(\cdot),q}.$$

Therefore, by Proposition 5.2, there exists a constant $C$ such that

$$\left\|St_\rho^{(\alpha)}\left(\mathcal{M}_0 f\right)\right\|_{p(\cdot),q}\leq C\ \left\|St_\rho^{(\alpha)}f\right\|_{p(\cdot),q}$$

and so, taking the supremum over all $\rho>0$, we get

$$\left\|\mathcal{M}_0 f\right\|_{p(\cdot),q,\alpha}\leq C\ \left\|f\right\|_{p(\cdot),q,\alpha}.$$

This ends the proof. □

We also prove the following result for the predual spaces.

**Theorem 5.5.** *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^d)$ and $p(\cdot)\leq\alpha\leq q<\infty$. Then the Hardy-Littlewood maximal operator $\mathcal{M}_0$ is bounded on $\mathcal{H}(p'(\cdot),q',\alpha')$.*

Proof. Let $g\in\mathcal{H}(p'(\cdot),q',\alpha')$ and $\{(c_n,\rho_n,g_n)\}_{n\geq1}$ be a $\mathfrak{h}$-decomposition of $g$; that is $g=\sum\limits_{n\geq1}c_nSt_{\rho_n}^{(\alpha')}g_n$ with $\|g_n\|_{p'(\cdot),q'}\leq1$ for any integer $n\geq1$.

Thanks to the homogeneity of $\mathcal{M}_0$ and Lemma 5.3, we have

$$\mathcal{M}_0\left(c_nSt_{\rho_n}^{(\alpha')}g_n\right)=|c_n|St_{\rho_n}^{(\alpha')}\left(\mathcal{M}_0(g_n)\right),\qquad n\geq1\qquad(*)$$

and by Proposition 5.2, there exists a constant $C > 0$ such that

$$\|\mathcal{M}_0(g_n)\|_{p'(\cdot),q'} \leq C\ \|g_n\|_{p'(\cdot),q'} \leq C\ , \qquad n \geq 1. \qquad (**)$$

Let us set $f = \sum_{n\geq 1} |c_n| St_{\rho_n}^{(\alpha')}\left(\mathcal{M}_0(g_n)\right)$. It is clear that

$$f = \sum_{n\geq 1} C\,|c_n| St_{\rho_n}^{(\alpha')}\left(C^{-1}\mathcal{M}_0(g_n)\right)$$

and by $(**)$, we have

$$\left\|C^{-1}\,\mathcal{M}_0(g_n)\right\|_{p'(\cdot),q'} \leq 1.$$

Thus $\{(C\,c_n, \rho_n, C^{-1}\,\mathcal{M}_0(g_n))\}_{n\geq 1}$ is a $\mathfrak{h}$-decomposition of $f$ and consequently

$$\|f\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq C \sum_{n\geq 1} |c_n|. \qquad (***)$$

Furthermore, it is well known that $\mathcal{M}_0$ is a sublinear operator. This and $(*)$ imply that

$$\mathcal{M}_0 g = \mathcal{M}_0\left(\sum_{n\geq 1} c_n St_{\rho_n}^{(\alpha')} g_n\right) \leq \sum_{n\geq 1} \mathcal{M}_0\left(c_n St_{\rho_n}^{(\alpha')} g_n\right) = \sum_{n\geq 1} |c_n| St_{\rho_n}^{(\alpha')}\left(\mathcal{M}_0(g_n)\right) = f.$$

Therefore, from the solidity of $\mathcal{H}(p'(\cdot), q', \alpha')$ (see Proposition 3.1) and $(***)$, we get

$$\|\mathcal{M}_0 g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq \|f\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq C \sum_{n\geq 1} |c_n|.$$

By taking the infimum with respect to all $\mathfrak{h}$-decompositions of $g$, we obtain

$$\|\mathcal{M}_0 g\|_{\mathcal{H}(p'(\cdot),q',\alpha')} \leq C\|g\|_{\mathcal{H}(p'(\cdot),q',\alpha')}.$$

This completes the proof. □

## 6. Calderón-Zygmund operators of type $\omega$

Let $\omega$ be a nonnegative nondecreasing function on $(0, \infty)$ such that

$$\int_0^1 \omega(t)\, t^{-1} dt < \infty$$

and $T$ a Calderón-Zygmund operator of type $\omega$ : $T$ is a linear bounbed map of $\mathcal{C}_c^\infty$ into $L^2$ and there is a continuous map $k$ of $\mathbb{R}^d \times \mathbb{R}^d \backslash \left\{(x, x)\ /\ x \in \mathbb{R}^d\right\}$ into $\mathbb{C}$ such that :

