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Almost convex-valued perturbation to second order sweeping process

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Abstract. This paper deals with nonconvex set-valued perturbation to second order nonlinear evolution system governed by the so-called nonconvex sweeping process for a class of subsmooth moving sets depending on state and velocity. Making use of our recent paper obtained for a convex valued perturbation, we prove a new existence result when the perturbations are *almost convex* . Furthermore, we apply our result in the study of an optimal control problem known as a *minimum time problem*.

1. Introduction

The second order perturbed sweeping process has been widely studied lately. It consists of a differential inclusion, governed by a normal cone subject to external forces called perturbations. This type of problem finds its applications in nonsmooth mechanics, quasistatics, planning procedures in mathematical economy, game theory, crowd motion among others. The first results were obtained for convex sets, then generalized to the non-convex case for uniformly prox-regular sets and then for *subsmooth sets* , see e.g. [**7**, **8**, **9**, **12**, **14**, **15**, **24**] and the references therein. In these works, the perturbation was convex valued. In [**16**], the authors introduced the notion of almost convex sets which generalizes the convex sets and for which, they obtained an existence result when the right hand side of the differential inclusion is upper semicontinuous. By applying such almost convex perturbations to the sweeping process, a number of results have been obtained, see [**1**, **2**, **3**, **4**, **5**, **6**].

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The second order perturbed nonconvex sweeping process takes the following general form:

$$(\mathcal{SP})\left\{\begin{array}{l} -\ddot{y}(t)\in N_{D(t,y(t),\dot{y}(t))}(\dot{y}(t))+H(t,y(t),\dot{y}(t)) \;\text{ a.e. }\; t\in[0,T] \\ y(0)=u_0;\;\; \dot{y}(0)=v_0\in D(0,u_0,v_0), \end{array}\right.$$

$N_{D(t,y(t),\dot{y}(t))}(\dot{y}(t))$ denotes the normal cone at $\dot{y}(t)$ to the moving set $D(t,y(t),\dot{y}(t))$ and $H:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\rightharpoonup\mathbb{R}^d$ plays the role of a perturbation to the problem and is upper semicontinuous with closed convex values unnecessarily bounded and without any compactness assumption. $(\mathcal{SP})$ was solved first for convex compact sets $D(y(t))$ and $F\equiv 0$ by [**13**], then for time and state-dependent nonconvex sets $D(t,y(t))$, see for instance [**15**] and the references therein; where the authors proved the existence of solution to $(\mathcal{SP})$ for nonconvex uniformly prox-regular sets $D(t,y(t)))$. For other approaches, we refer to [**20, 22, 23**].

In the present paper, using our recent result for the perturbed second order sweeping process $(\mathcal{SP})$, we give a new result for the following "autonomous problem"

$$(\mathcal{ASP})\left\{\begin{array}{c} -\ddot{y}(t)\in N_{D(y(t),\dot{y}(t))}(\dot{y}(t))+H(y(t),\dot{y}(t)) \quad \text{a.e }\; t\in[0,T]; \\ y(0)=u_0;\;\; \dot{y}(0)=v_0\in D(u_0,v_0), \end{array}\right.$$

with an almost convex perturbation. We study topological properties of the trajectories set and we establish the existence of a minimum time control problem.

The rest paper is organized as follows. After some preliminaries and notation, in Section 3, we study topological properties of the solution set and the admissible set of $(\mathcal{SP})$, when the perturbation is upper semicontinuous with nonempty closed convex values unnecessarily bounded and the sets D depends jointly on time, state and velocity. The existence of solution of $(\mathcal{ASP})$ under weaker hypotheses on the perturbation is considered in section 4. In the last section we study a time optimal problem.

