**Mathematical Analysis and its Contemporary Applications**
Volume 5, Issue 3, 2023, 35–67
doi: 10.30495/maca.2023.2005470.1079
ISSN 2716-9898

# A survey on $\Theta$-contractions and fixed point theorems

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Abstract. In this work, a collection of various fixed point results of $\theta$-contractions are examined. Important results from when the concept was introduced up to the recent developments are discussed. Hence, the aim of this paper is to collate and analyze the advances of fixed point results in the setting of $\theta$-contractions which will be helpful and handy for researchers in fixed point theory and related domains.

## 1. Introduction

The fixed point theorem, generally known as the Banach contraction principle, appeared in explicit form in Banach's thesis in 1922 [**3**], where it was used to establish the existence of a solution to some classes of integral equations. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. The Banach contraction principle has been generalized in many ways over the years. Some of the improvements are in crisp setting (e.g. [**13, 14, 20**]) and others are non-crisp (see, for example, [**37, 17, 21, 18, 19**]). In 2014, Jleli and Samet [**16**] introduced a new type of contractive mapping called the $\theta$-contraction in the setting of Branciari metric spaces and established some fixed point theorems which extended some existing ones, including the Banach contraction mapping principle.

Following the introduction of the idea of $\theta$-contraction, more than a handful of results have been presented, discussing various conditions under which such mappings have either a unique or some fixed points in the underlying space. However, the available literature reveals that up to this time, there is no single monograph

2020 *Mathematics Subject Classification.* 47H10; 4H25; 54H25.
*Key words and phrases.* fixed point, metric space, $\theta$-contraction.

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containing the progress of fixed point theory in the mentioned direction. Therefore, this manuscript focuses on highlighting distinct and remarkable fixed point extensions of $\theta$-contraction in effort to provide researchers in the area of fixed point theory with a glimpse into the advancements of fixed point theorems of contraction mappings.

## 2. Preliminaries

In this section, we introduce some definitions, fundamental notations, and terminology that will be deployed subsequently. Throughout this paper, every set $X$ is considered non-empty, and $\mathbb{N}$ is the set of natural numbers. We begin with the definition of generalized metric space due to Branciari [5].

**Definition 2.1.** [5] Let $X$ be a non-empty set and $d : X \times X \longrightarrow [0,\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from $x$ and $y$, we have

$(i)$ $d(x,y) = 0 \Leftrightarrow x = y$;
$(ii)$ $d(x,y) = d(y,x)$;
$(iii)$ $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then, $(X,d)$ is called a generalized metric space (or g.m.s. for short).

**Definition 2.2.** [16] Let $(X,d)$ be a g.m.s., $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $X$ and $x \in X$. We say that $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x$ if and only if $d(x_n,x) \longrightarrow 0$ as $n \longrightarrow \infty$.

**Definition 2.3.** [16] Let $(X,d)$ be a g.m.s. We say that $(X,d)$ is complete if and only if every Cauchy sequence in X converges to some element in X.

**Lemma 2.1.** [15] *Let $(X,d)$ be a g.m.s. and $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X,d)$ such that $d(x_n,x) \longrightarrow 0$ as $n \longrightarrow \infty$ for some $x \in X$. Then $d(x_n,y) \longrightarrow d(x,y)$ as $n \longrightarrow \infty$ for all $y \in X$. In particular, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to $y$ if $y \neq x$.*

**Definition 2.4.** [30] Let (X,d) be a metric space and let $T$ be a self-mapping on $X$. We say that $T$ is a Banach contraction if for all $x, y \in X$, there exists $\lambda \in [0,1)$ such that

$$d(Tx,Ty) \leq \lambda d(x,y).$$

**Definition 2.5.** [30] Let (X,d) be a metric space and let $T$ be a self-mapping on $X$. We say that $T$ is a Kannan contraction if for all $x, y \in X$, there exists $\gamma \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq \gamma[d(x,Tx) + d(y,Ty)].$$

**Definition 2.6.** [38] Let (X,d) be a metric space and let $T$ be a self-mapping on $X$. We say that $T$ is a Suzuki contraction if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) < d(x, y) \;\Rightarrow\; d(Tx, Ty) < d(x, y).$$

**Definition 2.7.** [30] Let (X,d) be a metric space and let $T$ be a self-mapping on $X$. We say that $T$ is a Chatterjea contraction if for all $x, y \in X$, there exists $\gamma \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)].$$

**Definition 2.8.** [30] Let $(X, d)$ be a metric space and let $T$ be a self mapping on $X$. We say that $T$ is a Hardy-Rogers contraction if for all $x, y \in X$, we have

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Tx) + \alpha_4 d(y, Ty) + \alpha_5 d(x, Ty),$$

where $\sum_{i=1}^{4} \alpha_i + \alpha_5 = 1$.

**Definition 2.9.** [30] Let (X,d) be a metric space and let $T$ be a self-mapping on $X$. We say that $T$ is a Ćirić contraction if for all $x, y \in X$, we have

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4[d(x, Ty) + d(y, Tx)],$$

where $\sum_{i=1}^{3} \alpha_i + 2\alpha_4 < 1$.

**Definition 2.10.** [30] Let $X$ be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0, +\infty)$ is a b-metric on $X$ if, for all $x, y, z \in X$, the following assertions hold:

($b_1$) $d(x, y) = 0$ if and only if $x = y$;
($b_2$) $d(x, y) = d(y, x)$;
($b_3$) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X,d) is called a $b$-metric space.

**Definition 2.11.** [1] For a non-empty set $X$ and a mapping $\theta : X \times X \longrightarrow [0, +\infty)$. We say that a function $d_e : X \times X \longrightarrow [0, +\infty)$ is called an extended $b$-metric (in short, $d_e$-metric) if it satisfies:

($i$) $d_e(x, y) = 0$ if and only if $x = y$;
($ii$) $d_e(x, y) = d_e(y, x)$;
($iii$) $d_e(x, y) \leq \theta(x, y)[d_e(x, z) + d_e(z, y)]$,

for all $x, y, z \in X$. Then, $(X, d_e)$ is a $d_e$-metric space.

**Remark 2.12.** It is clear that in the case of $\theta(x, y) = s$, for $s \geq 1$, the extended $b$-metric becomes the standard $b$-metric.

**Definition 2.13.** [25] Let $(X, \preceq)$ be an ordered set and let d be a metric on X. Then we say that the tripled $(X, \preceq, d)$ is an ordered metric space. If $(X, d)$ is complete, then $(X, \preceq, d)$ is called an ordered complete metric space.

**Definition 2.14.** [2] Let $\mathbb{R}^m$ be the set of $m \times 1$ real matrices, $X$ be a nonempty set and $d : X \times X \longrightarrow \mathbb{R}^m$ be a mapping. $(X, d)$ is said to be a vector-valued metric space if the following properties are satisfied: for all $x, y, z \in X$, we have

($i$) $d(x, y) = 0$ if and only if $x = y$;
($ii$) $d(x, y) = d(y, x)$;
($iii$) $d(x, y) \preceq d(x, z) + d(z, y)$;

where 0 is the zero $m \times 1$ matrix and $\preceq$ is the coordinate-wise ordering on $\mathbb{R}^m$.

**Definition 2.15.** [36] A self-mapping $T$ on a metric space (X, d) is said to be a Picard operator if it has a unique fixed point $z \in X$ and $z = \lim_{n\to\infty} T^n x$ for all $x \in X$.

**Definition 2.16.** [29] An operator $T : X \longrightarrow X$ is called a Banach $G$-contraction or simply $G$-contraction if:

(a) $T$ preserves edges of $G$; that is, for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(T(x), T(y)) \in E(G)$;
(b) $T$ decreases weights of edges of $G$; that is, there exists $k \in [0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have $d(T(x), T(y)) \leq kd(x, y)$.

**Definition 2.17.** [41] Let $X$ be a non-empty set and let $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be a mapping satisfying:

($G_1$) $G(x, y, z) = 0$ if $x = y = z$;
($G_2$) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
($G_3$) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
($G_4$) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$ (symmetry in all three variables);
($G_5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the mapping $G$ is called a generalized metric, or more specifically, a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

**Definition 2.18.** [11] Let $X$ be a nonempty set. A nonempty subset $\mathcal{R}$ of $X^2$ is said to be a binary relation on $X$. Accordingly, for $x, y \in X$ with $(x, y) \in \mathcal{R}$, we say that $x$ is related to $y$ or $x$ relates to $y$ under $\mathcal{R}$. Sometimes, we write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$. If $(x, y) \neq \mathcal{R}$, we say $x$ is not related to $y$. If $x\mathcal{R}y$, $y\mathcal{R}z$ implies $x\mathcal{R}z$, then $\mathcal{R}$ is called transitive.

**Definition 2.19.** [11] A binary relation $\mathcal{R}$ on $X$ is said to be $T$-closed if, for any $x, y \in X$, $(x, y) \in \mathcal{R}$ implies $(T(x), T(y)) \in \mathcal{R}$.

**Definition 2.20.** [11] A sequence $\{x_n\} \subseteq X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ $\forall n \in \mathbb{N}$. It is called $\mathcal{R}$-preserving Cauchy sequence if it is a Cauchy sequence and $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}$.

