Mathematical Analysis and its Contemporary Applications
Volume 5, Issue 2, 2023, 29–38
doi: 10.30495/maca.2023.2002074.1076
ISSN 2716-9898

Evaluation of integrals and series with uncommon methods

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Abstract. Definite integrals and series are calculated using uncommon methods: Feynman's parametrization, Schwinger's representation, Cauchy complex integration for series.

1. Introduction and some integrals

While derivation is a mechanical operation, integration is a sort of art. There are various techniques used for performing this operation, together with the solutions found in some special books [**1, 2, 6, 7**], and the increasing usage in recent times of computer algebra systems - CAS, to cite a few [**9**]-[**12**], not to forget their ancestors like Analitik (1968) and Macsyma (1968). Using a different toolbox from the usual one helps to achieve better results.

We shall start with the evaluation of the following integral:

$$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x} dx \ . \tag{1}$$

Going on the complex plane, observing that $\Im e^{ix} = \sin x$, we could choose a closed contour P formed by a semicircle on the upper half-plane, C_+, which contribution goes to zero as its radius approaches infinity because of the exponential function, then the part $\mathbb{R} \setminus \{0\}$, that is the principal value of I, in order to avoid the pole at the origin, and finally a small semicircle of radius ε that will eventually go to zero in the upper half-plane centred at the origin, c_+. Because there are no

2020 *Mathematics Subject Classification.* Primary 30E20; Secondary 33B15, 33B20.
Key words and phrases. Integration, Integrals of Cauchy, Gamma Function, Series

poles inside the contour P,

$$\int_P \frac{e^{iz}}{z} dz = 0 \; , \tag{2}$$

therefore one ends up with the sole contribution of the principal value integral and the contribution of the small semicircle c_+,

$$\text{p.v.} \quad \int_{-\infty}^{+\infty} \frac{e^{iz}}{z} dz + \int_{c_+} \frac{e^{iz}}{z} dz = 0 \; . \tag{3}$$

Clearly,

$$\int_{c_+} \frac{e^{iz}}{z} dz = i\pi \; , \tag{4}$$

and when c_+ shrinks to zero one obtains

$$\lim_{\varepsilon \to 0^+} \Im \int_{c_+} \frac{e^{iz}}{z} dz = I = \pi \; . \tag{5}$$

As a side product, formula (4) tells us also that p.v.$\int_{-\infty}^{+\infty} \frac{\cos x}{x} dx = 0$. A more difficult integral, akin to the previous one is

$$\int_0^{+\infty} dx \left(\frac{\sin x}{x} \right)^n \; . \tag{6}$$

Going again on the complex plane, and shifting the pole at the origin one could write

$$\begin{gathered}
\int_0^{+\infty} dx \left(\frac{\sin x}{x} \right)^n = \lim_{\varepsilon \to 0^+} \frac{1}{2} \int_0^{+\infty} dx \left(\frac{\sin x}{x - i\varepsilon} \right)^n = \\
\lim_{\varepsilon \to 0^+} \frac{1}{2} \int_0^{+\infty} dx \frac{1}{(x - i\varepsilon)^n} \left(\frac{\exp(ix) - \exp(-ix)}{2i} \right)^n = \\
\lim_{\varepsilon \to 0^+} \frac{1}{2} \frac{1}{(2i)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^{+\infty} dx \frac{\exp(ix(n-2k))}{(x - i\varepsilon)^n}
\end{gathered} \tag{7}$$

The integral obtained in (7) will be treated in the same manner as discussed previously, going on the complex plane. If $n - 2k \geq 0$ one closes the contour in the upper half–plane, picking the residue in $x = i\varepsilon$, otherwise closes the contour in the lower half–plane where are no residues. The upper bound of the sum is $\lfloor n/2 \rfloor$.

Using the Cauchy residues formula the result equals to

$$\frac{1}{2}\frac{1}{(2i)^n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\frac{2\pi i}{(n-1)!}\left(\frac{d}{dx}\right)^{n-1}\exp(ix(n-2k))\bigg|_{x=0} =$$

$$\frac{1}{2}\frac{1}{(2i)^n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\frac{2\pi i}{(n-1)!}(i(n-2k))^{n-1} =$$

$$\frac{\pi}{2^n(n-1)!}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}(n-2k)^{n-1}\ . \tag{8}$$

The result of the integral (6) is π times a rational number. For $n = 2, 3, 4, 5, 6$, its value is $\pi/2, 3\pi/8, \pi/3, 115\pi/384, 11\pi/40$ respectively.