2. Notation and preliminaries

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, $\overline{\mathbf{B}}$ the unit closed ball of $\mathbb{R}^d$, $B(a,r)$ (resp. $\overline{B}(a,r)$) the open (resp. closed) ball of center $a\in\mathbb{R}^d$ and radius $r>0$, and $L^1_{\mathbb{R}^d}([0,T])$ the space of all Lebesgue integrable $\mathbb{R}^d$-valued mappings defined on $[0,T]$. We denote by $\mathcal{C}_{\mathbb{R}^d}([0,T])$ the Banach space of all continuous mappings $u:[0,T]\to\mathbb{R}^d$ endowed with the norm of uniform convergence $\|\cdot\|_{\mathcal{C}}$, $\mathcal{C}^1_{\mathbb{R}^d}([0,T])$ the Banach space of all mappings $u\in\mathcal{C}_{\mathbb{R}^d}([0,T])$ having an absolutely continuous derivatives, equipped with the norm $\|u\|_{\mathcal{C}^1}=\max\{\|u\|_{\mathcal{C}},\|\dot{u}\|_{\mathcal{C}}\}$ and $W^{2,1}_{\mathbb{R}^d}([0,T])$, the space of all continuous mappings in $\mathcal{C}_{\mathbb{R}^d}([0,T])$ such that their first derivatives are absolutely continuous and their second weak derivatives belong to $L^1_{\mathbb{R}^d}([0,T])$. Let $t\in[0,T]$, we denote by

$$\mathcal{S}_t(u_0,v_0)=\{y\in W^{2,1}_{\mathbb{R}^d}([0,t]):\; y \text{ is solution of } (\mathcal{SP})\}$$

the trajectories set of the differential inclusion $(\mathcal{SP})$ on the interval $[0,t]$ and

$$\mathcal{A}_{(u_0,v_0)}(t) = \{y(t) :\ y \in \mathcal{S}_t(u_0,v_0)\}$$

the attainable set of $(\mathcal{SP})$ at the time t.

For a nonempty closed subset A of $\mathbb{R}^d$, we denote by $\operatorname{Proj}_A(u)$ the projection of u onto A defined by

$$\operatorname{Proj}_A(u) = \big\{y \in A :\quad d(u,A) = \|u-y\|\big\}.$$

We denote by $co(A)$ the convex hull of A and $\overline{co}(A)$ the closed convex hull, characterized by

$$\overline{co}(A) = \{x \in \mathbb{R}^d :\ \forall x' \in \mathbb{R}^d,\ \langle x', x\rangle \le \delta^*(x', A)\},$$

where $\delta^*(x',A) = \sup\limits_{y\in A}\big\langle x', y\big\rangle$ stands for the support function of A at $x' \in \mathbb{R}^d$. Recall that for a closed convex subset A, we have

$$d(x,A) = \sup_{x'\in\overline{\mathbf{B}}}\bigg(\big\langle x', x\big\rangle - \delta^*(x',A)\bigg).$$

If g is a real-valued locally-Lipschitz function defined on $\mathbb{R}^d$, the Clarke subdifferential $\partial g(x)$ of g at x is the nonempty convex compact subset of $\mathbb{R}^d$, given by

$$\partial g(x) = \{\xi \in \mathbb{R}^d :\ g^\circ(x;v) \ge \big\langle \xi, v\big\rangle, \forall v \in \mathbb{R}^d\},$$

where

$$g^\circ(x;v) = \limsup_{z\to x,\ h\downarrow 0}\frac{g(z+hv)-g(z)}{h}$$

is the generalized directional derivative of g at x in the direction v (see [**17**]). Let S be a nonempty closed subset of $\mathbb{R}^d$, the Clarke normal cone to S at x is defined by $N_S(x) = \partial\Psi_S(x)$ ([**17**]), where Ψ_S denotes the indicator function of S, i.e. $\Psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. An important concept of Fréchet subdifferential will be also needed. A vector $u \in \mathbb{R}^d$ is in the Fréchet subdifferential $\partial^F g(x)$ of g at x ([**19**]) provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in B(x,\delta)$ we have

$$\big\langle u, y-x\big\rangle \le g(y) - g(x) + \varepsilon\|y-x\|.$$

The Fréchet normal cone $N^F_S(x)$ of S at $x \in S$ is defined by $N^F_S(x) = \partial^F\Psi_S(x)$. It is known that for all $x \in S$ we have the following inclusions: $\partial^F g(x) \subset \partial g(x)$; $N^F_S(x) \subset N_S(x)$ and

$$\partial^F d(x,S) = N^F_S(x) \cap \overline{\mathbf{B}}. \tag{1}$$

One has also,

$$\text{when } y \in \operatorname{Proj}_S(x) \text{ then } x-y \in N^F_S(y). \tag{2}$$

Now, let recall the definition of equi-uniform subsmoothness for a family of sets, it is an extension of convexity and prox-regularity of a set. In this way, the result concerning existence of solution of the second-order differential inclusion is more general. We begin with some basic definitions from subsmoothness while referring the reader to [11].