**Definition 2.21.** [11] Let $(X, d)$ be a metric space and $\mathcal{R}$ a binary relation on $X$.

(i) $\mathcal{R}$ is called $d$-self-closed if, for any $\mathcal{R}$-preserving sequence $\{x_n\}$ converging to $x$, there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ with $(x_{n_k}, x) \in \mathcal{R}$.
(ii) $(X, d)$ is called $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in $X$ converges in $X$.
(iii) A mapping $T : X \longrightarrow X$ is called $\mathcal{R}$-continuous at $x \in X$ if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ converging to $x$, we have $Tx_n \longrightarrow Tx$. Moreover, $T$ is $\mathcal{R}$-continuous on $X$ if it is $\mathcal{R}$-continuous at each point of $X$.

**Definition 2.22.** [24] A partial metric on a nonempty set $X$ is a mapping $p : X \times X \longrightarrow \mathbb{R}^+$ such that, for all $x, y, z \in X$:

$(P_1)$ $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$;
$(P_2)$ $p(x, x) \leq p(x, y)$;
$(P_3)$ $p(x, y) = p(y, x)$;
$(P_4)$ $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair $(X, p)$ is called a partial metric space.

**Definition 2.23.** [26] Let $(X, d, \preceq)$ be a partially ordered metric space. Assume $f, g : X \longrightarrow X$ are two mappings. Then,

$(i)$ $x, y \in X$ are said to be comparable if $x \preceq y$ or $y \preceq x$ holds.
$(ii)$ $f$ is said to be nondecreasing if $x \preceq y$ implies $fx \preceq fy$.
$(iii)$ $f, g$ are called weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.
$(iv)$ $f$ is called weakly increasing if $f$ and $I$ are weakly increasing, where $I$ is denoted as the identity mapping on $X$.

**Definition 2.24.** [8] Let $T$ be a self mapping on $X$ and let $\alpha : X \times X \longrightarrow [0, +\infty)$ be a function. We say that $T$ is an $\alpha$-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

**Definition 2.25.** [8] Let $T$ be a self mapping on $X$ and $\alpha, \eta : X \times X \longrightarrow [0, +\infty)$ be two functions. We say that $T$ is an $\alpha$-admissible mapping with respect to $\eta$ if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Rightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 2.25. Also, if we take $\alpha(x, y) = 1$, then we say that $T$ is an $\eta$-subadmissible mapping.

**Definition 2.26.** [34] A self mapping $T$ on a metric space $(X, d)$ is called expansive mapping if for all $x, y \in X$, there exists $k \in (1, +\infty)$ such that

$$d(Tx, Ty) \geq kd(x, y).$$

**Definition 2.27.** [30] Let $(X, d)$ be a metric space. Let $\alpha : X \times X \longrightarrow [0, \infty)$ and $T : X \longrightarrow X$ be mappings. We say that $T$ is an $\alpha$-continuous mapping on (X,d) if, for given $x \in X$ and sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, $x_n \longrightarrow x$ as $n \to \infty$ yields that $Tx_n \longrightarrow Tx$.

**Definition 2.28.** [8] Let $(X,d)$ be a metric space. Let $\alpha,\eta : X\times X \longrightarrow [0,+\infty)$ and $T : X \longrightarrow X$ be mappings. Then we say that $T$ is an $\alpha$-$\eta$ continuous mapping on $(X,d)$, if for given $x\in X$ and sequence $\{x_n\}$ with

$$x_n \longrightarrow x \ as\ n \longrightarrow \infty,\ \alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})\ \forall n\in\mathbb{N} \Rightarrow Tx_n \longrightarrow Tx.$$

A mapping $T : X \longrightarrow X$ is called orbitally continuous at $p\in X$ if $\lim_{n\to\infty} T^n x = p$ implies that $\lim_{n\to\infty} TT^n x = Tp$. The mapping $T$ is orbitally continuous on $X$ if $T$ is orbitally continuous for all $p\in X$.

**Remark 2.29.** [8] $T : X \longrightarrow X$ be a self mapping on an orbitally $T$ complete metric space $X$. Define $\alpha,\eta : X\times X \longrightarrow [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} 3 \text{ if } \ x,y\in O(w),\ and\ \eta(x,y)=1; \\ 0 \text{ otherwise}, \end{cases}$$

where $O(w)$ is an orbit of a point $w\in X$. If $T : X \longrightarrow X$ is an orbitally continuous map on $(X,d)$, then $T$ is $\alpha$-$\eta$ continuous on $(X,d)$.

**Definition 2.30.** [27] Let $(X,d)$ be a metric space and $A_0 \neq \emptyset$. We say that the pair $(A,B)$ has the weak $P$-property if

$$\begin{cases} d(x_1,y_1) = d(A,B); \\ d(x_2,y_2) = d(A,B) \quad \Rightarrow d(x_1,x_2) \leq d(y_1,y_2), \end{cases}$$

for all $x_1,x_2\in A$ and $y_1,y_2\in B$.

**Definition 2.31.** [27] Let $(X,d)$ be a metric space and $A,B$ be two subsets of $X$. A non-self mapping $T : A \longrightarrow B$ is called $\alpha$-proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 1; \\ d(u_1,Tx_1) = d(A,B); \\ d(u_2,Tx_2) = d(A,B) \quad \Rightarrow \alpha(u_1,u_2) \geq 1, \end{cases}$$

for all $x_1,x_2,u_1,u_2\in A$, where $\alpha : A\times A \longrightarrow [0,\infty)$.

**Definition 2.32.** [30] Let $(X,d)$ be a metric space and let $T$ be a self-mapping on $X$. We say that $T$ is a Hardy-Rogers contraction if for all $x,y\in X$, we have

$$\begin{aligned} &d(Tx,Ty) \\ &\leq \alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Tx) + \alpha_4 d(y,Ty) + \alpha_5 d(x,Ty) \end{aligned}$$

where $\sum_{i=1}^{3}\alpha_1 + 2\alpha_4 < 1$.

**Definition 2.33.** [6] Let $X$ be a nonempty set. An $S$-metric on $X$ is a mapping $S : X^3 \longrightarrow [0.\infty)$ that satisfies the following conditions: for each $x,y,z,a\in X$,

$(S1)$ $S(x,y,z)\geq 0$;
$(S2)$ $S(x,y,z)=0$ if and only if $x=y=z$;

$(S3)$ $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

Then, the pair $(X, S)$ is called an $S$-metric space.

**Definition 2.34.** [6] Let $(X, S)$ be an $S$-metric space.

(i) A sequence $\{x_n\} \subset X$ is said to converge to $x \in X$ if $S(x_n, x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$. We write $x_n \longrightarrow x$ for brevity.

(ii) A sequence $\{x_n\} \subset X$ is called Cauchy sequence if $S(x_n, x_n, n_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$.

(iii) The $S$-metric space $(X, S)$ is said to be complete if every Cauchy sequence is a convergent sequence.

**Lemma 2.2.** [6] *Let $(X, S)$ be an $S$-metric space. If $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then, $S(x_n, x_n, y_n) \longrightarrow S(x, x, y)$.*

**Definition 2.35.** [40] Let $A, B$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : A \longrightarrow B$ is said to be a Geragthy-contraction if there exists $\beta \in \mathfrak{X}$ such that, for any $x, y \in A$, $d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$,
where $\mathfrak{X}$ denote the set of functions $\beta : [0, \infty) \longrightarrow [0, 1)$ satisfying:

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

**Definition 2.36.** [43] Let $(X, d)$ be a metric space and $T : X \longrightarrow X$ be a mapping. $T$ is said to be a Chatterjea contraction if there exists $k \in (0, 1)$ such that for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx)).$$

**Definition 2.37.** [43] Let $(X, d)$ be a metric space and $T : X \longrightarrow X$ be a mapping. $T$ is said to be a weakly Chatterjea contraction if for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi(d(x, Ty), d(y, Tx)),$$

where $\psi : [0, +\infty)^2 \longrightarrow [0, +\infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Recall that $T : X \to X$ is said to be a non-decreasing mapping if $x \leq y \Rightarrow Tx \leq Ty$ for all $x, y \in X$. Consistent with Jleli and Samet [16], we denote by $\Theta$ the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

$(\Theta_1)$ $\theta$ is non-decreasing: that is for all $x, y \in (0, \infty)$, if $x \leq y$, then $\theta(x) \leq \theta(y)$;

$(\Theta_2)$ for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{t\to\infty}(t_n) = 0^+$;

$(\Theta_3)$ there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

**Example 2.38.** [16] Let $\theta : (0, \infty) \longrightarrow (1, \infty)$ be defined by

(1) $\theta(t) := e^{\sqrt{t}}$.
(2) $\theta(t) := e^{\sqrt{te^t}}$.
(3) $\theta(t) := 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\alpha})$, $0 < \alpha < 1, t > 0$.

Then (1)-(3) satisfy all the properties of $\Theta$.

**Definition 2.39.** [16] Let $(X, d)$ be a g.m.s. A mapping $T : X \longrightarrow X$ is called $\theta$-contraction if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$for\ all\ x, y \in X, d(Tx, Ty) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k. \tag{1}$$

The first fixed point result in $\theta$-contraction was obtained in 2014 by Jleli and Samet[16]. It was shown that a self-mapping $T$ on a complete generalized-metric space satisfying certain contractive conditions has a unique fixed point.

