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Fixed point theorems in bi-b-metric spaces

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Abstract. In this paper, we have proved the existence and uniqueness of common fixed point results in bi-b-metric space.

1. Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, an interesting concept of generalized b-metric spaces was introduced by Bakhtin [2].Many researchers generalized the Banach fixed point theorem in b-metric space.The existence and uniqueness theorems in b-metric space was presented by Agrawal [1]. In 1968 Maia generalized the result of well known Banach Contraction Principle by taking two metrices on a set Mishra [13] generalized the Maia's fixed point theorem in bi-metric spaces.In 1993 Czerwik [7]extended the results of b-metric spaces.

We want to extend some fixed point theorems in bi-metric spaces which are also valid in bi-b-metric spaces. Chopade [6] obtained common fixed point theorems for contractive type mappings in metric space. Tomonari [11] obtained some basic inequalities on a b-metric space and it's applications. Roshan [10] gave the common fixed point of four maps in b-metric space.

2. Preliminaries

Definition 2.1. Let X be a non-empty set. A function $\rho : X \times X \to R$ is called as a metric provided that for all $u, v, w \in X$,

(1) $\rho(u, v) \geq 0$,
(2) $\rho(u, v) = 0$ if and only if $u = v$,
(3) $\rho(u, v) = \rho(v, u)$,
(4) $\rho(u, v) \leq \rho(u, w) + \rho(w, v)$.

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A pair (X, ρ) is called a metric space.

Definition 2.2. Let X be a non-empty set and $s \geq 1$ be a given real number.A function $\rho : X \times X \to R$ is called as a b-metric provided that for all $u, v, w \in X$,

(1) $\rho(u, v) \geq 0$,
(2) $\rho(u, v) = 0$ if and only if $u = v$,
(3) $\rho(u, v) = \rho(v, u)$,
(4) $\rho(u, v) \leq s\{\rho(u, w) + \rho(w, v)\}$.

A pair (X, ρ) is called a b-metric space. It is clear that definition of b-metric space is an extension of usual metric space.

Example 2.3. By [4], Let M $= \{0, 1, 2\}$ and $\rho : M \times M \to R$ is defined as,

$$\rho(0, 2) = \rho(2, 0) = m \geq 2,$$

$$\rho(0, 1) = \rho(1, 0) = \rho(1, 2) = \rho(2, 1) = 1,$$

$$\rho(0, 0) = \rho(1, 1) = \rho(2, 2) = 0.$$

Here $\rho(u, v)$ is a b-metric on M with $s = \frac{m}{2}$.

Example 2.4. Let (X, ρ) be a b-metric space and $\rho(u, v) = (d(u, v))^p$,where $p > 1$ is a real number. Clearly, $\rho(u, v)$ is b-metric with $s = 2^{p-1}$.

Definition 2.5. Let (X, ρ) be a b-metric space, $\{u_n\}$ be a sequence in X and $u \in X$. If for every $\epsilon > 0$,

(1) there exist $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$ we have, $\rho(u_n, u) < \epsilon$, then $\{u_n\}$ is said to be convergent.In this case we write $\rho(u_n, u) \to 0$ as $n \to \infty$.
(2) there is $n(\epsilon) \in \mathbb{N}$ such that for all $n, m \geq n(\epsilon)$we have,$\rho(u_n, u_m) < \epsilon$, then $\{u_n\}$ is said to be cauchy sequence in X.
(3) a b-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.1. [11] *Let (X, ρ) be a complete b-metric space and let $\{u_n\}$ be a sequence in X.Assume that there exist $r \in [0, 1)$ satisfying*

$$\rho(u_{n+1}, u_{n+2}) \leq r\rho(u_n, u_{n+1})$$

for any $n \in N$.Then $\{u_n\}$ is Cauchy sequence in X.