Let S be a closed subset of $\mathbb{R}^d$, we say that S is subsmooth at $x \in S$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big\langle \xi_1 - \xi_2, x_1 - x_2 \big\rangle \geq -\varepsilon \|x_1 - x_2\| \tag{3}$$

whenever $x_i \in \overline{B}(x, \delta) \cap S$ and $\xi_i \in N_S(x_i) \cap \overline{\mathbf{B}}$, $i = 1, 2$. The set S is subsmooth if it is subsmooth at each point of S. We say further that S is uniformly subsmooth, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that (3) holds for all $x_i \in S$, satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_S(x_i) \cap \overline{\mathbf{B}}$.
The following subdifferential regularity of the distance function also holds true for subsmooth sets:

Proposition 2.1. [21] *Let S be a closed set of a Hilbert space. If S is subsmooth at $x \in S$, then $N_S(x) = N_S^F(x)$ and $\partial d(x, S) = \partial^F d(x, S)$.*

Definition 2.1. Let $\big(S(q)\big)_{q\in I}$ be a set of closed sets of $\mathbb{R}^d$ with parameter $q \in I$. It is called equi-uniformly subsmooth, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $q \in I$, the inequality (3) holds, for all $x_i \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and for all $\xi_i \in N_{S(q)}(x_i) \bigcap \overline{\mathbf{B}}$, $i = 1, 2$.

Proposition 2.2. [18] *Let $\big\{S(t, v) \ : \ (t, v) \in [0, T] \times \mathbb{R}^d\big\}$ be a family of nonempty closed sets of $\mathbb{R}^d$ which is equi-uniformly subsmooth and let a real number $\eta > 0$. Assume that there exist real constants $L_1 > 0$ and $L_2 > 0$ such that, for any $x, y, u, v \in \mathbb{R}$ and $s, t \in [0, T]$*

$$|d\big(x, S(t, u)\big) - d\big(y, S(s, v)\big)| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Then the following assertions hold:

(a) *for all $(s, v, y) \in \mathrm{Gph}(S)$, $\eta\partial d\big(y, S(s, v)\big) \subset \eta\overline{\mathbf{B}}$;*
(b) *$\partial d\big(\cdot, S(\cdot, \cdot)\big)$ is an upper semicontinuity set-valued mapping, that is, for any sequences $(s_n, v_n)_n \subset [0, T] \times \mathbb{R}$ converging to (s, v), $(y_n)_n$ converging to $y \in S(s, v)$ with $y_n \in S(s_n, v_n)$ and for all $\xi \in \mathbb{R}$, we have*

$$\limsup_{n\to\infty} \sigma\bigg(\xi, \eta\partial d\big(y_n, S(s_n, v_n)\big)\bigg) \leq \sigma\bigg(\xi, \eta\partial d\big(y, S(s, v)\big)\bigg).$$

In the next, we give the definition of the almost convex sets.

Definition 2.2. [16] For a vector space X, a set $Q \subset X$ is called almost convex if for every $\xi \in co(Q)$ there exist λ_1 and λ_2, $0 \leq \lambda_1 \leq 1 \leq \lambda_2$, such that $\lambda_1\xi \in Q$ and $\lambda_2\xi \in Q$.

Trivially, any convex set is almost convex since $Q = co(K)$. If K is a convex set not containing the origin, $Q = \partial K$ is almost convex, and if the convex set K contains the origin, one take $Q = \{0\} \cup \partial K$.

3. Main results

3.1. Sweeping process with convex perturbation. In this section we study topological properties of the set of trajectories and the admissible set of the problem $(\mathcal{SP})$.

Theorem 3.1. *Let $D : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \rightharpoonup \mathbb{R}^d$ be a set-valued mapping with nonempty closed values satisfying:*

$(\mathcal{D}_1)$ $\left\{D(t,x,x') : (t,x,x') \in [0,T] \times \mathbb{R} \times \mathbb{R}^d\right\}$ *is equi-uniformly subsmooth;*

$(\mathcal{D}_2)$ *there are three constants $\Lambda_1 \in]0,1[$, $\Lambda_2 > 0$ and $L > 0$ such that, for all $t_i, x_i, x_i', z \in \mathbb{R}^d (i = 1,2)$*

$$|d(z, D(t_1,x_1,x_1')) - d(z, D(t_2,x_2,x_2'))| \leq L|t_1 - t_2| + \Lambda_1\|x_1 - x_2\| + \Lambda_2\|x_1' - x_2'\|.$$