**Theorem 2.3.** [16] *Let $(X, d)$ be a complete g.m.s. and $T : X \longrightarrow X$ be a given map. Suppose that there exit $\theta \in \Theta$ and $k \in (0, 1)$ such that (1) holds. Then, $T$ has a unique fixed point.*

Proof. Let $x \in X$ be an arbitrary point in X. If for some $p \in \mathbb{N}$, we have $T^P x = T^{p+1}x$, then $T^p x$ will be a fixed point of $T$. So, without restriction of the generality, we can suppose that $d(T^n x, T^{n+1}x) > 0$ for all $n \in \mathbb{N}$. From (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\theta(d(T^n x, T^{n+1}x)) &\leq [\theta(d(T^{n-1}x, T^n x))]^k \\ &\leq [\theta(d(T^{n-2}x, T^{n-1}x))]^{k^2} \\ &\leq \cdots \leq [\theta(d(T^n x, T^{n+1}x))]^{k^n}.\end{aligned}$$

Thus, we have

$$1 \leq \theta(d(T^n x, T^{n+1}x)) \leq [\theta(d(x, Tx))]^{k^n} \quad \text{for all n} \in \mathbb{N}. \tag{2}$$

Letting $n \longrightarrow \infty$ in (2), we obtain

$$\theta(d(T^n x, T^{n+1}x)) \to 1 \quad as \quad n \longrightarrow \infty,$$

which implies from $(\Theta_2)$ that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{3}$$

From condition $(\Theta_3)$, there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(T^n x, T^{n+1}x)) - 1}{d(T^n x, T^{n+1}x)^r} - \ell \right| \leq B \text{ for all n} \geq \mathrm{n}_0.$$

This implies that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{d(T^n x, T^{n+1}x)^r} \geq \ell - B = B, \text{ for all n} \geq \mathrm{n}_0.$$

Then,

$$n[d(T^n x, T^{n+1}x)]^r \leq An[\theta(d(T^n x, T^{n+1}x)) - 1],$$

for all $n \geq n_0$, where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{(d(T^n x, T^{n+1}x)]^r} \geq B, \text{ for all n} \geq \mathrm{n}_0.$$

This implies that

$$n[d(T^n x, T^{n+1}x)]^r \leq An[\theta(d(T^n x, T^{n+1}x)) - 1], \ for \ all \ n \geq n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d(T^n x, T^{n+1}x)]^r \leq An[\theta(d(T^n x, T^{n+1}x)) - 1], \ for \ all \ n \geq n_0.$$

Using (2), we obtain

$$n[d(T^n x, T^{n+1}x)]^r \leq An([\theta(d(T^n x, T^{n+1}x))]^{k^n} - 1), \quad \text{for all n} \geq \mathrm{n}_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} n[d(T^n x, T^{n+1}x)]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(T^n x, T^{n+1}x) \leq \frac{1}{n^{\frac{1}{r}}}, \text{ for all n} \geq \mathrm{n}_1. \tag{4}$$

Now, we shall prove that $T$ has a periodic point. Suppose that it is not the case, then $T^n x \neq T^m x$, for every $n, m \in \mathbb{N}$ such that $n \neq m$. Using (1), we obtain

$$\begin{aligned} \theta(d(T^n x, T^{n+2}x)) &\leq [\theta(d(T^{n-1}x, T^{n+1}x))]^k \\ &\leq [\theta(d(T^n x, T^{n+1}x))]^{k^2} \\ &\leq \cdots \leq [\theta(d(T^n x, T^{n+1}x))]^{k^n}. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and using $(\Theta_2)$, we obtain

$$\lim_{n \to \infty} d(T^n x, T^{n+2}x) = 0. \tag{5}$$

Similarly, from condition $(\Theta_3)$, there exists $n_2 \in \mathbb{N}$ such that

$$d(T^n x, T^{n+1} x) \leq \frac{1}{n^{\frac{1}{r}}}, \text{ for all } \mathrm{n} \geq \mathrm{n}_2. \tag{6}$$

Let $N = \max\{n_0, n_1\}$. We consider two cases.

Case 1. If $m > 2$ is odd, then writing $m = 2l+1$, $l \geq 1$. Using (4), for all $n \geq N$, we obtain

$$\begin{aligned} d(T^n x, T^{n+m} x) \leq & d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{n+2l} x, T^{n+2l+1} x) \\ \leq & \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2l)^{\frac{1}{r}}} \\ \leq & \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Case 2. If $m > 2$ is even, then writing $m = 2l$, $l \geq 2$. Using (4) and (6), for all $n \geq N$, we obtain

$$\begin{aligned} d(T^n x, T^{n+m} x) \leq & d(T^n x, T^{n+2} x) + d(T^{n+2} x, T^{n+3} x) + \cdots + d(T^{n+2l-1} x, T^{n+2l} x) \\ \leq & \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2l-1)^{\frac{1}{r}}} \\ \leq & \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Thus, combining all the cases, we have

$$d(T^n x, T^{n+m} x) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \text{ for all } \mathrm{n} \geq \mathrm{N}, \mathrm{m} \in \mathbb{N}.$$

From the convergence of the series $\sum_i \frac{1}{i^{\frac{1}{r}}}$ (since $\frac{1}{r} > 1$), we deduce that $T^n x$ is a Cauchy sequence. Since $(X, d)$ is complete, there is $z \in X$ such that $T^n x \longrightarrow z$. On the other hand, observe that $T$ is continuous, indeed, if $Tx \neq Ty$, then we have from (1)

$$\ln[\theta(d(Tx, Ty))] \leq k \ln[\theta(d(x, y))] \leq \ln[\theta(d(x, y))],$$

which implies from $(\Theta_1)$ that

$$d(Tx, Ty) \leq d(x, y), \text{ for all } x, y \in X.$$

From this observation, for all $n \in \mathbb{N}$, we have

$$d(T^{n+1} x, Tz) \leq d(T^n x, z).$$

Letting $n \longrightarrow \infty$ in the above inequality, we get $T^{n+1} \longrightarrow Tz$. And so $z = Tz$, which is a contradiction with the assumption: $T$ does not have a periodic point.

Thus $T$ has a periodic point, say z, of period q. Suppose that the set of fixed points of $T$ is empty. Then we have

$$q > 1 \quad and \quad d(z, Tz) > 0.$$

Using (1), we obtain

$$\theta(d(z, Tz)) = \theta(T^n z, T^{n+1} z) \leq [\theta(d(z, Tz))]^{k^n} < \theta(d(z, Tz)),$$

which is a contradiction. Thus, the set of fixed points of $T$ is non-empty, that is, $T$ has at least one fixed point. Now, suppose that $z, u \in X$ are two fixed points of $T$ such that $d(z, u) = d(Tz, Tu) > 0$. Using (1), we obtain

$$\theta(d(z, u)) = \theta(d(Tz, Tu)) \leq [\theta(d(z, u))]^k < \theta(d(z, u)),$$

which is a contradiction. Hence, $u = z$. □

Consequence of Theorem 2.3 is that, if $(X, d)$ is a complete metric space and $T : X \longrightarrow X$ is a given mapping, and suppose that there exist $k \in (0, 1)$ and $\theta \in \Theta$ such that (1) hold. Then, the fixed point of $T$ is unique.
It follows immediately that the Banach contraction principle is sustained. That is, if $T$ is a Banach contraction, there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X,$$

then we have

$$e^{d(Tx,Ty)} \leq [e^{d(x,y)}]^k,$$

which is equivalent to (1) with $\theta(t) = e^t$.

## 3. Sequent of Jleli and Samet's Result

In this section, we highlight some of the important extensions of the results of Jleli and Samet [**16**]. One of the earliest generalizations was given by Jleli et al [**15**].

**3.1. Jleli, Karapınar and Samet (2014).** Consistent with Jleli and Samet [**16**], Jleli et al. [**15**] introduced a new auxillary condition to the set of function $\theta : (0, \infty) \to (1, \infty)$ satisfying:

($\Theta_4$) $\theta$ is continuous.

Subsequently, $\Omega$ denote the set of functions satisfying $\Theta_1, \Theta_2$ and $\Theta_4$.

**Theorem 3.1.** [**15**] *Let $(X, d)$ be a complete g.m.s and $T : X \longrightarrow X$ be a given map. Suppose that there exist $\theta$ satisfying $(\Theta_1) - (\Theta_4)$ and $k \in (0, 1)$, such that*

$$x, y \in X, d(Tx, Ty) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^k, \tag{7}$$

*where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

*Then $T$ has a unique fixed point.*

It is easy to check the fact that every $\theta$-contraction (1) satisfies (7). Infact, suppose (1) holds. Then, by applying condition $(\Theta_1)$, for all $x, y \in X$, we have $d(Tx, Ty) \neq 0$ implies

$$\begin{aligned}\theta(d(Tx,Ty)) &\leq [\theta(d(x,y))]^k,\\ &\leq [\theta(M(x,y))]^k,\end{aligned}$$

where $k \in (0,1)$.