3. Main Results

Theorem 3.1. *Let (X, ρ_1, s) and (X, ρ_2, t) be a bi-b-metric space, where $s \geq 1$ and $t \geq 1$ such that,*

(1) *$\rho_1(u, v) \leq \rho_2(u, v)$ for all $u,\ v, \in X$.*
(2) *(X,ρ_1) is a complete space.*

(3) *$A : X \to X$ and $B : X \to X$ be any two selfmaps on X satisfying,*

$$\begin{aligned}
&\rho_2(Au, Bv) + \alpha \min\{\rho_2(Au, Bv), \rho_2(Av, Bu), \rho_2(u, v)\}\\
&\leq \beta \frac{\rho_2(u, Au)\rho_2(v, Bv)}{\rho_2(v, Bv) + \rho_2(v, Au)}\\
&+ \gamma \frac{[p + \rho_2(u, Au)]\,[\rho_2(v, Av)]^r\,[\rho_2(v, Au)]^q}{1 + \lambda\rho_2(v, Au) + \mu\rho_2(u, Bv) + \rho_2(u, v)}\\
&+ \delta \frac{\rho_2(u, Bv)\left[1 + \sqrt{\rho_2(u, v)\rho_2(u, Au)} + \sqrt{\rho_2(u, v)\rho_2(v, Au)}\right]}{1 + \rho_2(u, v) + \rho_2(u, Bu)\rho_2(u, Bv)\rho_2(v, Au)\rho_2(v, Av)}\\
&+ \eta \max\left\{\rho_2(u, v), \frac{\rho_2(u, Au)\rho_2(v, Bv)}{1 + \rho_2(Au, Bv)}\right\} \qquad (1)
\end{aligned}$$

where $p, q, r, \lambda, \mu \in R^+$ and $\alpha, \beta, \gamma, \delta, \eta \in [0, 1)$ are such that $\beta+2\delta t+\eta-\alpha < 1$.

(4) *Both the mappings A and B are continuous in (X, ρ_1). Then A and B have unique common fixed point in X.*

Proof. Let $u_0 \in X$ be an arbitrary and define a sequence $\{u_n\}$ in X such that

$$A\,(u_{2n}) = u_{2n+1} \qquad \textit{and} \qquad B\,(u_{2n-1}) = u_{2n}, \quad n = 1, 2, \cdots . \qquad (2)$$

Using equation (1) and (2) we obtain that,

$$\begin{aligned}
&\rho_2(Au_{2n}, Bu_{2n+1}) + \alpha \min\{\rho_2(Au_{2n}, Bu_{2n+1}), \rho_2(Au_{2n+1}, Bu_{2n}), \rho_2(u_{2n}, u_{2n+1})\}\\
&\leq \beta \frac{\rho_2(u_{2n}, Au_{2n})\rho_2(u_{2n+1}, Bu_{2n+1})}{\rho_2(u_{2n+1}, Bu_{2n+1}) + \rho_2(u_{2n+1}, Au_{2n})}\\
&+ \gamma \frac{[p + \rho_2(u_{2n}, Au_{2n})]\,[\rho_2(u_{2n+1}, Au_{2n+1})]^r\,[\rho_2(u_{2n+1}, Au_{2n})]^q}{1 + \lambda\rho_2(u_{2n+1}, Au_{2n}) + \mu\rho_2(u_{2n}, Bu_{2n+1}) + \rho_2(u_{2n}, u_{2n+1})}\\
&+ \delta \frac{\rho_2(u_{2n}, Bu_{2n+1})\left[1 + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n}, Au_{2n})} + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, Au_{2n})}\right]}{1 + \rho_2(u_{2n}, u_{2n+1}) + \rho_2(u_{2n}, Bu_{2n})\rho_2(u_{2n}, Bu_{2n+1})\rho_2(u_{2n+1}, Au_{2n})\rho_2(u_{2n+1}, Au_{2n+1})}\\
&+ \eta \max\left\{\rho_2(u_{2n}, u_{2n+1}), \frac{\rho_2(u_{2n}, Au_{2n})\rho_2(u_{2n+1}, Bu_{2n+1})}{1 + \rho_2(Au_{2n}, Bu_{2n+1})}\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\rho_2(u_{2n+1}, u_{2n+2}) + \alpha \min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n+2}, u_{2n+1}), \rho_2(u_{2n}, u_{2n+1})\}\\
&\leq \beta \frac{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+2})}{\rho_2(u_{2n+1}, u_{2n+2}) + \rho_2(u_{2n+1}, u_{2n+1})}\\
&+ \gamma \frac{[p + \rho_2(u_{2n}, u_{2n+1})]\,[\rho_2(u_{2n+1}, u_{2n+2})]^r\,[\rho_2(u_{2n+1}, u_{2n+1})]^q}{1 + \lambda\rho_2(u_{2n+1}, u_{2n+1}) + \mu\rho_2(u_{2n}, u_{2n+2}) + \rho_2(u_{2n}, u_{2n+1})}\\
&+ \delta \frac{\rho_2(u_{2n}, u_{2n+2})\left[1 + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n}, u_{2n+1})} + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+1})}\right]}{1 + \rho_2(u_{2n}, u_{2n+1}) + \rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n}, u_{2n+2})\rho_2(u_{2n+1}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+2})}\\
&+ \eta \max\left\{\rho_2(u_{2n}, u_{2n+1}), \frac{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+2})}{1 + \rho_2(u_{2n+1}, u_{2n+2})}\right\}.
\end{aligned}$$