And let $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \rightharpoonup \mathbb{R}^d$ be a set-valued mapping with nonempty closed convex values, upper semicontinuous on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$(\mathcal{H})$ *for some real $\kappa > 0$*

$$d\big(0, H(t,x,x')\big) \leq \kappa\big(1 + \|x\| + \|x'\|\big), \quad \textit{for all } (t,x,x') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

Then, for every $(u_0,v_0) \in \mathbb{R}^d \times \mathbb{R}^d$ with $v_0 \in D(0,u_0,v_0)$, the set of trajectories $\mathcal{S}_t(u_0,v_0)$ is nonempty and compact in $W^{2,1}_{\mathbf{R}^d}([0,t])$.

Proof. 1) A simple adjustment of Theorem 3.1 in [9] gives the existence of solution of $(\mathcal{SP})$, with

$$\|\dot{y}(t)\| \leq \Delta \quad \text{and} \quad \|\ddot{y}(t)\| \leq \Theta, \quad \text{a.e. } t \in [0,T],$$

where

$$\Delta = \Big(\|u_0\| + T\big(\frac{2\kappa(2 + 2\|v_0\| + \|u_0\|) + 2L}{1 - \Lambda_1}\big)\Big) \exp\Big(T\big(\frac{\Lambda_2 + 2\kappa(1+T)}{1 - \Lambda_1}\big)\Big),$$

$$\Theta = \frac{L + \Lambda_2\Delta + 2\kappa(1 + \Delta + \Upsilon)}{1 - \Lambda_1} + L + 2\kappa(1 + \|u_0\| + \|v_0\|).$$

2) **Compactness of $\mathcal{S}_t(u_0,v_0)$**. Let $(y_n)_n \subset \mathcal{S}_t(u_0,v_0)$ be a sequence of trajectories, so, for $\tilde{t} \in [0,t]$ and $n \in \mathbb{N}$,

$$||y_n(\tilde{t})|| \leq \Upsilon, \ \ ||\dot{y}_n(\tilde{t})|| \leq \Delta \quad \text{and} \ ||\ddot{y}_n(\tilde{t})|| \leq \Theta.$$

Then the sequences $(y_n(\tilde{t}))_n$ and $(\dot{y}_n(\tilde{t}))_n$ are relatively compact and equi-continuous, (y_n) is relatively compact in $(\mathcal{C}^1_{\mathbb{R}^d}([0,t]), \|.\|_{\mathcal{C}^1})$. By extracting a subsequence, (y_n) converges to some mapping $y \in (\mathcal{C}^1_{\mathbb{R}^d}([0,t]), \|.\|_{\mathcal{C}^1})$ and $(\ddot{y}_n)$ converges weakly to $\ddot{y}$ with $||\ddot{y}(\tilde{t})|| \leq \Theta$ a.e. $\tilde{t} \in [0,t]$.

For each $n \in \mathbb{N}$, let $h_n(t) \in H\big(t, y_n(t), \dot{y}_n(t)\big)$ be the element of minimal norm which is a measurable selection, by the hypothesis $(\mathcal{H})$, we get

$$\|h_n(\tilde{t})\| \leq \kappa(1 + \Delta + \Upsilon), \ \forall \tilde{t} \in [0, t],$$

so we can extract from the sequence $(h_n)_n$ a subsequence which converge in $\sigma(L^{\infty}_{\mathbb{R}^d}([0,t]), L^1_{\mathbb{R}^d}([0,t]))$ to some mapping $h \in L^1_{\mathbb{R}^d}([0,t])$. Let us prove now that y is solution of the problem $(\mathcal{SP})$. We have

$$d\big(\dot{y}_n(t), D(t, y(t), \dot{y}(t))\big) \leq d\big(\dot{y}_n(t), D(t, y(t), \dot{y}(t))\big) - d\big(\dot{y}_n(t), D(t, y_n(t), \dot{y}_n(t))\big)$$

$$\leq \Lambda_1 ||y_n(t) - y(t)|| + \Lambda_2 ||\dot{y}_n(t) - \dot{y}(t)||.$$

As $D\big(t, y(t), \dot{y}(t)\big)$ is closed, by passing to the limit in the preceding inequality, we get $y(t) \in D\big(t, y(t), \dot{y}(t)\big)$.