- $$Tf(x) = \int_{\mathbb{R}^d} k(x, y) f(y) dy\ , \qquad f \in \mathcal{C}_c^\infty \text{ and } x \in \mathbb{R}^d \backslash supp(f) \qquad (24)$$
- there is a real constant $A > 0$ such that, for any $x, y$ and $z$ in $\mathbb{R}^d$

$$|k(x, y)| \leq A\, |x - y|^{-d} \qquad \text{if } \ x \neq y\ , \qquad (25)$$

and

$$|k(x,y)-k(x,z)|+|k(y,x)-k(z,x)| \leq \frac{A}{|x-y|^d}\,\omega\left(\frac{|y-z|}{|x-y|}\right) \tag{26}$$

if $0 < 2\,|y-z| \leq |x-y|$.

Notice that these generalized Calderón-Zygmund operators have been introduced by Yabuta in [32]. They are related to the local sharp maximal operators $M^\sharp_\lambda$ $(0 < \lambda < 1)$ defined by

$$M^\sharp_\lambda f(x) = \sup_{Q\ni x}\inf_{c\in\mathbb{R}}\left[\left((f-c)\,\chi_Q\right)^*(\lambda|Q|)\right], \quad x\in\mathbb{R}^d\ , \quad f\in L^1_{\mathrm{loc}}$$

where the supremum is taken over all cube $Q$ of $\mathbb{R}^d$ containing $x$ and $g^*$ is the decreasing rearrangement function of $g\in L^1_{\mathrm{loc}}$, and defined by

$$g^*(t) = \inf\left\{s>0\ :\ \left|\left\{x\in\mathbb{R}^d\ :\ |g(x)|>s\right\}\right|\leq t\right\}, \quad 0<t<\infty.$$

The following result shows a link between Calderón-Zygmund operators of type $\omega$, the Hardy-Littlewood maximal operator and local sharp maximal operators.

**Proposition 6.1.** [10, Proposition 5.6]. *Let $T$ be a Calderón-Zygmund operator of type $\omega$ and $0<\lambda<1$. Then there is a real constant $c(\lambda)>0$ such that*

$$M^\sharp_\lambda(Tf)(x) \leq c(\lambda)\ \mathcal{M}_0 f(x)\ , \qquad f\in\mathcal{C}^\infty_c, \quad x\in\mathbb{R}^d. \tag{27}$$

Let us recall the following result of Lerner.

**Proposition 6.2.** [26] *There is an element $(c_d,\lambda_d)$ of $(0,\infty)\times(0,1)$ such that for any element $f$ of $L^1_{\mathrm{loc}}$ satisfying*

$$\left|\left\{x\in\mathbb{R}^d\ /\ |f(x)|>t\right\}\right| < \infty\ , \quad t>0,$$

*we have*

$$\int_{\mathbb{R}^d}|f(x)g(x)|dx \leq c_d\int_{\mathbb{R}^d}M^\sharp_{\lambda_d}f(x)\mathcal{M}_0 g(x)dx\ , \qquad g\in L^1_{\mathrm{loc}}. \tag{28}$$

From the above propositions, we deduce the following result which is a generalization of [10, Corollary 5.8] and its proof is just an adaptation of that given there.

**Theorem 6.3.** *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^d)$, $p(\cdot)\leq\alpha\leq q<\infty$ and $T$ be a Calderón-Zygmund operator of type $\omega$. Then $T$ has an unique bounded linear extension to $\mathcal{H}(p'(\cdot),q',\alpha')$.*

Proof. Let $(g,f)$ be an element of $\mathcal{C}^\infty_c\times L^1_{\mathrm{loc}}$.
By the definition of $T$, $Tg$ belongs to $L^2$ and this implies that

$$\left|\left\{x\in\mathbb{R}^d\ /\ |Tg(x)|>t\right\}\right| \leq \frac{1}{t^2}\|Tg\|_2^2 < \infty\ , \quad t>0.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^d}|Tg(x)f(x)|dx &\leq c_d\int_{\mathbb{R}^d}M^{\sharp}_{\lambda_d}(Tg)(x)\mathcal{M}_0f(x)dx && \text{(by (28))}\\ &\leq c_d\, c(\lambda_d)\int_{\mathbb{R}^d}\mathcal{M}_0g(x)\mathcal{M}_0f(x)dx && \text{(by (27))}\\ &\leq c_d\, c(\lambda_d)\left\|\mathcal{M}_0g\right\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\ \left\|\mathcal{M}_0f\right\|_{p(\cdot),q,\alpha} && \text{(by (12))}\\ &\leq C\ \left\|g\right\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\ \left\|f\right\|_{p(\cdot),q,\alpha} && \text{(by Theorem 5.4 and Theorem 5.5)}\end{aligned}$$

where $C$ is a real constant not depending on $f$ and $g$. Thus, by (15)

$$\left\|Tg\right\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\leq C\ \left\|g\right\|_{\mathcal{H}(p'(\cdot),q',\alpha')}\qquad g\in\mathcal{C}_c^{\infty}.$$