**3.2. Parvaneh, Golkarmanesh, Hussain and Salimi (2016).** Parvaneh et al. [**31**] modify the auxiliary conditions by introducing a new family of function in this way:
Let $\Delta_H$ denote the set of all functions $H : \mathbb{R}^{+4} \longrightarrow [0,\infty)$ satisfying:
$(G)$ for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1t_2t_3t_4 = 0$ there exists $k \in [0,1)$ such that

$$H(t_1,t_2,t_3,t_4) = k.$$

The authors also denote by $\Delta_\theta$ the set of functions $\theta : (0,\infty) \longrightarrow [1,\infty)$ satisfying the following conditions:

- $(\Theta_1)$ $\theta$ is strictly increasing: that is, if $x < y$ then $\theta(x) < \theta(y)$ for all $x, y \in (0,\infty)$;
- $(\Theta_2)$ for all sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\theta(t_n) = 1$ if and only if $\lim_{t\to\infty}(t_n) = 0^+$;
- $(\Theta_3)$ there exist $0 < r < 1$ and $l \in (0,\infty]$ such that $\lim\limits_{t\to 0^+}\dfrac{\theta(t)-1}{t^r} = \ell$.

**Definition 3.1.** [**31**] Let $(X,d)$ be a metric space and $T$ a self mapping on $X$. Also, suppose that $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ is a function. We say that $T$ is $\alpha$-$\eta$-$H\theta$-contraction if for all $x, y \in X$ with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$, we have

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^{H(d(x,Tx),d(y,Ty),d(x,Ty)d(y,Tx))}, \tag{8}$$

where $H \in \Delta_H$ and $\theta \in \Delta_\theta$.

Obviously, by defining the mapping $H : \mathbb{R}^{+4} \longrightarrow (0,1) \subset [0,\infty)$ as $H(t) = k$ for all $t \in \mathbb{R}^{+4}$. Then (8) reduces to (1).

**Theorem 3.2.** [**31**] *Let $(X,d)$ be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

- (i) *$T$ is $\alpha$-admissible mapping with respect to $\eta$;*
- (ii) *$T$ is $\alpha$-$\eta$-$H\theta$-contraction;*
- (iii) *there exists $x \in X$ such that $\alpha(x,Tx) \geq \eta(x,Tx)$;*
- (iv) *$T$ is $\alpha$-$\eta$-continuous.*

*Then $T$ has a fixed point. Moreover, $T$ has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$.*

**Definition 3.2.** [31] A self mapping $T$ is said to have the property $P$ if $Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$.

Note that $T^n$ is the $n$-th iterate of the mapping $T$

**Theorem 3.3.** [31] *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an $\alpha$-continuous self mapping. Assume that there exists $k \in [0, 1)$ such that*

$$\theta(d(Tx, T^2x)) \leq [\theta(d(x, Tx))]^k,$$

*holds for all $x \in X$ with $d(Tx, T^2x) > 0$, where $\theta \in \Delta_\theta$. If there exists $x_0 \in X$ such that $\alpha(x, Tx) \geq 1$, then $T$ has the property $P$.*

We note that by putting $\alpha(x, Tx) = 1$ and $y = Tx$ for all $x \in X$, Theorem 3.3 becomes Theorem 2.3. However, Theorem 3.3 is a proper extension of Theorem 2.3. For example see [31].

**3.3. Onsod, Saleewong, Ahmad, Al-Mazrooei and Kumam (2016).** Onsod et al. [29] introduced a new type of contraction called $\theta$-$G$-contraction on a metric space endowed with a graph and established some new fixed point theorems.

**Definition 3.3.** [29] A graph $G$ is connected if there is a directed path between any two vertices and it is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ denotes the undirected graph obtained from $G$ by ignoring the direction of the edges.

**Definition 3.4.** [29] We denote by $\xi = \{G : G$ *is a directed graph with* $V(G) = X$ *and* $\Delta \subseteq E(G)\}$. A self mapping $T : X \to X$ is said to be a $\theta$-$G$-contraction if there exist $\theta \in \Theta$ and $G \in \xi$, such that

(i) for all $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G);$$

(ii) there exists some $k \in (0, 1)$ such that

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $Tx \neq Ty$.

Their main result is the following.

**Theorem 3.4.** [29] *Let (X,d) be a complete metric space and $T : X \to X$ be a self mapping. Then, the following statements are equivalent:*

(i) *$G$ is weakly connected;*
(ii) *for any $\theta$-$G$-contraction, the sequences $\{T^n x\}$ and $\{T^n y\}$ are Cauchy and equivalent for all $x, y \in X$;*
(iii) *for any $\theta$-$G$-contraction and a mapping $T$, we have $card(FixT) \leq 1$.*

*Note: $card(FixT)$ is the cardinality of the fixed point of $T$.*

**3.4. Liu, Chang, Xiao and Zhao (2016).** Liu et al. [23] introduced the notions of $\theta$-type Suzuki contractions and they established some new fixed point theorems for these two kinds of mappings in the setting of complete metric spaces.

We denote by $\Theta'$ the set of function $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

($\Theta'_1$) $\theta$ is non-decreasing and continuous;
($\Theta'_2$) $\inf_{t\in(0,\infty)} \theta(t) = 1$.

**Definition 3.5.** [23] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping. $T$ is said to be a $\theta$-type suzuki contraction, if there exist $k \in (0, 1)$ and $\theta \in \Theta'$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^k,$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\right\}.$$

Without much difficulties, we can verify easily that every $\theta$-contraction (1) satisfies $\theta$-type Suzuki contraction (3.5). To see this, suppose (1) holds, then by applying ($\Theta_1$) for all $x, y \in X$, we have $\frac{1}{2}d(x, Tx) < d(x, y)$ and this implies that

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k \leq [\theta(M(x, y))]^k,$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\right\}.$$

We now state the main theorem of [23].

**Theorem 3.5.** [23] *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $\theta$-type Suzuki contraction, i.e., there exist $\theta \in \Theta'$ and $k \in (0, 1)$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^k,$$

*where*

$$M(x, y) = \max\left\{d(x, y), d(x, Ty), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\right\}.$$

*Then, $T^n x$ converges to $z$.*

**3.5. Parvaneh (2017).** Parvaneh [30] introduced a new class of control functions and the concept of $\alpha$-$\theta$-generalized Hardy-Rogers contraction and established some fixed point result in the setting of $b$-metric spaces. We denote by $\nabla_\theta$ the set of functions $\theta : [0,\infty) \longrightarrow [1,\infty)$ satisfying the following condition:

(i) $\theta$ is strictly increasing;
(ii) $\limsup_{n\to\infty} \theta(x_n) = \theta(\limsup_{n\to\infty} x_n)$ for real sequence $\{x_n\} \subseteq [0,\infty)$.

**Definition 3.6.** [30] Let $(X,d)$ be a $b$-metric space and $T$ be a self-mapping on $X$. Also suppose that $\alpha : X \times X \to [0,\infty)$ be a function. We say that $T$ is an $\alpha$-$\theta$-weakly Hardy-Rogers contraction if for all $x, y \in X$ with $1 \leq \alpha(x,y)$, we have

$$\begin{aligned}&\theta(s^4 d(Tx,Ty))\\ &\leq \Big[\theta(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Tx) + \alpha_4 d(y,Ty + \alpha_5 d(x,Ty)\\ &- \psi(d(x,y), d(x,Tx), d(y,Tx), d(y,Ty), d(x,Ty)))\Big]^k,\end{aligned}$$

where $s \geq 1$ is a constant, $\psi : [0,\infty)^5 \to [0,\infty)$ is a function such that $\psi(a,b,c,d,e) = 0$ yields that a=b=c=d=e=0 and
$\liminf_{n\to\infty} \psi(a_n, b_n, c_n, d_n, e_n) = \psi(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n, \liminf_{n\to\infty} c_n, \liminf_{n\to\infty} d_n, \liminf_{n\to\infty} e_n)$,
$k \in [0,1)$, $\sum_{i=1}^4 a_i + 2sa_5 = 1$ and $\theta \in \nabla_\theta$.

**Theorem 3.6.** [30] *Let $(X,d)$ be an $\alpha$-complete $b$-metric space and $T : X \to X$ be a self-mapping such that the following assertions hold:*

(i) *$T$ is a rectangular $\alpha$-admissible mapping;*
(ii) *$T$ is an $\alpha$-$\theta$-weakly Hardy-Rogers-contraction;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(vi) *$T$ is $\alpha$-continuous.*

*Then $T$ has a fixed point. Moreover, $T$ has a unique fixed point when $\alpha(x,y) \geq 1$ for all $x, y \in Fix(T)$.*

**3.6. Hussain, Al-Mazrooei and Ahmed (2017).** Hussain et al. [8] defined $(\alpha$-$\eta)$-$\theta$ contraction to the result of Jleli and Samet [16], they also deduced certain fixed and periodic point result for obitally continuous generalized $\theta$-contraction.

**Definition 3.7.** [8] Let $(X,d)$ be a metric space and $T$ be a self mapping on $X$. Also, suppose that $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be two functions. We say that $T$ is $\alpha$-$\eta$-$\theta$-contraction if for all $x, y \in X$ with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$, we have

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k,$$

where $\theta \in \Omega$ and $k \in (0,1)$.