Now, we have

$$\|\ddot{y}_n(t) + h_n(t)\| \leq \|\ddot{y}_n(t)\| + \|h_n(t)\| \leq \Theta + \kappa\big(1 + \Delta + \Upsilon\big) = l,$$

that is,

$$\ddot{y}_n(t) + h_n(t) \in l\overline{\mathbf{B}}, \text{ for all } n \in \mathbb{N}.$$

Since

$$\ddot{y}_n(t) + h_n(t) \in -N_{D\left(t, y_n(t), \dot{y}_n(t)\right)}\big(\dot{y}_n(t)\big), \text{ for all } n \in \mathbb{N}$$

we get by (1)

$$\ddot{y}_n(t) + h_n(t) \in -l\partial d\Big(\dot{y}_n(t), D\big(t, y_n(t), \dot{y}_n(t)\big)\Big).$$

Note that $(\ddot{y}_n + h_n, h_n)_n$ weakly converges in $L^1_{\mathbb{R}^d \times \mathbb{R}^d}\big([0,t]\big)$ to $(\ddot{y} + h, h)$. An application of the Mazur's Theorem to $(\dot{y}_n + h_n, h_n)_n$ and we follow the same demonstration as that of part 1) we get

$$\ddot{y}(t) + h(t) \in -l\partial d\Big(\dot{y}(t), D\big(t, y(t), \dot{y}(t)\big)\Big) \subset -N_{D\left(t, y(t), \dot{y}(t)\right)}\big(\dot{y}(t)\big), \quad \text{a.e } \tilde{t} \in [0, t]$$

and

$$h(t) \in H\big(t, y(t), \dot{y}(t)\big) \quad \text{for all } \tilde{t} \in [0, t].$$

This completes the proof of the Theorem. □

The following is a direct consequence of Theorem 3.1

Corollary 3.2. *Under the hypotheses of Theorem 3.1. For all $t \in [0, T]$, the attainable set $\mathcal{A}_{(u_0, v_0)}(t)$ at t for $(\mathcal{SP})$ is compact.*

3.2. Sweeping process with almost convex perturbation. Now we are going to announce an existence result for $(\mathcal{ASP})$ when the perturbation H takes almost convex values and with a weaker assumption on the upper semicontinuity.

Theorem 3.3. *Let $D : \mathbb{R}^d \times \mathbb{R}^d \rightharpoonup \mathbb{R}^d$ be with nonempty closed values satisfying:*

$(\mathcal{D}_1')$ *$\{D(x,y) : (x,y) \in \mathbb{R}^d \times \mathbb{R}^d\}$ is equi-uniformly subsmooth;*

$(\mathcal{D}_2')$ *there exist $\Lambda_1 \in]0,1[$ and $\Lambda_2 > 0$ s.t. for all $z, x_i, y_i \in \mathbb{R}^d (i = 1, 2)$*

$$|d(z, D(x_1, y_1)) - d(z, D(x_2, y_2))| \le \Lambda_1 \|x_1 - x_2\| + \Lambda_2 \|y_1 - y_2\|;$$

$(\mathcal{D}_3')$ *for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and every $\alpha > 0$, $\alpha D(x,y) \subseteq D(x, \alpha y)$.*

And let $H : \mathbb{R}^d \times \mathbb{R}^d \rightharpoonup \mathbb{R}^d$ be with almost convex compact values satisfying

$(\mathcal{H}_1)$ *$co\big(H(\cdot,\cdot)\big)$ is upper semicontinuous on $\mathbb{R}^d \times \mathbb{R}^d$;*

$(\mathcal{H}_2)$ *for some $\kappa > 0$*

$$d\big(0, co(H(x,y)\big) \le \kappa \Big(1 + \|x\| + \|y\|\Big), \ \forall (x,y,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d;$$

$(\mathcal{H}_3)$ *for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, and every $\beta > 0$, $H(x, \beta y) \subseteq \beta H(x,y)$.*

Then, for every $(u_0, v_0) \in \mathbb{R}^d \times \mathbb{R}^d$ with $v_0 \in D(u_0, v_0)$, there exists at least one solution u of $(\mathcal{ASP})$.