Therefore, by the density of $\mathcal{C}_c^{\infty}$ in $\mathcal{H}(p'(\cdot),q',\alpha')$ (see Proposition 3.4), $T$ has an unique bounded linear extension to $\mathcal{H}(p'(\cdot),q',\alpha')$. □

From what precedes we also deduce the following theorem whose proof ideal comes from [10, Corollary 5.9].

**Theorem 6.4.** *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^d)$, $p(\cdot)\leq\alpha\leq q<\infty$ and $T$ be a Calderón-Zygmund operator of type $\omega$. Then $T$ admits a bounded linear extension to $(L^{p(\cdot)},\ell^q)^{\alpha}$.*

Proof. We denote by $T_2$ the unique bounded linear extension of $T$ to $L^2$. Its transpose $T_2^t$, defined by

$$\int_{\mathbb{R}^d}T_2^tf(x)\,g(x)dx=\int_{\mathbb{R}^d}f(x)T_2\,g(x)dx\ ,\qquad f,g\in L^2$$

is a bounded linear operator on $L^2$. It is known (see [1]) that

$$T_2^tf(x)=\int_{\mathbb{R}^d}k(y,x)f(y)dy\ ,\qquad f\in\mathcal{C}_c^{\infty}\ ,\ x\in\mathbb{R}^d\backslash\text{supp}(f).$$

From this and inequalities (25) and (26) we deduce that the restriction $T^t$ of $T_2^t$ to $\mathcal{C}_c^{\infty}$ is a Calderón-Zygmund operator of type $\omega$ which, by Theorem 6.3, has an unique bounded linear extension $T^t_{p'(\cdot),q',\alpha'}$ to $\mathcal{H}(p'(\cdot),q',\alpha')$. The transpose $H$ of $T^t_{p'(\cdot),q',\alpha'}$ is a bounded linear operator on the dual space $(L^{p(\cdot)},\ell^q)^{\alpha}$ of $\mathcal{H}(p'(\cdot),q',\alpha')$. It satisfies

$$\int_{\mathbb{R}^d}Hf(x)\,g(x)dx=\int_{\mathbb{R}^d}f(x)T^t_{p'(\cdot),q',\alpha'}\,g(x)dx\ ,\qquad (f,g)\in(L^{p(\cdot)},l^q)^{\alpha}\times\mathcal{H}(p'(\cdot),q',\alpha')$$

and therefore, for any $f$ and $g$ in $\mathcal{C}_c^{\infty}$

$$\int_{\mathbb{R}^d}Hf(x)\,g(x)dx=\int_{\mathbb{R}^d}f(x)T_2^t\,g(x)dx=\int_{\mathbb{R}^d}T_2f(x)\,g(x)dx=\int_{\mathbb{R}^d}Tf(x)\,g(x)dx.$$

This shows that $H$ is an extension of $T$ to $(L^{p(\cdot)},\ell^q)^{\alpha}$. □

## 7. Generalized sublinear operators

We consider a class of sublinear operators $\mathcal{T}_\gamma$ $(0 \leq \gamma < d)$ satisfying the condition

$$|\mathcal{T}_\gamma f(x)| \leq C \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{d-\gamma}} dy \tag{29}$$

for any $f \in L^1$ with compact support $K$ and $x \in \mathbb{R}^d \backslash K$.

We point out that the condition (29) was first introduced by Soria and Weiss [30]. This condition is satisfied by many interesting operators in Harmonic Analysis. When $\gamma = 0$, such as the Hardy-Littewood maximal operator, Calderón-Zygmund singular integral operators, Bochner-Riesz operators at the critical index and so on. When $0 < \gamma < d$, such as fractional maximal operators, Riesz potential operators, fractional oscillatory singular integrals and so on.

Before announcing our result, we prove the following preparatory lemma.

