Mathematical Analysis and its Contemporary Applications
Volume 5, Issue 3, 2023, 23–33
doi: 10.30495/maca.2022.1965161.1063
ISSN 2716-9898

n-projective modules in n-abelian category

Feysal Hassani, Samira Hashemi, and Rasul Rasuli*

Abstract. In this paper, we introduce and clarify a new presentation between the n-exact sequence and the n-projective module. Also, we obtain some new results about them.

1. Introduction

Category theory formalizes mathematical structures and their concepts in terms of a labelled directed graph called a category, whose nodes are called objects, and their edges called arrows (or morphisms). This category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. The language of category theory has been employed to formalize concepts of other high-level abstractions such as sets, rings, and groups. Several terms were utilized in category theory, including "morphism" which is used differently from their usage in the rest of mathematics. In category theory, morphisms obey specific conditions of theory. Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations in 1942-45 in their study of algebraic topology, to understand the processes that preserve the mathematical structure. Category theory has practical applications in programming language theory, for example, the usage of monads in functional programming. It may also be used as an axiomatic foundation for mathematics, as an alternative to set theory and other proposed foundations. In mathematics, an abelian category is a category in which morphisms and objects can be added and in which kernels and cokernels exist and have desirable properties. The motivating prototype example of

2010 *Mathematics Subject Classification.* 46A22, 12G05 , 46M18, 18Gxx.

Key words and phrases. n-ker and n-coker,n-exact sequence, n-abelian category, n-pushout diagram, n-pullback diagram

*Corresponding author

an abelian category is the category of abelian groups, Ab. The theory originated to unify several cohomology theories by Alexander Grothendieck and independently in the slightly earlier work of David Buchsbaum. Abelian categories are very stable categories. For example, they are regular and satisfy the snake lemma. The class of Abelian categories is closed under several categorical constructions, for instance, the category of chain complexes of an Abelian category, or the category of functors from a small category to an Abelian category are Abelian as well. These stability properties make them inevitable in homological algebra and beyond. This theory has significant applications in algebraic geometry, cohomology, and pure category theory. The Abelian categories are named after Niels Henrik Abel. An exact sequence is a concept in mathematics, especially in group theory, ring, and module theory, homological algebra, as well as in differential geometry. An exact sequence is a sequence, either finite or infinite, of objects and morphisms between them such that the image of one morphism equals the kernel of the next. Homological algebra is the branch of mathematics that studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincare and David Hilbert. The development of homological algebra has closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. One quite useful and ubiquitous concept in mathematics is that of chain complexes, which can be studied both through their homology and cohomology. Homological algebra affords the means to extract information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. A powerful tool for doing this is provided by spectral sequences. From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline that draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes. This paper is organized as follows. The authors [**6, 7, 8**] investigated some properties of n-algebraic structures.

In this paper, we show to prove the important theorems of n-projective modules . Finally, we recall the definition of n-projective module, and we give an open problem about some theorems of n-projective modules.

2. Preliminaries

In this paper all rings R in this paper are assumed to have an identity element 1 (or unit) (where $r1 = r = 1r$ for all $r \in R$). We do not insist that $1 \neq 0$; however, should $1 = 0$, then R is the zero ring having only one element. let $\mathcal{C}$ be an additive category of left R-modules, and P be a projective left R-module. In this section, we recall some of the fundamental concepts and definitions, which are necessary for this paper. For details, we refer to [4,6,7,9,10,11].

Definition 2.1. A category $\mathcal{C}$ is abelian if

(1) $\mathcal{C}$ has a zero object.

(2) For every pair of objects there is a product and a sum.

(3) $\mathcal{C}$ Every map has a kernel and cokernel.

(4) $\mathcal{C}$ Every monomorphism is a kernel of a map.

(5) $\mathcal{C}$ Every epimorphism is a cokernel of a map.

Definition 2.2. Let $\mathcal{C}$ be an additive category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. A weak cokernel of f is a morphism $g : B \longrightarrow C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{\hat{g}} \mathcal{C}(B, C') \xrightarrow{\hat{f}} \mathcal{C}(A, C')$$

is exact. Equivalently, g is a weak cokernel of f if $fg = 0$ and for each morphism $h : B \longrightarrow C'$ such that $fh = 0$ there exists a (not necessarily unique) morphism $p : C \longrightarrow C'$ such that $h = gp$. These properties are subsumed in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "g"] \arrow[d, "\forall h"] & C \arrow[dl, dashed, "\exists p"] \\
& C' &
\end{tikzcd}$$

Clearly, a weak cokernel g of f is a cokernel of f if and only if g is an epimorphism.The concept of weak kernel is defined dually.