Proof. 1) Thanks to Theorem 3.1, there is at least one solution $x : [0,T] \to \mathbb{R}^d$, of the convexified problem

$$(\mathcal{ASP}_{co}) \begin{cases} -\ddot{y}(t) \in N_{D\big(y(t),\dot{y}(t)\big)}(\dot{y}(t)) + co\big(H(y(t), \dot{y}(t))\big) \text{ a.e } \ t \in [0,T]; \\ y(0) = u_0; \ \dot{y}(0) = v_0 \in D(u_0, v_0). \end{cases}$$

2) By the almost convexity of the set $H(x(t), \dot{x}(t))$, for any $t \in [0,T]$, there exist two sets

$$\Gamma_1(t) = \{\theta_1 \in [0,1] : \ \theta_1 \operatorname{Pro}_{co(H(x(t),\dot{x}(t))}(0) \in H\big(x(t), \dot{x}(t)\big)\}$$

and

$$\Gamma_2(t) = \{\theta_2 \in [1, +\infty[: \ \theta_2 \operatorname{Pro}_{co(H(x(t),\dot{x}(t))}(0) \in H(x(t), \dot{x}(t))\}.$$

Let $[a,b] \subset [0,T]$ and assume that $\lambda_1 \in \Gamma_1(t) \setminus \{0\}$ and $\lambda_2 \in \Gamma_2(t)$. Using Theorem 2 in [5], we conclude that there are two measurable subsets of $[a,b]$ having characteristic functions χ_1 and χ_2 such that $\chi_1 + \chi_1 = \chi_{[a,b]}$ and an absolutely continuous function $s : [a,b] \to [a,b]$ such that

$$\dot{s}(\tau) = \frac{1}{\lambda_1} \chi_1(\tau) + \frac{1}{\lambda_2} \chi_2(\tau)$$

and $s(b) - s(a) = b - a$.

3) Considering the closed set

$$C = \{\tau \in [0,T] : 0 \in co\big(H(x(t), \dot{x}(t))\big)\},$$

if C is empty, then $\lambda_1 \neq 0$, so, we can apply part 2) on $[0,T]$. Putting $s(\tau) = \int_0^\tau \dot{s}(\omega)d\omega$, s is increasing and $\big(s(0), s(T)\big) = (0,T)$, so, s maps $[0,T]$ onto itself. Let $t : [0,T] \to [0,T]$ be its inverse, then $\big(t(0), t(T)\big) = (0,T)$ and $\frac{d}{d\tau}s(t(\tau)) = \dot{s}(t(\tau))\dot{t}(\tau) = 1$, then,

$$\dot{t}(\tau) = \lambda_1 \mathcal{X}_1(t(\tau)) + \lambda_2 \mathcal{X}_2(t(\tau)),$$

and $\ddot{t}(\tau) = 0$. Now, considering the mapping $\tilde{x}(\tau) = x(t(\tau))$, we have $\frac{d}{d\tau}\tilde{x}(\tau) = \dot{t}(\tau)\dot{x}(t(\tau))$, then

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau) = (\dot{t}(\tau))^2\ddot{x}(t(\tau)) + \ddot{t}(\tau)\dot{x}(t(\tau)) = \ddot{x}(t(\tau))(\dot{t}(\tau))^2,$$

and we have

$$-\frac{1}{\dot{t}(\tau)}\frac{d^2}{d\tau^2}\tilde{x}(\tau) = -\ddot{x}(t(\tau))(\dot{t}(\tau)) = -\ddot{x}(t(\tau))\Big(\lambda_1 \mathcal{X}_1(t(\tau)) + \lambda_2 \mathcal{X}_2(t(\tau))\Big)$$

$$\in \Big(N_{D(x(t(\tau)),\dot{x}(t(\tau)))}(\dot{x}(t(\tau))) + \mathrm{Pro}_{co(H(x(t),\dot{x}(t))}(0)\Big)\Big(\lambda_1 \mathcal{X}_1(t(\tau)) + \lambda_2 \mathcal{X}_2(t(\tau))\Big),$$

using normal cone properties, the fact that $\lambda_1 \in \Gamma_1(t) \setminus \{0\}$ and $\lambda_2 \in \Gamma_2(t)$, we obtain

$$-\frac{1}{\dot{t}(\tau)}\frac{d^2}{d\tau^2}\tilde{x}(\tau) \in N_{D\left(\tilde{x}(\tau), \frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau)\right)}\big(\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau)\big) + H\big(\tilde{x}(\tau), \frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau)\big),$$