**Theorem 3.7.** [8] *Let $(X, d)$ be a complete metric space. Let $T : X \longrightarrow X$ be a self-mapping satisfying the following assertions:*

(i) *$T$ is $\alpha$-admissible mapping with respect to $\eta$;*
(ii) *$T$ is $\alpha$-$\eta$-$\theta$-contraction;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*
(iv) *$T$ is $\alpha$-$\eta$-continuous.*

*Then $T$ has a fixed point. Moreover, $T$ has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.*

**Theorem 3.8.** [8] *Let $(X, d)$ be a complete metric space and $T : X \longrightarrow X$ be an $\alpha$-continuous self mapping. Assume that there exists $k \in (0, 1)$ such that*

$$\theta(d(Tx, T^2x)) \leq [\theta(d(x, Tx))]^k, \tag{9}$$

*holds for all $x \in X$ with $d(Tx, T^2x) > 0$ where $\theta \in \Omega$. If $T$ is an $\alpha$-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then $T$ has a the property $P$.*

We clearly see that Theorem 3.8 and Theorem 3.3 are the same but while $\theta$ is nondecreasing in 3.8, $\theta$ is strictly increasing in Theorem 3.3. And by putting $\alpha(x, Tx) = 1$ and $y = Tx$ for all $x \in X$, their inequality both reduces to (1).

**3.7. Zheng, Cai and Wang (2017).** Zheng et al. [42] introduced the notions of $\theta$-$\phi$ contraction and define $\theta$-$\phi$ Suzuki and $\theta$-$\phi$ kannan contraction and established some new fixed point theorems for these mappings in the setting of complete metric spaces.

We denote by $\Phi$ the set of functions $\phi : [1, \infty) \to [1, \infty)$ satisfying the following conditions:

($\Phi_1$) $\phi : [1, \infty) \to [1, \infty)$ is nondecreasing;
($\Phi_2$) For each $t > 1$, $\lim_{n\to\infty} \phi^n(t) = 1$;
($\Phi_3$) $\phi$ is continuous on $[1, \infty)$.

**Lemma 3.9.** [42] *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.*

**Definition 3.8.** [42] Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping.

(1) $T$ is said to be a $\theta$-$\phi$ contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for any $x, y \in X$,
$$d(Tx, Ty) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq \phi[\theta(N(x, y))].$$
where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.
(2) $T$ is said to be a $\theta$-$\phi$ Kannan-type contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for any $x, y \in X$, $Tx \neq Ty$,
$$\theta(d(Tx, Ty)) \leq \phi\left[\theta\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right)\right].$$

(3) $T$ is said to be a $\theta$-$\phi$ Suzuki contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for any $x, y \in X$, $Tx \neq Ty$,

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \theta(d(Tx,Ty)) \leq \phi[\theta(N(x,y))],$$

where $N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Observe easily that $\theta$-$\phi$ contractions and $\theta$-$\phi$ Kannan-type contractions are $\theta$-$\phi$ Suzuki contractions.

**Theorem 3.10.** [42] *Suppose (X,d) is a complete metric space and $T : X \to X$ is $\theta$-$\phi$ suzuki contraction, i.e, there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for any $x, y \in X$, $Tx \neq Ty$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \theta(d(Tx,Ty)) \leq \phi[\theta(N(x,y))],$$

*where*

$$N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

*Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.*

It follows from the definition of $\theta$-contraction and Theorem 3.10, that fixed point theorems for $\theta$-$\phi$ contraction and $\theta$-$\phi$ Kannan-type contraction can be obtained. See [42]

**3.8. Minak and Altun (2017).** Minak and Altun introduced the concept of ordered $\theta$-contraction on ordered metric spaces, where they obtained some fixed point theorems and show some examples.

**Definition 3.9.** [25] We say that X is regular if for an ordered metric space $(X, \preceq, d)$, $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in $X$, then $x_n \preceq x$ for all $n$.

**Definition 3.10.** [25] Let $(X, \preceq, d)$ be an ordered metric space. Let $T : X \to X$ be a mapping and $\theta \in \Theta$. We say $T$ is an ordered $\theta$-contraction if there exists $k \in (0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k, \text{ such that}$$

for all $x, y \in S$, where

$$S = \{(x,y) \in X \times X : x \preceq y, d(Tx,Ty) > 0\}.$$

**Theorem 3.11.** [25] *Let $(X, \preceq, d)$ be an ordered complete metric space and let $T : X \to X$ be an ordered $\theta$-contraction. Suppose that $T$ is a non-decreasing mapping and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If $T$ is continuous or regular, then $T$ has a fixed point.*

**Corollary 3.12.** [25] *Let $(X, \preceq, d)$ be an ordered complete metric space and let $T : X \longrightarrow X$ be a non-decreasing mapping such that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Suppose that there exists $L \in (0,1)$ such that*

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)} \leq L < 1,$$

*for all $(x,y) \in S$,*
*where*

$$S = \{(x,y) \in X \times X : x \preceq y, d(Tx,Ty) > 0\}.$$

*If $T$ is continuous or is regular, then $T$ has a fixed point in $X$.*

**3.9. Zheng and Wang (2017).** Zheng and Wang [44] used $\theta$-$\phi$-contraction to answers the open question see [4] of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the continuity of $T^2$ at the fixed point.
Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Definition 3.11.** [44] Let $(X, d)$ be a metric space and $T : X \to X$ be a self mapping. $T$ is said to be weak $\theta$-$\phi$-contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for any $x, y \in X, T^2x \neq T^2y$,

$$\theta(d(T^2x, T^2y)) \leq \phi[\theta(N(Tx,Ty))],$$

where

$$N(Tx,Ty) = \max\{d(Tx,Ty), d(Tx,T^2x)d(Ty,T^2y), \frac{1}{2}d(Tx,T^2y) + d(Ty,T^2x)\}.$$

**Theorem 3.13.** [44] *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a weak $\theta$-$\phi$-contraction, i.e, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X, T^2x \neq T^2y$,*

$$\theta(d(T^2x, T^2y)) \leq \phi[\theta(N(Tx,Ty))],$$

*where*

$$N(Tx,Ty) = \max\{d(Tx,Ty), d(Tx,T^2x)d(Ty,T^2y), \frac{1}{2}(d(Tx,T^2y) + d(Ty,T^2x))\}.$$

*Then, $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^nx\}$ converges to $x^*$ for every $x \in X$. Moreover, $T$ is discontinuous at $x^*$ iff $\lim_{x\to x^*} N(Tx,x^*) = \lim_{x\to x^*} N(Tx,Tx^*) \neq 0$.*

**3.10. Imdad, Alfaqih and Khan (2018).** Imdad et al. [10] extended the work of Jleli and Samet [16] by relaxing $\Theta_1$ thereby defining weak $\theta$-contraction and proved some fixed point results.

**Proposition 3.14.** [10] *Let $(X, d)$ be a metric space and $T : X \longrightarrow X$ be a mapping. If $T$ satisfies (1) for some $\theta \in \Theta_2$ and $k \in (0,1)$, then, $T$ is continuous.*

**Remark 3.12.** [10] Observe that condition $\Theta_1$ can be withdrawn, and Theorem 2.3 still survives.

The above proposition and remark was used to define weak $\theta$-contraction. The definition is stated below.

**Definition 3.13.** [10] Let $(X, d)$ be a metric space and $T : X \longrightarrow X$. We say that $T$ is a weak $\theta$-contraction if there exist $\theta \in (\Theta_2, \Theta_3)$ or $\theta \in (\Theta_2, \theta_4)$ and $k \in (0, 1)$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

for all $x, y \in X$.

**Theorem 3.15.** [10] *Every weak $\theta$-contraction on a complete metric space is a Picard operator.*

**Theorem 3.16.** *Let $(X, d)$ be a complete metric space and $T : X \longrightarrow X$ be a given mapping. If there exists $n \in \mathbb{N}$ such that $T^n$ is a weak $\theta$-contraction, then $T$ is a Picard operator.*

**3.11. Hu, Zheng and Zhou (2018).** Hu et al. used the notion of $\theta$-$\phi$ contraction on partial metric spaces and establish some new fixed point theorems. They also gave example to illustrate their result.
Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Definition 3.14.** [7] Let $(X, p)$ be a partial metric space and let $T : X \longrightarrow X$ be a mapping.

(i) $T$ is said to be a $\theta$-$\phi$ contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that, for any $x, y \in X$,

$$\theta(p(Tx, Ty)) \leq \phi[\theta(p(x, y))].$$

(ii) $T$ is said to be $\theta$-$\phi$ Kannan-type contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that, for any $x, y \in X, Tx, \neq Ty$,

$$\theta(p(Tx, Ty)) \leq \phi\left[\theta\left(\frac{p(x, Tx) + p(y, Ty)}{2}\right)\right].$$

**Theorem 3.17.** [7] *Suppose $(X, p)$ is a complete partial metric space and $T : X \longrightarrow X$ is a $\theta$-$\phi$ contraction. Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.*

**Theorem 3.18.** [7] *Let $(X, p)$ is a complete partial metric space and suppose $T : X \longrightarrow X$ is a $\theta$-$\phi$ Kannan-type contraction. Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$*

**3.12. Zheng (2018).** Zheng [41] introduced the notion of generalized $\theta$-$\phi$ contraction and established some new fixed point theorems for this contraction in the setting of complete $G$-metric spaces.
Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Definition 3.15.** [41] Let (X,G) be a $G$-metric space. A mapping $T : X \longrightarrow X$ is said to be a generalized $\theta$-$\phi$ contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$, such that, for any $x, y, z \in X$,

$$G(Tx, Ty, Tz) \neq 0 \Rightarrow \theta(G(Tx, Ty, Tz)) \leq \phi[\theta(N(x, y, z))],$$

where

$$\begin{aligned} N(x, y, z) = \max\Big\{ & G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ & \frac{1}{2}G(x, Ty, Ty), \frac{1}{2}G(y, Tz, Tz), \frac{1}{2}G(z, Tx, Tx), \\ & \frac{1}{3}\left(G(x, Tx, Ty), +G(y, Tz, Tz) + G(z, Tx, Tx)\right) \Big\}. \end{aligned}$$

Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.