Hence,

$$\rho_2(u_{2n+1}, u_{2n+2}) + \alpha \min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n}, u_{2n+1})\}$$
$$\leq \beta\rho_2(u_{2n}, u_{2n+1}) + \delta\rho_2(u_{2n}, u_{2n+2}) + \eta\rho_2(u_{2n}, u_{2n+1})$$
$$\leq \beta\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n+1}, u_{2n+2}) + \eta\rho_2(u_{2n}, u_{2n+1}). \quad (3)$$

Case 1: $\min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n}, u_{2n+1})\} = \rho_2(u_{2n}, u_{2n+1})$. From equation (3)

$$\begin{aligned}
\rho_2(u_{2n+1}, u_{2n+2}) + \alpha\rho_2(u_{2n}, u_{2n+1}) &\leq (\beta + \delta t + \eta)\,\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n+1}, u_{2n+2})\\
(1 - \delta t)\,\rho_2(u_{2n+1}, u_{2n+2}) &\leq (\beta + \delta t + \eta - \alpha)\,\rho_2(u_{2n}, u_{2n+1})\\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq \frac{(\beta + \delta t + \eta - \alpha)}{(1 - \delta t)}\rho_2(u_{2n}, u_{2n+1})\\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq k\rho_2(u_{2n}, u_{2n+1}),
\end{aligned}$$

where k $= \frac{(\beta+\delta t+\eta-\alpha)}{(1-\delta t)} < 1$. In general for all $n \in N$,

$$\rho(u_{n+1}, u_{n+2}) \leq k\rho(u_n, u_{n+1}) \quad (4)$$

Case 2: $\min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n}, u_{2n+1})\} = \rho_2(u_{2n+1}, u_{2n+2})$. From equation (3)

$$\begin{aligned}
\rho_2(u_{2n+1}, u_{2n+2}) + \alpha\rho_2(u_{2n+1}, u_{2n+2}) &\leq (\beta + \delta t + \eta)\,\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n+1}, u_{2n+2})\\
(1 - \delta t + \alpha)\,\rho_2(u_{2n+1}, u_{2n+2}) &\leq (\beta + \delta t + \eta)\,\rho_2(u_{2n}, u_{2n+1})\\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq \frac{(\beta + \delta t + \eta)}{(1 - \delta t + \alpha)}\rho_2(u_{2n}, u_{2n+1})\\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq k\rho_2(u_{2n}, u_{2n+1}),
\end{aligned}$$