Definition 2.3. Let R be a commutative ring with 1. Let S be a set. A free R-module M on generators S is an R-module M and a set map $i : S \longrightarrow M$ such that, for any R-module N and any set map $f : S \longrightarrow N$, there is a unique R-module homomorphism $\bar{f} : M \longrightarrow N$ such that $\bar{f}oi = f : S \longrightarrow N$. The elements of $i(S)$ in M are an R-basis for M.

Definition 2.4. A covariant functor $T :_R Mod \longrightarrow Ab$ is an exact functor if, for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0,$$

the sequence

$$0 \longrightarrow T(A) \xrightarrow{T(i)} T(B) \xrightarrow{T(p)} T(C) \longrightarrow 0$$

is also exact. A contravariant functor $T :_R Mod \longrightarrow Ab$ is an exact functor if there is always exactness of

$$0 \longrightarrow T(C) \xrightarrow{T(p)} T(B) \xrightarrow{T(i)} T(A) \longrightarrow 0.$$

Definition 2.5. A left R-module F is a free left R-module if F is isomorphic to a direct sum of copies of R: that is, there is a (possibly infinite) index set B with $F = \bigoplus_{b \in B} R_b$, where $R_b =< b >\cong R$ for all $b \in B$. We call B a basis of F.

By the definition of direct sum, each $m \in F$ has a unique expression of the form

$$m = \sum_{b \in B} r_b b,$$

where $r_b \in R$ and almost all $r_b = 0$. It follows that $F =< B >$.

Definition 2.6. Chain maps $f, g : (C_\bullet, d_\bullet) \longrightarrow (C'_\bullet, d'_\bullet)$ are homotopic, denoted by $f \simeq g$, if, for all n, there is a map $s = (s_n) : C_\bullet \longrightarrow C'_\bullet$ of degree $+1$ with

$$f^n - g^n = {d'}^{n+1} s^n + s^{n-1} d^n.$$

$$\begin{tikzcd}
{} \arrow[r] & C^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "f^{n+1}"] & C^n \arrow[r, "d^n"] \arrow[d, "f^n"] \arrow[dl, "s^n"'] & C^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] \arrow[dl, "s^{n-1}"'] & {}. \\
{} \arrow[r] & C'^{n+1} \arrow[r, "d'^{n+1}"'] & C'^n \arrow[r, "d'^n"'] & C'^{n-1} \arrow[r] & {}.
\end{tikzcd}$$

A chain maps $f : (C_\bullet, d_\bullet) \longrightarrow (C'_\bullet, d'_\bullet)$ is null-homotopic if $f \simeq 0$, where 0 is the zero chain map.

Definition 2.7. A left R-module P is projective if, whenever p is surjective and h is any map, there exists a lifting g; that is, there exists a map g making the following diagram commute:

$$\begin{tikzcd}
& P \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\
A \arrow[r, "p"'] & A'' \arrow[r] & 0.
\end{tikzcd}$$

Definition 2.8. Let $\mathcal{C}$ be an additive category and $d^0 : X^0 \longrightarrow X^1$ a morphism in $\mathcal{C}$. An n-coker of d^0 is a sequence

$$(d^1, ..., d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

such that, , for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{\hat{d}^n} \mathcal{C}(X^n, Y) \xrightarrow{\hat{d}^{n-1}} ... \xrightarrow{\hat{d}^1} \mathcal{C}(X^1, Y) \xrightarrow{\hat{d}^0} \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, ..., d^n)$ is an n-coker of d^0 if for all $1 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. In this case, we say the sequence

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

is right n-exact.

Remark 2.1. *When we say n-cokernel we always means that n is a positive integer. We note that the notion of 1-cokernel is the same as cokernel. we can define n- kernel and left n-exact sequence dually.*

Definition 2.9. Let $\mathcal{C}$ be an additive category. An n-exact sequence in $\mathcal{C}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \tag{1}$$

in $Ch^n(\mathcal{C})$ such that $(d^0, ..., d^{n-1})$ is an n-ker of d^n, and $(d^1, ..., d^n)$ is an n-coker of d^0. The sequence (3.1) is called n-exact if it is both right n-exact and left n-exact.