by $(\mathcal{D}_3')$, $(\mathcal{H}_2)$ and the properties of the normal cone we can write

$$-\frac{1}{\dot{t}(\tau)}\frac{d^2}{d\tau^2}\tilde{x}(\tau) \in N_{\frac{1}{\dot{t}(\tau)}D\left(\tilde{x}(\tau),\dot{\tilde{x}}(\tau)\right)}\big(\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau)\big) + \frac{1}{\dot{t}(\tau)}H\big(\tilde{x}(\tau), \dot{\tilde{x}}(\tau)\big)$$

$$\in N_{D\left(\tilde{x}(\tau),\dot{\tilde{x}}(\tau)\right)}\big(\dot{\tilde{x}}(\tau)\big) + \frac{1}{\dot{t}(\tau)}H\big(\tilde{x}(\tau), \dot{\tilde{x}}(\tau)\big)$$

then

$$-\frac{d^2}{d\tau^2}\tilde{x}(\tau) \in N_{D\left(\tilde{x}(\tau),\dot{\tilde{x}}(\tau)\right)}\big(\dot{\tilde{x}}(\tau)\big) + H\big(\tilde{x}(\tau), \dot{\tilde{x}}(\tau)\big).$$

If C is nonempty, let $c = \sup\{\tau;\ \tau \in C\} \in C$, since C is closed. The complement of C is open relative to $[0,T]$, it consists of at most countably many nonoverlapping open intervals (a_i, b_i), with the possible exception of one of the form $[a_{i_i}, b_{i_i})$ with $a_{i_i} = 0$ and one $(a_{i_f}, b_{i_f}]$ with $a_{i_f} = c$. For each i apply the part 2) to the interval (a_i, b_i) to infer the existence of K_1^i and K_2^i, two subsets of (a_i, b_i) with characteristic functions $\mathcal{X}_1^i(.)$, $\mathcal{X}_2^i(.)$ such that $\mathcal{X}_1^i + \mathcal{X}_2^i = \mathcal{X}_{[a_i,b_i]}$, setting

$$\dot{s}(\tau) = \mathcal{X}_1^i(\tau)\frac{1}{\lambda_1} + \mathcal{X}_2^i(\tau)\frac{1}{\lambda_2}$$

we obtain

$$\int_{a_i}^{b_i} \dot{s}(\omega)d\omega = b_i - a_i.$$

a) On $[0,c]$, set

$$\dot{s}(\tau)=\frac{1}{\lambda_2}\mathcal{X}_C(\tau)+\sum\left(\mathcal{X}_1^i(\tau)\frac{1}{\lambda_1}+\mathcal{X}_2^i(\tau)\frac{1}{\lambda_2}\right),$$

where the sum is over all intervals contained in $[0,c]$, i.e., with the exception of $]c,T]$, we obtain

$$\int_0^c \dot{s}(\omega)d\omega=\kappa\le c;$$

bercause $\lambda_2\ge 1$ and $\int_{a_i}^{b_i}\dot{s}(\omega)d\omega=b_i-a_i$. Set $s(\tau)=\int_0^\tau \dot{s}(\omega)d\omega$, s is an invertible mapping from $[0,c]$ to $[0,\kappa]$. Define $t:[0,\kappa]\to[0,c]$ the inverse of $s(\cdot)$ and extend $t(\cdot)$ to $[0,c]$ as an absolutely continuous mapping $\tilde{t}(\cdot)$ with $\dot{\tilde{t}}(s)=0$ for $s\in]\kappa,c]$. Let prove that $\tilde{x}(\tau)=x(\tilde{t}(\tau))$ is a solution of $(\mathcal{ASP})$ on the interval $[0,c]$ and satisfies $\tilde{x}(c)=x(c)$. Indeed, we have that for $\tau\in[0,\kappa]$, $\tilde{t}(\tau)=t(\tau)$ is invertible, such that

$$\dot{t}(\tau)=\lambda_2\mathcal{X}_C(\tau)+\sum\left(\mathcal{X}_1^i(\tau)\lambda_1+\mathcal{X}_2^i(\tau)\lambda_2\right),$$

since

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau)=(\dot{t}(\tau))^2\ddot{x}(t(\tau))+\ddot{t}(\tau)\dot{x}(t(\tau))=\ddot{x}(t(\tau))(\dot{t}(\tau))^2,$$