where $k = \frac{(\beta+\delta t+\eta)}{(1-\delta t+\alpha)} < 1$. In general, for all $n \in N$,

$$\rho(u_{n+1}, u_{n+2}) \leq k\rho(u_n, u_{n+1}) \quad (5)$$

Therefore by Lemma 2.1 the sequence $\{u_n\}$ is Cauchy sequence in X. Since the cauchy sequence $\{u_n\}$ defined by equation (2) has convergent subsequence $\{u_{nk}\}$ in (X,ρ_1) converging to u^* in (X,ρ_1), the sequence $\{u_n\}$ also converges to u^* in (X,ρ_1). Hence,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} u_{2n+1} = u^*.$$

Now, we show that u^* is fixed point of both the mappings A and B. As A and B are continuous in (X, ρ_1), therefore,

$$A\,(u^*) = A\left[\lim_{n\to\infty} u_{2n}\right] = \lim_{n\to\infty} [Au_{2n}] = u^*.$$

Similarly,

$$B\,(u^*) = B\left[\lim_{n\to\infty} u_{2n-1}\right] = \lim_{n\to\infty} [Bu_{2n-1}] = u^*.$$

Thus u^* is common fixed point of the mappings A and B.

Suppose that u^* and v^* be two common fixed points of the mappings A and B. Therefore $Au^* = Bu^* = u^*$ and $Av^* = Bv^* = v^*$. Consider

$$\begin{aligned}
&\rho_2(Au^*, Bv^*) + \alpha \min\{\rho_2(Au^*, Bv^*), \rho_2(Av^*, Bu^*), \rho_2(u^*, v^*)\}\\
&\leq \beta \frac{\rho_2(u^*, Au^*)\rho_2(v^*, Bv^*)}{\rho_2(v^*, Bv^*) + \rho_2(v^*, Au^*)}\\
&+ \gamma \frac{[p + \rho_2(u^*, Au^*)]\,[\rho_2(v^*, Av^*)]^r\,[\rho_2(v^*, Au^*)]^q}{1 + \lambda\rho_2(v^*, Au^*) + \mu\rho_2(u^*, Bv^*) + \rho_2(u^*, v^*)}\\
&+ \delta \frac{\rho_2(u^*, Bv^*)\left[1 + \sqrt{\rho_2(u^*, v^*)\rho_2(u^*, Au^*)} + \sqrt{\rho_2(u^*, v^*)\rho_2(v^*, Au^*)}\right]}{1 + \rho_2(u^*, v^*) + \rho_2(u^*, Bu^*)\rho_2(u^*, Bv^*)\rho_2(v^*, Au^*)\rho_2(v^*, Av^*)}\\
&+ \eta \max\left\{\rho_2(u^*, v^*), \frac{\rho_2(u^*, Au^*)\rho_2(v^*, Bv^*)}{1 + \rho_2(Au^*, Bv^*)}\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\rho_2(u^*, v^*) + \alpha \min\{\rho_2(u^*, v^*), \rho_2(v^*, u^*), \rho_2(u^*, v^*)\}\\
&\leq \beta \frac{\rho_2(u^*, u^*)\rho_2(v^*, v^*)}{\rho_2(v^*, v^*) + \rho_2(v^*, u^*)}\\
&+ \gamma \frac{[p + \rho_2(u^*, u^*)]\,[\rho_2(v^*, v^*)]^r\,[\rho_2(v^*, u^*)]^q}{1 + \lambda\rho_2(v^*, u^*) + \mu\rho_2(u^*, v^*) + \rho_2(u^*, v^*)}\\
&+ \delta \frac{\rho_2(u^*, v^*)\left[1 + \sqrt{\rho_2(u^*, v^*)\rho_2(u^*, u^*)} + \sqrt{\rho_2(u^*, v^*)\rho_2(v^*, u^*)}\right]}{1 + \rho_2(u^*, v^*) + \rho_2(u^*, u^*)\rho_2(u^*, v^*)\rho_2(v^*, u^*)\rho_2(v^*, v^*)}\\
&+ \eta \max\left\{\rho_2(u^*, v^*), \frac{\rho_2(u^*, u^*)\rho_2(v^*, v^*)}{1 + \rho_2(u^*, v^*)}\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\rho_2(u^*, v^*) + \alpha\rho_2(u^*, v^*) &\leq \delta\rho_2(u^*, v^*) + \eta\rho_2(u^*, v^*)\\
\rho_2(u^*, v^*) &\leq (\delta + \eta - \alpha)\,\rho_2(u^*, v^*).
\end{aligned}$$