**Lemma 7.1.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$, $0 \leq \gamma < \frac{d}{p_+}$ and $\frac{1}{p_*(\cdot)} = \frac{1}{p(\cdot)} - \frac{\gamma}{d}$. Then for any integer $k \in \mathbb{N}$ and every ball $B$ in $\mathbb{R}^d$, we have*

$$\left\|\chi_{2^{k+1}B}\right\|_{p'(\cdot)} \left\|\chi_B\right\|_{p_*(\cdot)} \lesssim \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p(\cdot)}-1} |B|^{1-\frac{\gamma}{d}}. \tag{30}$$

Proof. Let $k \in \mathbb{N}$ and $B$ be any ball in $\mathbb{R}^d$. We have by (5)

$$\begin{aligned}
\left\|\chi_{2^{k+1}B}\right\|_{p'(\cdot)} \left\|\chi_B\right\|_{p_*(\cdot)} &\lesssim \frac{|2^{k+1}B|}{\left\|\chi_{2^{k+1}B}\right\|_{p(\cdot)}} \left\|\chi_B\right\|_{p_*} \\
&\lesssim |2^{k+1}B|^{1-\frac{1}{p(\cdot)}} \frac{\left\|\chi_B\right\|_{p_*(\cdot)}}{|2^{k+1}B|^{-\frac{1}{p(\cdot)}} \left\|\chi_{2^{k+1}B}\right\|_{p(\cdot)}} \\
&\lesssim \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p(\cdot)}-1} \frac{|B|^{1-\frac{1}{p(\cdot)}} \left\|\chi_B\right\|_{p_*(\cdot)}}{|2^{k+1}B|^{-\frac{1}{p(\cdot)}} \left\|\chi_{2^{k+1}B}\right\|_{p(\cdot)}} \\
&\lesssim \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p(\cdot)}-1} |B|^{1-\frac{\gamma}{d}} \frac{|B|^{-\frac{1}{p_*(\cdot)}} \left\|\chi_B\right\|_{p_*(\cdot)}}{|2^{k+1}B|^{-\frac{1}{p(\cdot)}} \left\|\chi_{2^{k+1}B}\right\|_{p(\cdot)}} \\
&\lesssim \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p(\cdot)}-1} |B|^{1-\frac{\gamma}{d}},
\end{aligned}$$

where the last inequality follows from Lemma 4.1.6 and Corollary 4.5.9 in [12]. □

Our result can be stated as follows.

**Theorem 7.2.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$, $p(\cdot) \leq \alpha < q \leq \infty$, $0 \leq \gamma < d\left(\frac{1}{\alpha} - \frac{1}{q}\right)$, $\frac{1}{p_*(\cdot)} = \frac{1}{p(\cdot)} - \frac{\gamma}{d}$ and $\frac{1}{\alpha_*} = \frac{1}{\alpha} - \frac{\gamma}{d}$. If $\mathcal{T}_\gamma$ is a sublinear operator which is bounded from $L^{p(\cdot)}$ to $L^{p_*(\cdot)}$ and satisfy the condition (29) then $\mathcal{T}_\gamma$ is bounded from $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$ to $\left(L^{p_*(\cdot)}, \ell^q\right)^{\alpha_*}$.*

Proof. We fix $r > 0$, $x \in \mathbb{R}^d$ and set $B := B(x,r)$, $\lambda B := B(x, \lambda r)$ for all $\lambda > 0$. Let $f$ be an element of $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$. We have

$$f = f\chi_{2B} + \sum_{k=1}^{\infty} f\chi_{(2^{k+1}B)\setminus(2^k B)}.$$

By the sublinearity of $\mathcal{T}_\gamma$ and the condition (29) we obtain

$$|(\mathcal{T}_\gamma f)| \lesssim |\mathcal{T}_\gamma(f\chi_{2B})| + \sum_{k=1}^{\infty} \left|2^{k+1}B\right|^{\frac{\gamma}{d}-1} \int_{2^{k+1}B} f(y)dy$$

and so by application of Hölder's inequality (see Proposition 2.1), we have

$$|(\mathcal{T}_\gamma f)| \lesssim |\mathcal{T}_\gamma(f\chi_{2B})| + \sum_{k=1}^{\infty} \left|2^{k+1}B\right|^{\frac{\gamma}{d}-1} \|f\chi_{2^{k+1}B}\|_{p(\cdot)} \|\chi_{2^{k+1}B}\|_{p'(\cdot)}.$$

Taking the $L^{p_*(\cdot)}$-norm on the ball $B$ and using the boundedness of $\mathcal{T}_\gamma$ from $L^{p(\cdot)}$ to $L^{p_*(\cdot)}$, we obtain

$$\|(\mathcal{T}_\gamma f)\chi_B\|_{p_*(\cdot)} \lesssim \|f\chi_{2B}\|_{p(\cdot)} + \sum_{k=1}^{\infty} \left|2^{k+1}B\right|^{\frac{\gamma}{d}-1} \|f\chi_{2^{k+1}B}\|_{p(\cdot)} \|\chi_{2^{k+1}B}\|_{p'(\cdot)} \|\chi_B\|_{p_*(\cdot)}.$$