Definition 2.10. Let n be a positive integer. An n-abelian category is an additive category $\mathcal{C}$ which satisfies the following axioms;
(A^0) The category $\mathcal{C}$ is idempotent complete.
(A^1) Every morphism in $\mathcal{C}$ has n-ker and n-coker.
(A^2) For every monomorphism $f^0 : X^0 \longrightarrow X^1$ in $\mathcal{C}$ and, for every n-cokernel $(f^0, f^1, ..., f^{n-1})$ of f^0, the following sequence n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} ... \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

$(A^{2^{op}})$ For every epimorphism $g^n : X^n \longrightarrow X^{n+1}$ in $\mathcal{C}$ and, for every n-kernel $(g^0, g^1, ..., g^{n-1})$ of g^n, the following sequence n-exact:

$$X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} ... \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}.$$

Definition 2.11. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a R-homomorphism in $\mathcal{C}$. An R-module P is n-projective if the sequence of R-module in $\mathcal{C}$ is left n-exact

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

if there is $P \in \mathcal{C}$ the induced sequence of abelian groups

$$Hom_{\mathcal{C}}(P, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(P, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(P, Y^2) \xrightarrow{\hat{f^2}}$$

$$... \xrightarrow{\hat{f^{n-2}}} Hom_{\mathcal{C}}(P, Y^{n-1}) \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(P, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(P, Y^{n+1}) \longrightarrow 0$$

is left n-exact.

Remark 2.2. *The functors $Hom_R(X, \square)$ and $Hom_R(\square, Y)$ almost preserve exact sequences; they are left exact functors. Similarly, the functors $\square \otimes_R Y$ and $X \otimes_R \square$ almost preserve exact sequences; they are right exact functors.*

Proposition 2.3. ***(Extending by Linearity)*** *Let R be a ring and let F be the free left R-module with basis X. If M is any left R-module and if $f : X \longrightarrow M$ is any function, then there exists a unique R-map $\widetilde{f} : F \longrightarrow M$ with $\widetilde{f}\mu = f$, where $\mu : X \longrightarrow F$ is the inclusion; that is, $\widetilde{f} = f(x)$ for all $x \in X$, so that $\widetilde{f}$ extends f.*

$$\begin{tikzcd} F \arrow[dr, dashed, "\widetilde{f}"] & \\ X \arrow[u, "\mu"] \arrow[r, "f"'] & M \end{tikzcd}$$

Proposition 2.4. *Every left R-module M is a quotient of a free left R-module F. Moreover, M is finitely generated if and only if F can be chosen to be finitely generated.*

Proposition 2.5. *Let $\mathcal{C}$ be an additive category and X a complex in $Ch^n(\mathcal{C})$ such that $(d^1, ..., d^n)$ is an n-cokernel of d^0. Then, d^0 is a split monomorphism if and only if X is a contractible n-exact sequence.*

Proposition 2.6. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a morphism in $\mathcal{C}$. A direct sum of R-modules $\oplus_{i \in \mathbb{I}} P_i$ is n-projective if only if P_i is n-projective for every $i \in \mathbb{I}$ and $\mathbb{I}$ is finite.*

3. n-projective modules

Let C be an additive category. An n-exact sequence in C is a

Remark 3.1. *The functors $Hom_R(X, \square)$ and $Hom_R(\square, Y)$ almost preserve n-exact sequences; they are left exact functors. Similarly, the functors $\square \otimes_R Y$ and $X \otimes_R \square$ almost preserve n-exact sequences; they are right exact functors.*

Theorem 3.2. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$. Let F be a free left R-module. If Y a complex in $Ch^n(\mathcal{C})$ such that $(f^0, ..., f^{n-1})$ is an n-kernel of d^n, there is $F \in \mathcal{C}$ the induced sequence of abelian groups*

$$Hom_{\mathcal{C}}(F, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(F, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(F, Y^2) \xrightarrow{\hat{f^2}}$$

$$... \xrightarrow{\hat{f^{n-2}}} Hom_{\mathcal{C}}(F, Y^{n-1}) \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(P, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(F, Y^{n+1}) \longrightarrow 0$$

is left n-exact.