2. Feynman's technique and Schwinger's representation

The integrals met so far are difficult, yet there was just employed the usual technique of integration on the complex plane. We shall now use a novel set of tools for solving definite integrals.

A very powerful integration technique was invented by Richard Feynman, a famous theoretical physicist of 20^{th} century, Nobel laureate, known for many important discoveries, among others the Feynman's diagrams for quantum field theory (see for example [4]), the development of the path integral (see for example [5]), the fundamental contribution to Quantum Electrodynamics - QED, and many others.

His technique is essentially a clever application of differentiation under the sign of integral. Let us start with a simple example:

$$I(a) = \int_0^{+\infty} e^{-ax}dx = a^{-1}\ . \tag{9}$$

This integral is convergent for $a > 0$, therefore it is possible to derive with respect to a inside the integral, obtaining

$$\left(-\frac{d}{da}\right)I(a) = \int_0^{+\infty} xe^{-ax}dx = a^{-2}\ . \tag{10}$$

By induction one has the following result ($n \in \mathbb{N}$)

$$\left(-\frac{d}{da}\right)^n I(a) = \int_0^{+\infty} x^n e^{-ax}dx = a^{-n-1}n!\ . \tag{11}$$

For $a = 1$ in (11) one recovers the usual Gamma function

$$\int_0^{+\infty} x^n e^{-x}dx = \Gamma(n+1) = n!\ . \tag{12}$$

It is possible to calculate other integrals involving the exponential function using this method. Consider

$$I(a) = \int_0^{+\infty} \exp(-ax^2)dx = \frac{\sqrt{\pi}}{2}a^{-\frac{1}{2}} \tag{13}$$

for $a > 0$. By differentiation,

$$\left(-\frac{d}{da}\right) I(a) = \int_0^{+\infty} x^2 \exp(-ax^2)dx = \frac{\sqrt{\pi}}{4}a^{-\frac{3}{2}} , \tag{14}$$

and by induction one obtains

$$\left(-\frac{d}{da}\right)^n I(a) = \int_0^{+\infty} x^{2n} \exp(-ax^2)dx = (-1)^n \frac{\pi}{2} a^{-\frac{2n+1}{2}} \frac{1}{\Gamma\left(-n+\frac{1}{2}\right)} . \tag{15}$$

A simple differentiation allows us to calculate momenta of Gaussian integral, for example $\int_0^{+\infty} x^8 \exp(-ax^2)dx = a^{-9/2}105\sqrt{\pi}/72$.

Calculate the more complicated integral

$$I = \int_0^{+\infty} \exp - \left(x^2 + \frac{1}{x^2}\right) dx , \tag{16}$$

expressing it in terms of a Feynman parameter:

$$I(a) = \int_0^{+\infty} \exp - \left(x^2 + \frac{a^2}{x^2}\right) dx . \tag{17}$$

Therefore,

$$\frac{d}{da}I(a) = \int_0^{+\infty} \exp - \left(x^2 + \frac{a^2}{x^2}\right) \cdot \left(-\frac{2a}{x^2}\right) dx . \tag{18}$$

Define the variable u such that $u = a/x$, that is $du = -(a/x^2)dx$, rewriting (18) as

$$\frac{d}{da}I(a) = -2 \int_0^{+\infty} \exp - \left(u^2 + \frac{a^2}{u^2}\right) du , \tag{19}$$

and comparing this result with (16) one obtains the differential equation

$$\frac{d}{da}I(a) = -2I(a) , \tag{20}$$

which is solved using also the result of (13) by the expression

$$I(a) = I(0) \exp(-2a) = \frac{\sqrt{\pi}}{2} \exp(-2a) . \tag{21}$$

Combining the techniques encountered so far, one could face this difficult integral

$$\int_0^{+\infty} \frac{\sin(x^n)}{x^n} . \tag{22}$$

Rewrite it first in Feynman's style

$$I(a) = \int_0^{+\infty} \frac{\sin(ax^n)}{x^n} , \tag{23}$$

then differentiate

$$\frac{dI(a)}{da} = \int_0^{+\infty} \cos(ax^n)dx = a^{-\frac{1}{n}} \int_0^{+\infty} \cos(x^n)dx \ . \tag{24}$$