But $(\delta + \eta - \alpha) < 1$. Therefore we have,$\rho_2(u^*, v^*) < \rho_2(u^*, v^*)$. This is a contradiction. Hence, A and B have unique common fixed point in X. This completes the proof. □

Theorem 3.2. *Let (X, ρ_1, s) and (X, ρ_2, t) be a bi-b-metric space,where $s \geq 1$and $t \geq 1$. Let P= $\{T_i : i \in I, thesetofpositiveintegers\}$ be a family of mappings on X such that the following conditions holds*

(1) *$\rho_1(u, v) \leq \rho_2(u, v)$ for all u, v, $\in X$.*
(2) *(X,ρ_1)is complete space.*

(3) *for each* $T_j : X \to X \in P$ *there exist* $T_i : X \to X \in P$ *such that,*

$$\begin{aligned}
&\rho_2(T_i^m u, T_j^n v) + \alpha \min\{\rho_2(T_i^m u, T_j^n v), \rho_2(T_i^m v, T_j^n u), \rho_2(u,v)\} \\
&\leq \beta \frac{\rho_2(u, T_i^m u)\rho_2(v, T_j^n v)}{\rho_2(v, T_j^n v) + \rho_2(v, T_i^m u)} \\
&+ \gamma \frac{[p + \rho_2(u, T_i^m u)]\,[\rho_2(v, T_i^m v)]^r\,[\rho_2(v, T_i^m u)]^q}{1 + \lambda\rho_2(v, T_i^m u) + \mu\rho_2(u, T_j^n v) + \rho_2(u,v)} \\
&+ \delta \frac{\rho_2(u, T_j^n v)\left[1 + \sqrt{\rho_2(u,v)\rho_2(u, T_i^m u)} + \sqrt{\rho_2(u,v)\rho_2(v, T_i^m u)}\right]}{1 + \rho_2(u,v) + \rho_2(u, T_i^m u)\rho_2(u, T_j^n v)\rho_2(v, T_i^m u)\rho_2(v, T_i^m v)} \\
&+ \eta \max\left\{\rho_2(u,v), \frac{\rho_2(u, T_i^m u)\rho_2(v, T_j^n v)}{1 + \rho_2(T_i^m u, T_j^n v)}\right\} \qquad (6)
\end{aligned}$$

where,$p, q, r, \lambda, \mu \in R^+$ *also m,n are positive integers and* $\alpha, \beta, \gamma, \delta, \eta \in [0,1)$ *are such that* $\beta + 2\delta t + \eta - \alpha < 1$.