PROOF. Let B be a basis of F. For each $b \in B$, and for all $1 \leqslant k \leqslant n-2$ the R-homomorphism f^{k-1} is a weak kernel of d^k, and f^{n-1} is moreover a kernel of d^n. In the following diagram

$$\begin{tikzcd} & F \arrow[dl, dashed, "\exists\theta"'] \arrow[d, "g"] \arrow[dr, "0"] & \\ Y^{k-1} \arrow[r, "f^{k-1}"] & Y^k \arrow[r, "f^k"] & Y^{k+1}, \end{tikzcd}$$

for each $b \in B$, the element $g(b) \in Y^k$ has the form $g(b) = f^{k-1}(y_{k-1})$ for some $y_{k-1} \in Y^{k-1}$, because the R-homomorphism f^{k-1} is a weak kernel of f^k, there is $d^k g(b) = 0$ and $f^k f^{k-1}(y_{k-1}) = d^k g(b) = 0$; by the Axiom of Choice, there is a function $u : B \longrightarrow Y^{k-1}$ with $u(b) = y_{k-1}$ for all $b \in B$. By Proposition (2.3) gives an R-homomorphism $\theta : F \longrightarrow A$ with $\theta(b) = y_{k-1}$ for all $b \in B$. Now

$$f^{k-1}\theta(b) = f^{k-1}(y_{k-1}) = g(b),$$

so that $d^{k-1}\theta$ agrees with θ on the basis B; since $< B >= F$, we have $d^{k-1}\theta = g$.

By F be a free left R-module and, $Hom(F, \square)$ is an additive functor ${}_RMod \longrightarrow Ab$, so F the induced sequence of abelian groups

$$Hom_{\mathcal{C}}(F, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(F, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(F, Y^2) \xrightarrow{\hat{f^2}}$$

$$... \xrightarrow{\hat{f^{n-2}}} Hom_{\mathcal{C}}(F, Y^{n-1}) \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(P, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(F, Y^{n+1}) \longrightarrow 0$$

is left n-exact. □

Theorem (3.3) says that every free left R-module is n-projective.

Proposition 3.3. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$. An left R-module P is n-projective if and only if $Hom_R(P, \square)$ is an n-exact functor.*

PROOF. If P is an n-projective, then given $g : P \longrightarrow Y^k$, there exists a R-homomorphism $\theta : P \longrightarrow Y^{k-1}$ whit $f^{k-1}\theta = g$. Thus, if $g \in Hom_R(F, Y^k)$, then

$$g = f^{k-1}\theta = \hat{f}^{k-1}(\theta) \in Im\hat{f}^{k-2},$$

□

and so f^{k-1} is a weak kernel of $\hat{f}^{k-2}$. Hence, $Hom(P, \square)$nis an left n-exact functor.

For the converse, assume that $Hom(P, \square)$ is an n-exact functor, so that f^{k-1} is a weak kernel of $\hat{f}^{k-2}$: if $g \in Hom_(P, Y^{k-1})$ with $g = \hat{f}^{k-1}(\theta) = d^{k-1}\theta$. This says that given d^{k-1} and g, there is exists a R-homomorphism θ making the diagram commutative,

$$\begin{tikzcd} & & P \arrow[dl, dashed, "\exists\theta"'] \arrow[d, "g"] \arrow[dr, "0"] & \\ & Y^{k-1} \arrow[r, "f^{k-1}"] & Y^{k} \arrow[r, "f^{k}"] & Y^{k+1}, \end{tikzcd}$$

that is, P is n-projective.

Remark 3.4. *Since $Hom_R(P, \square)$ is a left n-exact functor, The thrust of the Proposition (3.3) is that $\hat{f}^{k-1}$ is a weak kernel of $\hat{f}^{k-2}$, whenever f^{k-1} is a weak kernel of f^{k-2} for all $0 \le i \le n+1$.*

Definition 3.1. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \le i \le n+1$, and f^i for all $0 \le i \le n$ is a R-homomorphism in $\mathcal{C}$. For all Y^i, $1 \le i \le n+1$ there exists a submodule $S^i \in \mathcal{C}$ of left R-module Y^i such that in the complex of R-module in $\mathcal{C}$