The latter is a Fresnel's integral for the cosine function:

$$\int_0^{+\infty} \cos(x^n)dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(1+\frac{1}{n}\right) \ , \tag{25}$$

$I(0) = 0$, so to find the result of (22) one has to integrate (24), thus obtaining

$$I(1) = \int_0^1 \frac{dI(y)}{dy}dy = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(1+\frac{1}{n}\right)\frac{n}{n-1} \ . \tag{26}$$

For instance, for $n = 5$, (22) is equal to $(5/4)\cdot\Gamma(6/5)\cdot\sqrt{5/8+\sqrt{5}/8}$, approximately 1.09.

About the definite Fresnel's integrals for functions sine and cosine,

$$\int_0^{+\infty} \sin(x^n)dx \tag{27}$$

and

$$\int_0^{+\infty} \cos(x^n)dx \ , \tag{28}$$

one could rewrite them as already done for (1),

$$\int_0^{+\infty} \sin(x^n) \text{ or } \cos(x^n)dx = \int_0^{+\infty} \Im \text{ or } \Re \exp(iax^n)dx \ , \tag{29}$$

a being the usual Feynman parameter. Then, by a change of variable like $u = x^n$, transform Fresnel's integrals to Gamma function (12), ending up with:

$$\int_0^{+\infty} \exp(iax^n)dx = \Gamma\left(\frac{1}{n}\right)\frac{1}{na^{\frac{1}{n}}}\exp\left(i\frac{\pi}{2n}\right) \ . \tag{30}$$

For $a = 1$, one eventually obtains the values of Fresnel's integrals:

$$\int_0^{+\infty} \sin(x^n)dx = \Im\int_0^{+\infty} \exp(iax^n)dx = \Gamma\left(\frac{1}{n}\right)\frac{1}{n}\sin\left(\frac{\pi}{2n}\right) \tag{31}$$

and

$$\int_0^{+\infty} \cos(x^n)dx = \Re\int_0^{+\infty} \exp(iax^n)dx = \Gamma\left(\frac{1}{n}\right)\frac{1}{n}\cos\left(\frac{\pi}{2n}\right) \ . \tag{32}$$

Julian Schwinger was another important theoretical physicist of the 20^{th} century, Nobel laureate, who greatly contributed to the foundation of Quantum Electrodynamics. He suggested the following representation

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)}\int_0^{+\infty} t^{n-1}\exp(-tA)dt \ , \tag{33}$$

which, among other things, allows the calculation of many Feynman's diagrams in simpler terms of Gaussian's function for the propagators (see for instance [4]). Using his method, we will calculate

$$I = \int_0^{+\infty} \frac{1}{(x^2+m^2)^n} dx \ , \tag{34}$$

where $n \in \mathbb{N}$, and m is a parameter, usually a mass in physics. Using his representation (33) in (34), one obtains a double integral

$$I = \frac{1}{\Gamma(n)} \int_0^{+\infty} \int_0^{+\infty} \left(t^{n-1} \exp(-t \cdot (x^2+m^2)) dt \right) dx \ , \tag{35}$$

immediately recognising the familiar Gamma function (12) in the variable x as well. As both integrals converge because of the exponential function, it is possible to exchange the order of integration and start with the dx part:

$$\frac{1}{\Gamma(n)} \int_0^{+\infty} t^{n-1} \exp(-t \cdot (x^2+m^2)) dx = \frac{\sqrt{\pi}}{2\Gamma(n)} t^{n-3/2} \exp(-tm^2) \ , \tag{36}$$

and then integrate in the variable t

$$\frac{\sqrt{\pi}}{2\Gamma(n)} \int_0^{+\infty} t^{n-3/2} \exp(-tm^2) dt = \frac{\sqrt{\pi}}{2\Gamma(n)} \Gamma\left(n - \frac{1}{2}\right) (m^2)^{-n+1/2} = I \ . \tag{37}$$

For instance, $\int_0^{+\infty} 1/(x^2+m^2)^4 dx = 5\sqrt{\pi}(m^2)^{-7/2}/32$.