(4) *mapping* T_i *is continuous in (X,*ρ_1*) for all* $i \in I$*,Then P has a unique common fixed point.*

PROOF. Let $u_0 \in X$ be an arbitrary.We define a sequence of iterates $\{u_n\}$ in X such that

$$u_{2n-1} = T_i^m(u_{2n-2}) \qquad and \qquad u_{2n} = T_j^n(u_{2n-1}), \quad n = 1, 2, ... \qquad (7)$$

Using equation (6) and (7)we obtain that,

$$\begin{aligned}
&\rho_2(T_i^m u_{2n}, T_j^n u_{2n+1}) + \alpha \min\{\rho_2(T_i^m u_{2n}, T_j^n u_{2n+1}), \rho_2(T_i^m u_{2n+1}, T_j^n u_{2n}), \rho_2(u_{2n}, u_{2n+1})\} \\
&\leq \beta \frac{\rho_2(u_{2n}, T_i^m u_{2n})\rho_2(u_{2n+1}, T_j^n u_{2n+1})}{\rho_2(u_{2n+1}, T_j^n u_{2n+1}) + \rho_2(u_{2n+1}, T_i^m u_{2n})} \\
&+ \gamma \frac{[p + \rho_2(u_{2n}, T_i^m u_{2n})]\,[\rho_2(u_{2n+1}, T_i^m u_{2n+1})]^r\,[\rho_2(u_{2n+1}, T_i^m u_{2n})]^q}{1 + \lambda\rho_2(u_{2n+1}, T_i^m u_{2n}) + \mu\rho_2(u_{2n}, T_j^n u_{2n+1}) + \rho_2(u_{2n}, u_{2n+1})} \\
&+ \delta \frac{\rho_2(u_{2n}, T_j^n u_{2n+1})\left[1 + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n}, T_i^m u_{2n})} + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, T_i^m u_{2n})}\right]}{1 + \rho_2(u_{2n}, u_{2n+1}) + \rho_2(u_{2n}, T_j^n u_{2n})\rho_2(u_{2n}, T_j^n u_{2n+1})\rho_2(u_{2n+1}, T_i^m u_{2n})\rho_2(u_{2n+1}, T_i^m u_{2n+1})} \\
&+ \eta \max\left\{\rho_2(u_{2n}, u_{2n+1}), \frac{\rho_2(u_{2n}, T_i^m u_{2n})\rho_2(u_{2n+1}, T_j^n u_{2n+1})}{1 + \rho_2(T_i^m u_{2n}, T_j^n u_{2n+1})}\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\rho_2(u_{2n+1}, u_{2n+2}) + \alpha \min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n+2}, u_{2n+1}), \rho_2(u_{2n}, u_{2n+1})\} \\
&\leq \beta \frac{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+2})}{\rho_2(u_{2n+1}, u_{2n+2}) + \rho_2(u_{2n+1}, u_{2n+1})} \\
&+ \gamma \frac{[p + \rho_2(u_{2n}, u_{2n+1})]\,[\rho_2(u_{2n+1}, u_{2n+2})]^r\,[\rho_2(u_{2n+1}, u_{2n+1})]^q}{1 + \lambda\rho_2(u_{2n+1}, u_{2n+1}) + \mu\rho_2(u_{2n}, u_{2n+2}) + \rho_2(u_{2n}, u_{2n+1})} \\
&+ \delta \frac{\rho_2(u_{2n}, u_{2n+2})\left[1 + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n}, u_{2n+1})} + \sqrt{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+1})}\right]}{1 + \rho_2(u_{2n}, u_{2n+1}) + \rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n}, u_{2n+2})\rho_2(u_{2n+1}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+2})} \\
&+ \eta \max\left\{\rho_2(u_{2n}, u_{2n+1}), \frac{\rho_2(u_{2n}, u_{2n+1})\rho_2(u_{2n+1}, u_{2n+2})}{1 + \rho_2(u_{2n+1}, u_{2n+2})}\right\}.
\end{aligned}$$

Hence,

$$\rho_2(u_{2n+1}, u_{2n+2}) + \alpha \min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n}, u_{2n+1})\}$$
$$\leq \beta\rho_2(u_{2n}, u_{2n+1}) + \delta\rho_2(u_{2n}, u_{2n+2}) + \eta\rho_2(u_{2n}, u_{2n+1})$$
$$\leq \beta\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n+1}, u_{2n+2}) + \eta\rho_2(u_{2n}, u_{2n+1}). \quad (8)$$