**Theorem 3.19.** [41] *Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a generalized $\theta$-$\phi$ contraction. Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.*

**Corollary 3.20.** [41] *Let $(X, G)$ be a complete $G$-metric space and let $T : X \longrightarrow X$ be a self mapping. Assume that there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that, for all $x, y \in X$.*

$$G(Tx, Ty, Ty) \neq 0 \Rightarrow \theta(G(Tx, Ty, Ty)) \leq \phi[\theta(G(x, y, y))].$$

*Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.*

**3.13. Chaipornjareansri (2018).** Chaipornjareansri [6] introduced the notion of generalized $\theta$-$\phi$-contraction in the setting of $S$-metric spaces.
Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Lemma 3.21.** [6] *Let $(X, S)$ be an $S$-metric space. Then, $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

**Remark 3.16.** [6] Let $(X, S)$ be an $S$-metric space

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z),$$

then

$$\frac{1}{3}S(x, x, z) \leq \max\{S(x, x, y), S(y, y, z)\}$$

for all $x, y, z \in X$

**Definition 3.17.** [6] Let $(X, S)$ be a $S$-metric space. A mapping $T : X \longrightarrow X$ is said to be a generalized $\theta$-$\phi$-contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that, for any $x, y \in X$,

$$S(Tx, Ty, Tz) \neq 0 \Rightarrow \theta(S(Tx, Ty, Tz)) \leq \phi[\theta(N(x, y, z))],$$

where

$$N(x, y, z) = \max\Bigg\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), \frac{1}{3}S(x, x, Ty), \frac{1}{3}S(y, y, Tz), \frac{1}{3}S(z, z, Tx), \frac{1}{6}\Bigg(S(x, x, Ty) + S(y, y, Tz) + S(z, z, Tx)\Bigg)\Bigg\}.$$

**Theorem 3.22.** *Let $(X, S)$ be a complete $S$-metric space and $T : X \longrightarrow X$ be a generalized $\theta$-$\phi$-contraction. Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.*

**Corollary 3.23.** *Let $(X, S)$ be a complete $S$-metric space and $T : X \longrightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$S(Tx, Tx, Ty) \neq 0 \Rightarrow \theta(S(Tx, Tx, Ty)) \leq \phi[\theta(S(x, x, y))].$$

*Then $T$ has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to $x^*$ for every $x \in X$.*

**3.14. Abduljawad, Agarwal, Karapinar and Kumari (2019).** Abdeljawad et al. [1] defined three (3) new notions: $\Theta_e$-contraction, a Hardy-Rogers-type $\Theta$-contraction and interpolative $\Theta$-contraction in the framework of extended $b$-metric space. Some fixed point result via these new notions was also studied.

**Definition 3.18.** A self mapping $T$ on an extended $b$-metric space $(X, d_e)$, is named a $\Theta_e$-contraction if there exists a function $\theta$ satisfying $(\Theta_1)$-$(\Theta_4)$ such that

$$\theta(d_e(Tx, Ty)) \leq [\theta(d_e(x, y))]^k \text{ if } d_e(Tx, Ty) \neq 0 \text{ for } x, y \in X,$$

where $k \in [0, 1)$ such that $\limsup\limits_{m,n\to\infty} T(x_n, x_m) < \dfrac{1}{k}$, here, $x_n = T^n x_0$ for $x_0 \in X$.

**Theorem 3.24.** *If a self-mapping $T$ defined on a complete extended $b$-metric space $(X, d_e)$ forms a $\Theta_e$-contraction, then, $T$ has a unique fixed point in $X$.*

**Definition 3.19.** A self-mapping $T$ on an extended $b$-metric space $(X, d_e)$, is called hardy-Rogers-type $\Theta$-contraction ($HR$-$\Theta$-contraction), if there exists a function $\theta$ satisfying $(\Theta_1)$-$(\Theta_4)$ and non-negative real number $k < 1$ such that

$$\theta(d_e(Tx, Ty)) \leq [M_{T,\theta}(x, y)]^k,$$

for all $x, y \in X$, where

$$M_{T,\theta} = \max\left\{\theta(d_e(x,y)), \theta(d_e(x,Tx)), \theta(d_e(y,Ty)), \theta\left(\frac{(d_e(x,Ty)) + d_e(y,Tx)}{2}\right)\right\},$$

where $\limsup_{m,n\to\infty} T(x_n, x_m) < \frac{1}{k}$, here, $x_n = T^n x_0$ for $x_0 \in X$ and $k \in (0,1)$.

**Theorem 3.25.** *If a self-mapping $T$ on a complete extended b-metric space $(X, d_e)$ forms on $HR$-$\Theta$-contraction, then the fixed point of $T$ in $X$ is unique.*

**3.15. Altun, Hussain, Qasim and Al-Sulami (2019).** Altun et al. [2] presented a new generalization of the Perov fixed point theorem on vector-valued metric space. They presented both a nontrivial comparative example and an application to semilinear operator system about the existence of solution.

**Definition 3.20.** [2] Let $(X, d)$ be a vector-valued metric space and $T : X \longrightarrow X$ be a map. If there exist $\theta \in \Theta$ and $k = (k_i)_{i=1}^m \in \mathbb{R}_+^m$ with $k_i < 1$ for all $i \in \{1, 2, \dots m\}$ such that

$$\theta(d(Tx, Ty)) \preceq [\theta(d(x, y))]^k,$$

for all $x, y \in X$ with $d(Tx, Ty) \succ 0$, then $T$ is called a Perov type $\theta$-contraction.

**Theorem 3.26.** [2] *Let $(X, d)$ be a complete vector-valued metric space and $T : X \longrightarrow X$ be a perov type $\theta$-contraction, then $T$ has a unique fixed point.*

**3.16. Parvaneh, Hussain, Mukheimer and Aydi (2019).** Parvaneh et al. [32] introduced some changes to the function by denoting $\Theta'$ the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ such that

- $(\Theta'_1)$ $\theta$ is continuous and strictly incresing;
- $(\Theta'_2)$ for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{t\to\infty}(t_n) = 0^+$.

**Definition 3.21.** [32] Let $T : X \longrightarrow X$ be a mapping on a metric space $(X, d)$. such $T$ is said to be $\mathfrak{p}$-contraction, whenever there are $\theta \in \Theta'$ and $\tau_1, \tau_2, \tau_3, \tau_4 \geq 0$ with $\tau_1 + \tau_2 + \tau_3 + \tau_4 < 1$ such that the following holds:

$$\theta(d(Tx, Ty)) \leq (\theta(d(x, y)))^{\tau_1} (\theta(d(x, Tx)))^{\tau_2} (\theta(d(y, Ty)))^{\tau_3} \left(\theta\left(\frac{(d(x, Ty) + d(y, Tx))}{2}\right)\right)^{\tau_4},$$

for all $x, y \in X$.

**Theorem 3.27.** [32] *Each $\mathfrak{p}$-contraction mapping on a complete metric space has a unique fixed point.*

**Remark 3.22.** [32] In Theorem 3.34, we can substitute the continuity of $\theta$ by the continuity of $T$

**Corollary 3.28.** [32] *By setting $\theta(t) = e^{\sqrt{t}}$, let $T : X \longrightarrow X$ be a mapping on a complete metric space $(X, d)$ such that the following holds:*

$$\sqrt[n]{d(Tx,Ty)} \leq \tau_1 \sqrt[n]{d(x,y)} + \tau_2 \sqrt[n]{d(x,Tx)} + \tau_3 \sqrt[n]{d(y,Ty)} + \tau_4 \sqrt[n]{\frac{(d(x,Ty) + d(y,Tx)}{2}},$$

*for all $x, y \in X$, where $\theta \in \mathfrak{p}$ and $\tau_1, \tau_2, \tau_3, \tau_4 \geq 0$ such that $\tau_1 + \tau_2 + \tau_3 + \tau_4 < 1$. Then $T$ has a unique fixed point.*

**3.17. Hussain and Adeel (2019).** Hussain and Adeel [9] point out some errors and omissions in the work of Parvaneh et al. [31]. Some examples where also given to substantiate their result.

**Theorem 3.29.** [9] *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the following assertions:*

*(i) $T$ is $\alpha$-admissible mapping;*
*(ii) $T$ is $\alpha$-$H\theta$-contraction;*
*(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iv) $T$ is $\alpha$-continuous function.*

*Then $T$ has a fixed point. Moreover, $T$ has a unique fixed point when $1 \leq \alpha(x, y)$ for all $x, y \in Fix(T)$.*

**Theorem 3.30.** [9] *Let $(X, d)$ be a complete metric space and $T : X \to X$ be $\alpha$-continuous self mapping. assume that there exist $k \in [0, 1)$ such that*

$$\theta(d(Tx, T^2x)) \leq [\theta(d(x, Tx))]^k,$$

*holds for all $x \in X$ with $d(Tx, T^2x) > 0$, where $\theta \in \Theta$. If $T$ is an $\alpha$-continuous self mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then $T$ has the property $P$.*

**3.18. Zhenhua, Hussain, Adeel, Hussain and Savas (2019).** The authors [27] introduced the notion of Ćirić type $\alpha$-$\psi$-$\theta$-contraction and prove best proximity result in the context of complete metric spaces and partially ordered complete metric spaces.