Now we shall see an integral solved via Feynman's technique that does not involve an exponential function.

Consider the integral

$$I = \int_0^1 \frac{x^2-1}{\log x} dx \ , \tag{38}$$

and introduce the function

$$I(a) = \int_0^1 \frac{x^a-1}{\log x} dx \tag{39}$$

which is a slight variation of (38), being equal when $a = 2$. It converges for $a > 0$, and the derivative in a is easy to compute:

$$\frac{dI(a)}{da} = \int_0^1 x^a = \frac{1}{1+a} \ . \tag{40}$$

To solve (38) one has to calculate $I(2)$, which is obtained by integration and using the fact that $I(0) = 0$:

$$I(2) = \int_0^2 \frac{dI(y)}{dy} dy = \int_0^2 \frac{dy}{1+y} = \log 3 \ , \tag{41}$$

and it is also clear that $I(a) = \log(a+1)$.

3. Summation of series

Using an integration on the complex plane and Cauchy's residue theorem it is possible to obtain a method for finding a sum of series.

The sketch of the proof for this method goes as follows. Suppose that $f(z)$ goes to zero for large $|z|$ like $1/z^2$ or faster. Define the auxiliary function

$$g(z) = \frac{\pi}{\tan(\pi z)} f(z) \ , \tag{42}$$

$g(z)$ is singular for every $z \in \mathbb{Z}$: it has a pole at every integer, besides those of $f(z)$.

On the complex plane, define a square S_n centred at the origin, with sides parallels to axes and length $2n+1$, $n \in \mathbb{N}$. We shall show that on this contour

$$\lim_{n\to+\infty} \int_{S_n} g(z)dz = 0 \ . \tag{43}$$

From the hypothesis on $f(z)$, we have that $|f(z)| \leq K/n^\alpha$, $\alpha \geq 2$ for large n. Also,

$$\left|\frac{1}{\tan(\pi z)}\right| = \left|1 + \frac{2}{(\exp(2\pi i z) - 1)}\right| \leq \left|1 + \frac{2}{(\exp(2\pi i z) - 1)}\right| \ . \tag{44}$$

For z on the border of the square S_n, the maximal value of the above expression is obtained for its closest point of to the point $(1, 0)$ on the complex plane, that is $z \in S_n = (n + 1/2, 0)$. Formula (44) becomes:

$$\left|1 + \frac{2}{(\exp(2\pi i(n + 1/2)) - 1)}\right| = 1 + \left|\frac{2}{(\exp(\pi i) - 1)}\right| = 1 + \frac{2}{2} = 2 \ , \tag{45}$$

because $\exp(2\pi i n) = 1$. Finally, $1/|\tan(\pi z)| \leq 2$. For z approaching infinity, that is $n \to +\infty$,

$$\lim_{n\to+\infty} \left|\int_{S_n} g(z)dz\right| \leq \lim_{n\to+\infty} \int_{S_n} \left|2\pi \frac{K}{n^\alpha}\right| dz \leq \lim_{n\to+\infty} \left(\frac{2\pi K}{n^\alpha} 4(2n+1)\right) = 0 \ . \tag{46}$$

The integral of equation (43) is zero. This also equals the sum of residues of the integrand, that is

$$\lim_{n\to+\infty} \left(\sum_{k=-n}^{+n} f(k)\right) + \sum_{\substack{\text{Residues of } g \text{ calculated} \\ \text{at singularities of } f}} \mathrm{Res}(g(z)) = 0. \tag{47}$$

Therefore,

$$\sum_{k=-\infty}^{+\infty} f(k) = - \sum_{\substack{\text{Residues of } g \text{ calculated} \\ \text{at singularities of } f}} \mathrm{Res}(g(z)) \ , \tag{48}$$

this means that the calculation of a series is converted to a finite sum over singularities of $f(z)$.