According to Lemma 7.1, we get

$$\begin{aligned}\|(\mathcal{T}_\gamma f)\chi_B\|_{p_*(\cdot)} &\lesssim \|f\chi_{2B}\|_{p(\cdot)} + \sum_{k=1}^{\infty} \left|2^{k+1}B\right|^{\frac{\gamma}{d}-1} \|f\chi_{2^{k+1}B}\|_{p(\cdot)} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p(\cdot)}-1} |B|^{1-\frac{\gamma}{d}} \\ &\lesssim \|f\chi_{2B}\|_{p(\cdot)} + \sum_{k=1}^{\infty} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p_*(\cdot)}} \|f\chi_{2^{k+1}B}\|_{p(\cdot)}.\end{aligned}$$

Therefore, since $\frac{1}{\alpha_*} - \frac{1}{p_*(\cdot)} - \frac{1}{q} = \frac{1}{\alpha} - \frac{1}{p(\cdot)} - \frac{1}{q}$, we obtain

$$\begin{aligned}|B|^{\frac{1}{\alpha_*}-\frac{1}{p_*(\cdot)}-\frac{1}{q}} \|(\mathcal{T}_\gamma f)\chi_B\|_{p_*(\cdot)} &\lesssim |B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2B}\|_{p(\cdot)} \\ &\quad + |B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \sum_{k=1}^{\infty} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p_*(\cdot)}} \|f\chi_{2^{k+1}B}\|_{p(\cdot)} \\ &=: I + II. \qquad (*)\end{aligned}$$

We have

$$\begin{aligned}I &= |2B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2B}\|_{p(\cdot)} \left(\frac{|B|}{|2B|}\right)^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \\ &\le |2B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2B}\|_{p(\cdot)} \left(\frac{|B|}{|2B|}\right)^{\frac{1}{\alpha}-\frac{1}{p_-}-\frac{1}{q}} \\ &\lesssim |2B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2B}\|_{p(\cdot)}. \qquad (**)\end{aligned}$$

Furthermore, we have

$$II = \sum_{k=1}^{\infty} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{p_*(\cdot)}} |B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2^{k+1}B}\|_{p(\cdot)}$$

$$\lesssim \sum_{k=1}^{\infty} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{\alpha_*}-\frac{1}{q}} \left|2^{k+1}B\right|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2^{k+1}B}\|_{p(\cdot)}. \qquad (***)$$

Combining $(*), (**)$ and $(***)$, we get

$$|B|^{\frac{1}{\alpha_*}-\frac{1}{p_*(\cdot)}-\frac{1}{q}} \|(\mathcal{T}_\gamma f)\chi_B\|_{p_*(\cdot)} \lesssim |2B|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2B}\|_{p(\cdot)} + \sum_{k=1}^{\infty} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{\alpha_*}-\frac{1}{q}} \left|2^{k+1}B\right|^{\frac{1}{\alpha}-\frac{1}{p(\cdot)}-\frac{1}{q}} \|f\chi_{2^{k+1}B}\|_{p(\cdot)}.$$

Taking the $L^q$-norm of both sides of the above inequality and therefore the supremum over all $r > 0$, we obtain

$$\widetilde{\|\mathcal{T}_\gamma f\|}_{p_*(\cdot),q,\alpha_*} \lesssim \left[1 + \sum_{k=1}^{\infty} \left(\frac{1}{2^{d(k+1)}}\right)^{\frac{1}{\alpha_*}-\frac{1}{q}}\right] \widetilde{\|f\|}_{p(\cdot),q,\alpha}.$$

This completes the proof because the series on the right hand side converges. □

Notice that Theorem 4.1 and Point 1) of Corollary 5.1 are particular cases of Theorem 7.2. Moreover, when we take $\gamma = 0$ in Theorem 7.2, we obtain the following result which extends Theorem 5.4 and Theorem 6.4. It is also a generalization of [16, Theorem 4.5].

**Corollary 7.3.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$ and $p(\cdot) \leq \alpha < q \leq \infty$. If $\mathcal{T}_0$ is a sublinear operator which is bounded on $L^{p(\cdot)}$ and satisfy the condition (29) then $\mathcal{T}_0$ is bounded on $\left(L^{p(\cdot)}, \ell^q\right)^\alpha$.*

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