We denote by $\Psi$ the functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

($\psi_1$) $\psi$ is nondecreasing;
($\psi_2$) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^n$ is the nth iterate of $\psi$ and $\psi(t) < t$ for any $t > 0$.

**Definition 3.23.** [27] Let $A, B$ be two subsets of a metric space $(X, d)$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be a function. A mapping $F : A \longrightarrow B$ is said to be Ćirić type $\alpha$-$\psi$-$\theta$-contraction if for all $\psi \in \Psi$, $\theta \in \Omega$, there exists $k \in (0,1)$ and for all $x, y \in X$ with $\alpha(x,y) \geq 1$ and $d(Tx,Ty) > 0$, we have

$$\alpha(x,y)\theta[d(Tx,Ty)] \leq [\psi(\theta(m(x,y)))]^k$$

where

$$m(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2} - d(A,B), \frac{d(x,Ty)+d(y,Tx)}{2} - d(A,B)\right\}.$$

**Theorem 3.31.** [27] *Let $A$ and $B$ be two closed subsets of a complete metric space $(X,d)$ with $A_0 \neq \emptyset$ and let $T : A \longrightarrow B$ be a Ćirić type $\alpha$-$\psi$-$\theta$-contraction satisfying*

(i) *$T$ is $\alpha$-proximal admissible;*
(ii) *$T(A_0) \subseteq B_0$ and the pair $(A,B)$ satisfies the weak $P$-property;*
(iii) *$T$ is continuous;*
(iv) *there exist $x_0, x_1 \in A_0$ with $d(x_1, Tx_0) = d(A,B)$ such that $\alpha(x_0,x_1) \geq 1$.*

*Then, there exists $u \in A$ such that $d(u,Tu) = d(A,B)$.*

**3.19. Yesilkaya and Aydin (2020).** Yesilkaya and Aydin [39] introduced the fixed point theorem for $\theta$-expansive mapping on ordered metric spaces.

**Definition 3.24.** [39] Let $(X, \preceq, d)$ be an ordered metric space. A mapping $T : X \to X$ is said to be surjective $\theta$-expansive if there exists $\theta \in \Theta$ and $\eta > 1$ such that

$$\theta(d(Tx,Ty)) \geq [\theta(d(x,y))]^\eta,$$

for all $(x,y) \in S$, where

$$S = \{(x,y) \in X \times X : x \preceq y, d(Tx,Ty) > 0\}.$$

**Theorem 3.32.** [39] *Let $(X, \preceq, d)$ be an ordered complete metric space, $T : X \to X$ is surjective $\theta$-expansive mapping and $T^*$ is $\preceq$-increasing. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq T^* x_0$. If $T$ is continuous or $X$ is regular, then $T$ has a fixed point.*

**Theorem 3.33.** [39] *Let $(X,d)$ be a complete metric space. Let $A$ and $B$ be weakly compatible self mappings of $X$ and $B(X) \subseteq A(X)$. Suppose that $\theta \in \Theta$ and there exists a constant $\eta \geq 1$ such that*

$$\theta(d(Ax,Ay)) \geq [\theta(d(Bx,By))]^\eta,$$

*for all $x, y \in X$. If one of the subspaces $B(X)$ or $A(X)$ is complete, then $A$ and $B$ have a unique common fixed point in $X$.*

**3.20. Imdad, Ali, Alfaqih, Sessa and Aldurayhim (2020).** Imdad et al. [**11**] introduced two new classes of auxiliary functions and utilizing the same to define $(\theta,\psi)_{\mathcal{R}}$-weak contractions. Also, they prove some fixed point theorems in the setting of relational metric spaces. Some examples where also given to substantiate their result.

**Definition 3.25.** [**11**] Let $\mho$ be the collection of all $\theta : (0,\infty) \longrightarrow (1,\infty)$ that satisfy the following conditions:

- $(\mho_1)$ for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$, $\lim_{n\to\infty}(x_n) = 0$ if and only if $\lim_{t\to\infty}\theta(x_n) = 1$;
- $(\mho_2)$ $\theta$ is lower semicontinuous.

**Definition 3.26.** [**11**] Let $\Psi$ be the collection of all $\psi : (0,\infty) \longrightarrow (1,\infty)$ that satisfies the following conditions:

- $(\Psi_1)$ for every sequence $\{x_n\} \subseteq (0,\infty)$, $\lim\limits_{n\to\infty} x_n = 0$ iff $\lim\limits_{n\to\infty} \psi(x_n) = 1$;
- $(\Psi_1)$ $\psi$ is right upper semicontinuous.

**Definition 3.27.** [**11**] Let $(X,d)$ be a metric space, $\mathcal{R}$ a binary relation on $X$ and $T : X \to X$. Then, $T$ is called a $(\theta,\psi)_{\mathcal{R}}$-weak contraction if there exist $k \in (0,1)$, $\theta \in \mho$ and $\psi \in \Psi$ with $\theta(t) \geq \psi$ for all $t > 0$ such that

$$\theta(d(Tx,Ty)) \leq (\psi(m(x,y)))^k,$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,T(x)), d(y,T(y)), \frac{d(x,T(y)) + d(y,T(x))}{2}\right\}.$$

for all $x,y \in X$ with $x\mathcal{R}y$ and $T(x) \neq T(y)$.

**Theorem 3.34.** [**11**] *Let $(X,d)$ be a metric space endowed with a transitive binary relation $\mathcal{R}$ and assume that:*

- (i) *$T$ is $\mathcal{R}$-complete;*
- (ii) *there exists $x_0 \in X$ such that $x_0\mathcal{R}T(x_0)$;*
- (iii) *$\mathcal{R}$ is $T$-closed;*
- (iv) *$T$ is $(\theta,\psi)_{\mathcal{R}}$-weak contraction;*
- (v) *$T$ is $\mathcal{R}$-continuous.*

*Then, $T$ has a fixed point.*

**Theorem 3.35.** [**11**] *The conclusion of Theorem 3.34 holds true if assumption $(v)$ is replaced by:*

- (v') *$\mathcal{R}$ is d-self-closed.*

**3.21. Kari, Rossafi, Marhrani and Aamri (2020).** Kari et al. [22] presented the notion of $\theta$-$\phi$-contraction in $b$-rectangular metric spaces inspired by the concept of $\theta$-$\phi$-contraction in metric spaces introduced by Zheng et al. [42] and study the existence and uniqueness of fixed point for the mappings in the space. Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Definition 3.28.** [22] Let (X,d) be a $b$-rectangular metric space with parameter $s > 1$ and $T : X \longrightarrow X$.

(1) $T$ is said to be a $\theta$-contraction if there exist $\theta \in \Omega$ and $k \in (0, 1)$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^2 d(Tx, Ty)] \leq \theta[m(x, y)]^k,$$

where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}.$$

(2) $T$ is said to be a $\theta$-$\phi$-contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^2 d(Tx, Ty)] \leq \phi[\theta(m(x, y))],$$

where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}.$$

(3) $T$ is said to be a $\theta$-$\phi$-Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that $d(Tx, Ty) > 0$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right)\right].$$

(4) $T$ is said to be a $\theta$-$\phi$-Reich-type contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that $d(Tx, Ty) > 0$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}\right)\right].$$

**Theorem 3.36.** [22] *Let $(X, d)$ be a complete b-rectangular metric space and let $T : X \longrightarrow X$ be a $\theta$-contraction, i.e, there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^2 d(Tx, Ty)] \leq [\theta(m(x, y))]^k.$$

*Then $T$ has a unique fixed point.*

There are other theorems given by Kari et al. [22] with respect to the different definitions of contractions on $b$-rectangular metric space given above.

**3.22. Rossafi, Kari, Marhrani and Aamri (2021).** Rossafi et al. [34] presented the notion of $\theta$-$\phi$-expansive mapping in complete rectangular metric spaces and studied various fixed point theorems for such mapping. Their work extend many exiting results in the literature.
Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Definition 3.29.** [34] Let (X,d) be a rectangular metric space and $T : X \longrightarrow X$ be a given mapping. $T$ is said to be generalized $\theta$-$\phi$-expansive mapping if there exists two functions $\theta \in \Omega$ and $\phi \in \Phi$ such that

$$M(x,y) > 0 \Rightarrow \phi[\theta(d(Tx,Ty))] \geq \theta(M(x,y)), \text{ for all } x,y \in X,$$

where

$$M(x,y) = \min\{d(x,y), d(x.Tx), d(y,Ty), d(x,Ty)\}.$$

**Theorem 3.37.** [34] *Let (X,d) be a $(\alpha,\eta)$-complete generalized metric space and $T : X \longrightarrow X$ be a bijective generalized $\theta$-$\phi$-expansive mapping satisfying the following conditions:*

(i) *$T^{-1}$ is a triangular $(\alpha,\eta)$-admissible mapping;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$ or $\eta(x_0, T^{-1}x_0) \leq 1$;*
(iii) *$T$ is a $(\alpha,\eta)$-continuous.*

*Then, $T$ has a fixed point.*

**3.23. Perveen, Alfaqih, Sessa and Imdad (2021).** Perveen et al. [33] introduced the notion of $\theta^*$-weak contraction which was used to prove some fixed point results.
Let $\hat{\theta}$ be the set of all functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ satisfying the following condition:

($\hat{\theta}_2$) for every sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0$.