Case 1: $\min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n}, u_{2n+1})\} = \rho_2(u_{2n}, u_{2n+1})$. From equation (8)

$$\begin{aligned}
\rho_2(u_{2n+1}, u_{2n+2}) + \alpha\rho_2(u_{2n}, u_{2n+1}) &\leq (\beta + \delta t + \eta)\,\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n+1}, u_{2n+2}) \\
(1 - \delta t)\,\rho_2(u_{2n+1}, u_{2n+2}) &\leq (\beta + \delta t + \eta - \alpha)\,\rho_2(u_{2n}, u_{2n+1}) \\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq \frac{(\beta + \delta t + \eta - \alpha)}{(1 - \delta t)}\rho_2(u_{2n}, u_{2n+1}) \\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq k\rho_2(u_{2n}, u_{2n+1})
\end{aligned}$$

where $k = \frac{(\beta+\delta t+\eta-\alpha)}{(1-\delta t)} < 1$. In general, for all $n \in N$,

$$\rho(u_{n+1}, u_{n+2}) \leq k\rho(u_n, u_{n+1}). \quad (9)$$

Case 2: $\min\{\rho_2(u_{2n+1}, u_{2n+2}), \rho_2(u_{2n}, u_{2n+1})\} = \rho_2(u_{2n+1}, u_{2n+2})$. From equation (8)

$$\begin{aligned}
\rho_2(u_{2n+1}, u_{2n+2}) + \alpha\rho_2(u_{2n+1}, u_{2n+2}) &\leq (\beta + \delta t + \eta)\,\rho_2(u_{2n}, u_{2n+1}) + \delta t\rho_2(u_{2n+1}, u_{2n+2}) \\
(1 - \delta t + \alpha)\,\rho_2(u_{2n+1}, u_{2n+2}) &\leq (\beta + \delta t + \eta)\,\rho_2(u_{2n}, u_{2n+1}) \\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq \frac{(\beta + \delta t + \eta)}{(1 - \delta t + \alpha)}\rho_2(u_{2n}, u_{2n+1}) \\
\rho_2(u_{2n+1}, u_{2n+2}) &\leq k\rho_2(u_{2n}, u_{2n+1})
\end{aligned}$$

where $k = \frac{(\beta+\delta t+\eta)}{(1-\delta t+\alpha)} < 1$. In general, for all $n \in N$,

$$\rho(u_{n+1}, u_{n+2}) \leq k\rho(u_n, u_{n+1}) \quad (10)$$

Therefore by Lemma 2.1 the sequence $\{u_n\}$ is Cauchy sequence in X and (X, ρ_1) is a complete space. Since the cauchy sequence $\{u_n\}$ defined by equation (7) has convergent subsequence $\{u_{nk}\}$ in (X, ρ_1) converging to u^* in (X,ρ_1), the sequence $\{u_n\}$ also converges to u^* in (X,ρ_1). Hence,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} u_{2n+1} = u^*.$$

Now we show that u^* is fixed point of both the mappings T_i^m and T_j^n. As T_i^m and T_j^n are continuous in (X,ρ_1), therefore,

$$T_i^m(u^*) = T_i^m\left[\lim_{n\to\infty} u_{2n}\right] = \lim_{n\to\infty}[T_i^m u_{2n}] = u^*.$$

Similarly,

$$T_j^n(u^*) = T_j^n\left[\lim_{n\to\infty} u_{2n-1}\right] = \lim_{n\to\infty}\left[T_j^n u_{2n-1}\right] = u^*.$$

Thus u^* is common fixed point of the mappings T_i^m and T_j^n.