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

there exists an R-homomorphism $j^i : Y^i \longrightarrow S^i$, called a n-retraction, with $j^i(s_i) = s_i$ for all $s_i \in S^i$, $1 \le i \le n+1$ as R-homomorphism chain maps $j = (j^i)$: $(Y_\bullet, f_\bullet) \longrightarrow (S_\bullet, k_\bullet)$ making the following diagram commute:

$$\begin{tikzcd} Y^0 \arrow[r, "f^0"] \arrow[d, "j^0"] & Y^1 \arrow[r] \arrow[d] & Y^2 \arrow[r] \arrow[d] & ... \arrow[r] & Y^n \arrow[r, "f^n"] \arrow[d] & Y^{n+1} \arrow[d, "j^{n+1}"] \\ S^0 \arrow[r, "k^0"] & S^1 \arrow[r] & S^2 \arrow[r] & ... \arrow[r] & S^n \arrow[r, "k^n"] & S^{n+1} \end{tikzcd}$$

Equivalently, j^i is a n-retraction if and only if $j^i i^i = 1_{S^i}$, where $i^i : S^i \longrightarrow Y^i$ is the inclusion.

Corollary 3.5. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \le i \le n+1$, and f^i for all $0 \le i \le n$ is a R-homomorphism in $\mathcal{C}$. For all Y^i, $1 \le i \le n+1$*

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1},$$

a submodule $S^i \in \mathcal{C}$ of a left R-module Y^i is a direct summand if only if there exists a n-retraction $j^i : Y^i \longrightarrow S^i$.

Proof. In this case, we let $i^i : S^i \longrightarrow Y^i$ be the inclusion. We show that $Y^i = S^i \oplus T^i$, where $T^i = kerj^i$ for all $1 \le i \le n+1$. If $y_i \in Y^i$, then $y_i = (y_i - j^i y_i) + j^i y_i$. Plainly, $j^i y_i \in imj^i = S^i$. On the other hand, $j^i(y_i - j^i y_i) = j^i y_i - j^i j^i y_i = 0$, because $j^i y_i \in S^i$ and so $j^i j^i y_i = j^i y_i$ for all $1 \le i \le n+1$. Therefore, $Y^i = S^i \oplus T^i$.

If $y_i \in S^i$, then $j^i y_i = y_i$; if $y_i \in T^i = \ker j^i$, then $j^i y_i = 0$ for all $1 \le i \le n+1$. Hence if $y_i \in S^i \cap T^i$, then $y_i = 0$. Therefore, $S^i \cap T^i = \{0\}$, and $Y^i = S^i \oplus T^i$ for all $1 \le i \le n+1$.

For the converse, if $Y^i = S^i \oplus T^i$, then each $y_i \in Y^i$ has unique expression of the form $y_i = S_i + t_i$, where $s_i \in S^i$ and $t_i \in T^i$ for all $1 \le i \le n+1$. It is easy to check that $j^i : Y^i \longrightarrow S^i$, defined by $j^i : s_i + t_i \longmapsto s^i$, and the chain maps $j = (j^i) : (Y_\bullet, f_\bullet) \longrightarrow (S_\bullet, k_\bullet)$ making the following diagram commute:

$$\begin{tikzcd}
Y^0 \arrow[r, "f^0"] \arrow[d, "j^0"] & Y^1 \arrow[r] \arrow[d] & Y^2 \arrow[r] \arrow[d] & ... \arrow[r] & Y^n \arrow[r, "f^n"] \arrow[d] & Y^{n+1} \arrow[d, "j^{n+1}"] \\
S^0 \arrow[r, "k^0"] & S^1 \arrow[r] & S^2 \arrow[r] & ... \arrow[r] & S^n \arrow[r, "k^n"] & S^{n+1}
\end{tikzcd}$$

is a n-retraction $Y^i \longrightarrow S^i$ for all $1 \leq i \leq n+1$. □

Definition 3.2. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a R-homomorphism in $\mathcal{C}$. A complex of left R-modules in $\mathcal{C}$

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

is n-split if there exists a R-homomorphism $d^i : Y^{i+1} \longrightarrow Y^i$ with $f^i d^i = 1_{Y^{i+1}}$ for all $1 \leq i \leq n$.