**Definition 3.30.** [33] Let $(X,d)$ be a metric space. A self mapping $T$ on $X$ is said to be a $\theta^*$-weak contraction if there exist $k \in (0,1)$ and $\theta \in \hat{\theta}$, such that

$$x,y \in X, d(Tx,Ty) > 0 \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(M(x,y))]^k,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Now, we state the main theorem of their result.

**Theorem 3.38.** [33] *Let $(X,d)$ be a complete metric space and $T : X \longrightarrow X$, a $\theta^*$-weak contraction. If $\theta$ is continuous, then*

(a) *$T$ has a unique fixed point (say $Z^* \in X$);*
(b) $\lim_{n\to\infty} T^n x = z^*, \forall z \in X$.

*Moreover, $T$ is continuous at $z^*$ if and only if* $\lim_{z\to z^*} d(z,z^*) = 0$.

**3.24. Javed, Arshad, Baazeem and Mlaiki (2022).** Javed et al. [12] introduced the notion of $\mathcal{R}\alpha$-$\theta$-contractions ($\alpha_{\mathcal{R}}$-$\theta_{\mathcal{R}} - contractions$) and prove some fixed point theorem in the sense of $\mathcal{R}$-complete metric spaces.

**Definition 3.31.** [12] Let $(X, d)$ be an $\mathcal{R}$-metric space and $T_{\mathcal{R}} : X \longrightarrow X$ be a mapping. We say that $T_{\mathcal{R}}$ is an $\alpha_{\mathcal{R}}$-$\theta_{\mathcal{R}}$-contraction if there exists $k \in (0, 1)$ and two functions $\alpha_{\mathcal{R}} : X \times X \longrightarrow [0, \infty)$ and $\theta \in \Theta$ such that

$$d(Tx, Ty) \neq 0 \Rightarrow \alpha_{\mathcal{R}}(x, y)\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k, \forall x, y \in X \textit{ with } x\mathcal{R}y.$$

Note that if $\alpha_{\mathcal{R}}(x, y) = 1$, (1) is satisfied.

**Definition 3.32.** [12] Let $T_{\mathcal{R}} : X \longrightarrow X$ and $\alpha_{\mathcal{R}} : X \times X \longrightarrow [0, \infty)$ be two functions. We say that $T_{\mathcal{R}}$ is $\alpha_{\mathcal{R}}$-admissible if for all $x, y \in X$ with $x\mathcal{R}y$,

$$\alpha_{\mathcal{R}}(x, y) \geq 1 \Rightarrow \alpha_{\mathcal{R}}(T_{\mathcal{R}}x, T_{\mathcal{R}}y) \geq 1.$$

We now state the main theorem as given by Javed et al [12].

**Theorem 3.39.** [12] *Let $(X, \mathcal{R}, d)$ be an $\mathcal{R}$-complete metric space and $T_{\mathcal{R}}$ be a self mapping, $\mathcal{R}$-preserving, $\mathcal{R}$-continuous and $\alpha_{\mathcal{R}} : X \times X \longrightarrow [0, \infty)$ be a function. Suppose that the below circumstances fulfill:*

(i) *suppose there exist $k \in (0, 1)$ and a function $\theta \in \Theta$ such that for all $x, y \in X$ with $x\mathcal{R}y$,*

$$d(Tx, Ty) \neq 0 \Rightarrow \alpha_{\mathcal{R}}(x, y)\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k;$$

(ii) *$T_{\mathcal{R}}$ is $\alpha_{\mathcal{R}}$-admissible;*
(iii) *$T_{\mathcal{R}}$ is $\mathcal{R}$-continuous;*
(iv) *there exists $x_0 \in X$ such that $x_0\mathcal{R}T_{\mathcal{R}}x_0$ and $\alpha_{\mathcal{R}}(x_0, T_{\mathcal{R}}x_0) \geq 1$.*

*Then $T_{\mathcal{R}}$ has a unique fixed point.*

**3.25. Onsod, Kumam and Saleewong (2023).** Onsod et al. [28] introduced the idea of Suzuki-Geraghty type $\theta$-contraction and they established some fixed point theorems in the setting of complete partial metric spaces.
We denote by $\mathfrak{X}$ the family of functions $\beta : [0, \infty) \longrightarrow [0, 1)$ which satisfies the following condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} (t_n) = 0.$$

Also, we denote by $\Theta'$ the set of function $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

($\Theta'_1$) $\theta$ is non-decreasing and continuous;
($\Theta'_2$) $\inf_{t\in(0,\infty)} \theta(t) = 1$.

The authors in [28] modify the definition of Suzuki contraction, Gerathty contraction and $\theta$-contraction as given bellow.

**Definition 3.33.** [28] Let $(X, d)$ be a partial metric space. A mapping $T : X \longrightarrow X$ is said to be Suzuki-Geraghty type $\theta$-contraction, if there exist $k \in (0,1), \theta \in \Theta'$ and $\beta \in \mathfrak{X}$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(\beta(d(x,y))d(x,y))]^k.$$

**Theorem 3.40.** [28] *Let $(X, d)$ be a complete partial metric space and $T : X \to X$ be a Suzuki-Geraghty type $\theta$-contraction, if there exist $k \in (0,1), \theta \in \theta'$ and $\beta \in \mathfrak{X}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(\beta(d(x,y))d(x,y))]^k.$$

*Then $T$ has a unique fixed point $z \in X$.*

**Definition 3.34.** [28] Let $(X, d)$ be a partial metric space. A mapping $T : X \to X$ is said to be Geraghty type $\theta$-contraction. If there exist $k \in (0,1), \theta \in \theta'$ and $\beta \in \mathfrak{X}$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$d(Tx,Ty) > 0 \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(\beta(d(x,y))d(x,y))]^k.$$

**Theorem 3.41.** [28] *Let $(X, d)$ be a complete partial metric space and $T : X \to X$ be a Geraghty type $\theta$-contraction. If there exist $k \in (0,1), \theta \in \tilde{\theta}$ and $\beta \in \mathfrak{X}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$d(Tx,Ty) > 0 \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(\beta(d(x,y))d(x,y))]^k.$$

*Then $T$ has a unique fixed point $z \in X$.*

**3.26. Rossafi, Kari, Park and Lee (2023).** Rossafi et al. [35] define $\theta$-$\phi$-contraction on $b$-metric space which is an extension of $\theta$-$\phi$-contraction introduced by Zheng et al. [42]
Based on the functions $\theta \in \Omega$ and $\phi \in \Phi$, we give the following definition.

**Definition 3.35.** [35] Let $(X, d)$ be a $b$-metric space with parameter $s > 1$ and $T : X \to X$ be a mapping.

(i) $T$ is said to be a $\theta$-contraction if there exist $\theta \in \Omega$ and $k \in (0,1)$ such that

$$d(Tx,Ty) > 0 \Rightarrow \theta[s^3 d(Tx,Ty)] \leq \theta[m(x,y)]^k,$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2s^2}\right\}.$$

(ii) $T$ is said to be a $\theta$-$\phi$-contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that

$$d(Tx,Ty) > 0 \Rightarrow \theta[s^3 d(Tx,Ty)] \leq \phi[\theta(m(x,y))],$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2s^2}\right\}.$$

(iii) $T$ is said to be a $\theta$-$\phi$-Kannan-type contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta[s^3 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right)\right].$$

(iv) $T$ is said to be a $\theta$-$\phi$-Reich-type contraction if there exist $\theta \in \Omega$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta[s^3 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}\right)\right].$$

We now state the main theorem of their work.

**Theorem 3.42.** [35] *Let $(X, d)$ be a complete b-metric space and $T : X \longrightarrow X$ be a $\theta$-contraction, i.e, there exist $\theta \in \Omega$ and $k \in (0, 1)$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3(d(Tx, Ty))] \leq \theta[m(x, y)]^k. \tag{10}$$

*Then $T$ has a unique fixed point.*

Rossafi et al. presented the following results as an immediate consequence of their main theorem.

**Corollary 3.43.** [35] *Let $(X, d)$ be a complete b-metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Omega$ and $k \in (0, 1)$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3(d(Tx, Ty))] \leq [\theta(d(x, y))]^k.$$

*Then $T$ has a unique fixed point.*

## 4. Conclusion

In this manuscript, some important transformation results of $\theta$-contraction as introduced by Jleli and Samet [16] are surveyed. Several authors have generalized this result in different directions. For Some of the results, the auxiliary conditions for the set of functions $\Theta$ are relaxed while in other cases, additional auxiliary conditions are introduced. We have observed therefore that the basic idea in developing the concepts of $\theta$-contractions and related fixed point results involved either modifying the auxiliary functions, improving the contractive inequalities or changing the ambient space.

## Competing Interests

The authors declare that they have no competing interests.

**Acknowledgment**

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and fruitful comments to improve this manuscript.

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*Received* : *June* 2023
*Accepted* : *September* 2023