Let us compute some values of Riemann's zeta function (see for instance [3] for more details), starting from $\zeta(2)$. In this case, $f(z) = 1/z^2$, which has a single pole at the origin, then

$$\zeta(2) = \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{1}{k^2} = -\frac{1}{2}\text{Res}\left(\frac{\pi}{z^2\tan(\pi z)}, z=0\right) . \tag{49}$$

Since

$$\frac{\pi}{\tan(\pi z)} = \frac{1}{z} - z\frac{\pi^2}{3} - z^3\frac{\pi^4}{45} - z^5\frac{2\pi^6}{945} - z^7\frac{\pi^8}{4725} + O(z^9) , \tag{50}$$

$$\zeta(2) = -\frac{1}{2}\text{Res}\left(\frac{\pi}{z^2\tan(\pi z)}, z=0\right) = \text{Res}\left(-\frac{1}{2z^3} + \frac{\pi^2}{6z} + O(z), z=0\right) = \frac{\pi^2}{6} . \tag{51}$$

In analogous fashion, in order to calculate $\zeta(4)$, it suffices to find out the term of $O(1/z)$ coming from the product of the expansion given in formula (50) and the function $1/z^4$, multiplied by $-1/2$, that is

$$\zeta(4) = \frac{\pi^4}{90} . \tag{52}$$

Another example is the evaluation of the series

$$\sum_{k=1}^{+\infty} \frac{1}{k^2+k^4} . \tag{53}$$

In this case, $f(z) = 1/(z^2+z^4)$ that has three poles, for $z = 0$, and $z = \pm i$. Therefore,

$$\sum_{k=1}^{+\infty} \frac{1}{k^2+k^4} = -\frac{1}{2}\sum_{\substack{z=0, \\ z=+i, \\ z=-i}} \text{Res}\left(\frac{\pi}{(z^2(1+z^2))\tan(\pi z)}\right) . \tag{54}$$

The residue of $z = 0$ is the same of the case $\zeta(2)$. Using the fact that $\pi/\tan(\pm i\pi) = \mp i\pi/\tanh(\pi)$ and summing over the residues, one arrives at the result

$$\sum_{k=1}^{+\infty} \frac{1}{k^2+k^4} = \frac{1}{2} + \frac{\pi^2}{6} - \frac{1}{2}\frac{\pi}{\tanh(\pi)} . \tag{55}$$

3.1. Products. When calculating products, there are seldom useful formulæ to treat them. Usually the best possible thing to do is to convert the expression to exponential form and deal with sums. Consider this example:

$$\prod_{n=1}^{+\infty}\left(\frac{1}{2^n}\right)^{\frac{1}{2^{n+1}}} , \tag{56}$$

in order to evaluate it, one shall transform the product to a sum and apply the known methods, that is:

$$\prod_{n=1}^{+\infty} \left(\frac{1}{2^n}\right)^{\frac{1}{2^{n+1}}} = \exp\left(\sum_{n=1}^{+\infty} \ln 2^{-\frac{n}{2(n+1)}}\right) = \exp\left(-\ln 2 \sum_{n=1}^{+\infty} n \left(\frac{1}{2}\right)^{n+1}\right) . \tag{57}$$

The series inside the exponential has value

$$\sum_{n=1}^{+\infty} n \left(\frac{1}{2}\right)^{n+1} = 1 \tag{58}$$

because, when $|x| < 1$, the series

$$\sum_{n=1}^{+\infty} x^n = \frac{1}{1-x} \tag{59}$$

converges and could be differentiated:

$$x^2 \frac{d}{dx}\left(\sum_{n=1}^{+\infty} x^n\right) = \sum_{n=1}^{+\infty} n x^{n+1} = x^2 \frac{d}{dx}\left(\frac{1}{(1-x)}\right) = \frac{x^2}{(x-1)^2} . \tag{60}$$

Therefore, as $x = 1/2$, (56) has the value

$$\prod_{n=1}^{+\infty} \left(\frac{1}{2^n}\right)^{\frac{1}{2^{n+1}}} = \frac{1}{2} . \tag{61}$$

4. Conclusions

The integrals shown here are, of course, just a tiny example of many more problems that could be solved using the presented techniques, normally tackled, often with great effort, by means of the usual complex plane contour integrals or with a series expansions. Learning those simple yet powerful unusual techniques allows one to be able to solve difficult problems of various kind that, to the present day, are still out of reach even for the most advanced software currently available.

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Received : *May* 2023
Accepted : *June* 2023