Note that $d^i f^i$ is a n-retraction $Y^i \longrightarrow imd^i$ for all $1 \leq i \leq n+1$.

Proposition 3.6. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a R-homomorphism in $\mathcal{C}$. A complex of left R-modules in $\mathcal{C}$*

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

is n-split, then $Y^i \cong Y^{i-1} \oplus Y^{i+1}$ for all $1 \leq i \leq n$.

Proof. We show that $Y^i = imf^{i-1} \oplus imd^i$, where $d^i : Y^{i+1} \longrightarrow Y^i$ satisfies $f^i d^i = 1_{Y^{i+1}}$ for all $1 \leq i \leq n$. If $y_i \in Y^i$, then $f^i y_i \in Y^{i+1}$ and $y_i - d^i(f^i y_i) \in kerf^i$, for $f^i(y_i - d^i(f^i y_i)) = f^i y_i - f^i d^i(f^i y_i) = 0$ because $f^i d^i = 1_{Y^{i+1}}$ for all $1 \leq i \leq n$. By exactness, there is $y_{i-1} \in Y^{i-1}$ with $f^{i-1} y_{i-1} = y_i - d^i(f^i y_i)$. It follows that $Y^i = imf^{i-1} + imd^i$ for all $1 \leq i \leq n$. It remains to prove $imf^{i-1} \cap imd^i = 0$. If $f^{i-1} y_{i-1} = y = d^i y_{i+1}$, then $f^i y = f^i f^{i-1} y_{i-1} = 0$, because $f^i f^{i-1} = 0$, whereas $f^{i-1} y = f^{i-1}(d^i y_{i+1}) = y_{i+1}$, because $f^i d^i = 1_{Y^{i+1}}$ for all $y \in Y^i$, $y_i \in Y^i$ and $y_{i+1} \in Y^{i+1}$, for all $1 \leq i \leq n$. Therefore, $y = d^i y_{i+1} = 0$, and so $Y^i = Y^{i-1} \oplus Y^{i+1}$. □

Proposition 3.7. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a R-homomorphism in $\mathcal{C}$, and a complex of left R-modules in $\mathcal{C}$*

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}.$$

A left R-module $P \in \mathcal{C}$ is n-projective if and only if n-exact sequence

$$\longrightarrow Y^{i-1} \xrightarrow{f^{i-1}} Y^i \xrightarrow{f^i} Y^{i+1}$$

for all $1 \leq i \leq n$, and $Y^{i+1} = P$, n-splits.

PROOF. If P is n-projective, then for a complex of left R-modules in $\mathcal{C}$ for all $1 \leq i \leq n$, and $Y^{i+1} = P$

$$\longrightarrow Y^{i-1} \xrightarrow{f^{i-1}} Y^i \xrightarrow{f^i} Y^{i+1}$$

the induced the sequence of abelian groups

$$\longrightarrow Hom_{\mathcal{C}}(P, Y^{i-1}) \xrightarrow{\hat{f}^{i-1}} Hom_{\mathcal{C}}(P, Y^i) \xrightarrow{\hat{f}^i} Hom_{\mathcal{C}}(P, Y^{i+1}) \longrightarrow 0$$

is left n-exact. Then the exists $\theta^i : P \longrightarrow Y^i$ with $1_P = Hom_{\mathcal{C}}(Y^i, P) = f^i\theta^i$, for all $1 \leq i \leq n$, that is $Y^{i+1} = P$, P is n-retract of Y^i. By corollary (3.5) now gives the result.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists\theta^i"'] \arrow[d, "1_P"] & \\
Y^i \arrow[r, "f^i"] & P \arrow[r] & 0.
\end{tikzcd}$$

Conversely, assume that every complex of left R-modules in $\mathcal{C}$ for all $1 \leq i \leq n$ ending with n-split. Consider the complex in $\mathcal{C}$

$$\longrightarrow Y^{i-1} \xrightarrow{f^{i-1}} Y^i \xrightarrow{f^i} C$$

there is $P \in \mathcal{C}$ the induced the sequence in abelian groups

$$\longrightarrow Hom_{\mathcal{C}}(P, Y^{i-1}) \xrightarrow{\hat{f}^{i-1}} Hom_{\mathcal{C}}(P, Y^i) \xrightarrow{\hat{f}^i} Hom_{\mathcal{C}}(P, C) \longrightarrow 0$$

is left n-exact. Let F be an free left R-module for which there exists a R-homomorphism $\beta : F \longrightarrow P$ (by the Theorem 2.4), and consider the augmented diagram

$$\begin{tikzcd}
F \arrow[r, shift left, "\beta"] \arrow[d, dashed, "k^i"] & P \arrow[l, shift left, "\alpha"] \arrow[d, "j^i"] & \\
Y^i \arrow[r, "f^i"] & C \arrow[r] & 0.
\end{tikzcd}$$