Suppose that u^* and v^* be two common fixed points of the mappings T_i^m and T_j^n. Therefore $T_i^m u^* = T_j^n u^* = u^*$ and $T_i^m v^* = T_j^n v^* = v^*$. Consider

$$\rho_2(T_i^m u^*, T_j^n v^*) + \alpha \min\{\rho_2(T_i^m u^*, T_j^n v^*), \rho_2(T_i^m v^*, T_j^n u^*), \rho_2(u^*, v^*)\}$$
$$\leq \beta \frac{\rho_2(u^*, T_i^m u^*)\rho_2(v^*, T_j^n v^*)}{\rho_2(v^*, T_j^n v^*) + \rho_2(v^*, T_i^m u^*)} + \gamma \frac{[p + \rho_2(u^*, T_i^m u^*)]\,[\rho_2(v^*, T_i^m v^*)]^r\,[\rho_2(v^*, T_i^m u^*)]^q}{1 + \lambda\rho_2(v^*, T_i^m u^*) + \mu\rho_2(u^*, T_j^n v^*) + \rho_2(u^*, v^*)}$$
$$+ \delta \frac{\rho_2(u^*, T_j^n v^*)\left[1 + \sqrt{\rho_2(u^*, v^*)\rho_2(u^*, T_i^m u^*)} + \sqrt{\rho_2(u^*, v^*)\rho_2(v^*, T_i^m u^*)}\right]}{1 + \rho_2(u^*, v^*) + \rho_2(u^*, T_j^n u^*)\rho_2(u^*, T_j^n v^*)\rho_2(v^*, T_i^m u^*)\rho_2(v^*, T_i^m v^*)}$$
$$+ \eta \max\left\{\rho_2(u^*, v^*), \frac{\rho_2(u^*, T_i^m u^*)\rho_2(v^*, T_j^n v^*)}{1 + \rho_2(T_i^m u^*, T_j^n v^*)}\right\}.$$

Therefore

$$\rho_2(u^*, v^*) + \alpha \min\{\rho_2(u^*, v^*), \rho_2(v^*, u^*), \rho_2(u^*, v^*)\}$$
$$\leq \beta \frac{\rho_2(u^*, u^*)\rho_2(v^*, v^*)}{\rho_2(v^*, v^*) + \rho_2(v^*, u^*)} + \gamma \frac{[p + \rho_2(u^*, u^*)]\,[\rho_2(v^*, v^*)]^r\,[\rho_2(v^*, u^*)]^q}{1 + \lambda\rho_2(v^*, u^*) + \mu\rho_2(u^*, v^*) + \rho_2(u^*, v^*)}$$
$$+ \delta \frac{\rho_2(u^*, v^*)\left[1 + \sqrt{\rho_2(u^*, v^*)\rho_2(u^*, u^*)} + \sqrt{\rho_2(u^*, v^*)\rho_2(v^*, u^*)}\right]}{1 + \rho_2(u^*, v^*) + \rho_2(u^*, u^*)\rho_2(u^*, v^*)\rho_2(v^*, u^*)\rho_2(v^*, v^*)}$$
$$+ \eta \max\left\{\rho_2(u^*, v^*), \frac{\rho_2(u^*, u^*)\rho_2(v^*, v^*)}{1 + \rho_2(u^*, v^*)}\right\}.$$

Hence

$$\rho_2(u^*, v^*) + \alpha\rho_2(u^*, v^*) \leq \delta\rho_2(u^*, v^*) + \eta\rho_2(u^*, v^*)$$
$$\rho_2(u^*, v^*) \leq (\delta + \eta - \alpha)\,\rho_2(u^*, v^*)$$

But $(\delta + \eta - \alpha) < 1$. Therefore we have,$\rho_2(u^*, v^*) < \rho_2(u^*, v^*)$. This is a contradiction. Hence, u^* is a unique common fixed point of T_i^m and T_j^n. The fixed point of T_i^m is a fixed point of T_i and the fixed point of T_j^n is fixed point of T_j.Therefore u^* is unique common fixed point of T_i and T_j. Hence u^* is unique common fixed point of P. This completes the proof. □

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