$\square$

4. One Open Problem

Proposition 4.1. *Let $\mathcal{C}$ be an category of R-modules, $Y^i, S^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$. Get R-homomorphism chain maps $j = (j^i) : (Y_\bullet, f_\bullet) \longrightarrow (S_\bullet, k_\bullet)$ making the following diagram commute:*

$$\begin{tikzcd}
Y^0 \arrow[r, "f^0"] \arrow[d, "j^0"] & Y^1 \arrow[r] \arrow[d] & Y^2 \arrow[r] \arrow[d] & \dots \arrow[r] & Y^n \arrow[r, "f^n"] \arrow[d] & Y^{n+1} \arrow[d, "j^{n+1}"] \\
S^0 \arrow[r, "k^0"] & S^1 \arrow[r] & S^2 \arrow[r] & \dots \arrow[r] & S^n \arrow[r, "k^n"] & S^{n+1}
\end{tikzcd}$$

Equivalently,

$$\begin{tikzcd}
{} \arrow[r] & Y^{i-1} \arrow[r, "f^{i-1}"] \arrow[d, "j^{i-1}"'] & Y^{i} \arrow[r] \arrow[d] & Y^{i+1} \arrow[d, "j^{i+1}"] \\
{} \arrow[r] & S^{i-1} \arrow[r, "k^{i-1}"] & S^{i} \arrow[r] & S^{i+1}
\end{tikzcd}$$

Y^i, S^i *are n-projective, then there is an isomorphism*

$$Y^{i-1} \oplus S^i = Y^i \oplus S^{i-1}.$$

References

[1] M. Auslander and S. O. Smalø, *Almost split sequences in subcategories*, J. Algebra, **69**(2) (1981), 426-454.

[2] M. P. Brodmann and R. Y. Sharp, *Local cohomology, An Algebraic Introduction with Geometric Applications*, Cambridge Studies in Advanced Mathematics, vol. 60, Cambridge University Press, Cambridge, 1998.

[3] T. Bühler, *Exact categories*, Exp. Math., **28**(1) (2010), 1-69.

[4] P. Fredy, *Abelian Categories, An Introduction to the Theory of Functors*, Harper's Series in Modern Mathematics, Harper and Row Publishers, New York, 1994.

[5] L. Frerick and D. Sieg, *Exact categories in functional analysis*, Preprint, 2010.

[6] S. Hashemi, F. Hassani, and R. Rasuli, *The new results in injective modules*, Earthline J. Math. Sci. **7** (2021), 145-159.

[7] S. Hashemi, F. Hassani and R. Rasuli, *The new results in n-injective Modules and n-projective modules*, Earthline J. Math. Sci. **7**(2) (2021), 271-285.

[8] S. Hashemi, F. Hassani and R. Rasuli, *The new results in n-abelian category*, Eur. J. Math. Appl. **1** (2021).

[9] G. Jasso, *n-Abelian and n-exact categories*, Math. Z., **283** (2016), 703–759.

[10] A. Neeman, *The derived category of an exact category*, J. Algebra, **135**(2) (1990), 388-394.

[11] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer, Berlin Heidelberg, 2010.

[12] J. J. Rotman, *An Introduction to Homological Algebra*, New York Springer Verlag, 1992.

[13] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.

Department of mathematics, Payame Noor University (PNU), P. OBox, 19395-4697 Tehran, Iran

Email address: `hassani@pnu.ac.ir`

Department of mathematics, Payame Noor University (PNU), P. OBox, 19395-4697 Tehran, Iran

Email address: `samirahashemi@student.pnu.ac.ir`

Department of mathematics, Payame Noor University (PNU), P. OBox, 19395-4697 Tehran, Iran

Email address: `rasuli@pnu.ac.ir,`

Received : August 2022
Accepted : November 2022