we get

$$\begin{aligned}\frac{1}{\dot{t}(\tau)}\frac{d^2\tilde{x}(\tau)}{d\tau^2}&=\ddot{x}(t(\tau))(\dot{t}(\tau))\\&=\left(\lambda_2\mathcal{X}_C(t(\tau))+\sum(\mathcal{X}_1^i(t(\tau))\lambda_1+\mathcal{X}_2^i(t(\tau))\lambda_2)\right)\ddot{x}(t(\tau))\\&\in N_{D(\tilde{x}(\tau),\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau))}(\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau))+H(\tilde{x}(\tau),\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau))\end{aligned}$$

consequently

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau)\in N_{D(\tilde{x}(\tau),\dot{\tilde{x}}(\tau))}(\dot{\tilde{x}}(\tau))+H(\tilde{x}(\tau),\dot{\tilde{x}}(\tau)).$$

In particular, since $t(k)=c$ and $\dot{t}(\tau)=0$, for all $\tau\in]k,c]$, one has

$$\tilde{t}(\tau)=\tilde{t}(k)=t(k),\ \forall\tau\in]k,c]$$

then

$$\tilde{x}(k)=x(\tilde{t}(k))=x(\tilde{t}(\tau))=\tilde{x}(\tau),\ \forall\tau\in]k,c]$$

then, $\tilde{x}$ is constant on $]\kappa,c]$, we cdeduce that $\dot{\tilde{x}}(\tau)=\dot{x}(\tau)$ is solution of $(\mathcal{ASP})$.

b) On $[c,T]$, $\lambda_1>0$, by (a), the construction of the part 2 can be repeated to find a solution to problem $(\mathcal{ASP})$. □

3.3. Time optimal problem. In this section we investigate the existence of solution to the following minimum time problem for the differential inclusion

$$(\mathcal{ASP}_h)\begin{cases} \ddot{y}(t) \in -N_{D\left(y(t),\dot{y}(t)\right)}(y(t)) + g\Big(y(t),\dot{y}(t),\nu(t)\Big) \;\text{ a.e. } t\in[0,T],\\ \nu(t)\in Z\big(y(t),\dot{y}(t)\big),\ \forall t\in[0,T],\\ y(t)\in D\Big(y(t),\dot{y}(t)\Big),\ \forall t\in[0,T],\\ y(0)=u_0,\ \dot{y}(0)=v_0 \end{cases}$$

Corollary 3.4. *Let $D:\mathbb{R}^d\times\mathbb{R}^d \rightharpoonup \mathbb{R}^d$ be with nonempty closed values satisfying $(\mathcal{D}'_1), (\mathcal{D}'_2)$ and $(\mathcal{D}'_3)$, $Z:\mathbb{R}^d\times\mathbb{R}^d\rightharpoonup\mathbb{R}^d$ an upper semicontinuous set-valued mapping with nonempty compact values and $g: Gph(Z)\to\mathbb{R}^n$ a continuous single-valued mapping satisfying:*

$(\mathcal{G}_1)$ *there is a nonnegative constant κ such that*

$$\|g(x,x',z)\| \le \kappa\big(1+\|x\|+\|x'\|\big),\ \forall (x,x',z)\in Gph(Z);$$

$(\mathcal{G}_2)$ *for all $(x,x',z)\in Gph(Z)$ and every $\beta>0$, $g(x,\beta x',z)=\beta g(x,x',z)$.*

Consider the set-valued mapping $H:\mathbb{R}^d\times\mathbb{R}^d\rightharpoonup\mathbb{R}^d$ defined by

$$H(x,x') = \{g(x,x',z)\}_{z\in Z(x,x')} \text{ for all } (x,x')\in\mathbb{R}^d\times\mathbb{R}^d.$$

Assume that $H(\cdot,\cdot)$ is almost convex and compact valued for every $(x,x')\in\mathbb{R}^d\times\mathbb{R}^d$ and assume that for given u_0,v_0,u_1 in $\mathbb{R}^d$, and for some $0\le t\le T$, $u_1\in\mathcal{A}_{(u_0,v_0)}(t)$. Then, the problem of reaching u_1 from u_0 in a minimum time admits a solution.

Proof. First we must show that for all $t\in[0,T]$ the attainable set at t, $\mathcal{A}_{(u_0,v_0}(t)$ coincides with $\mathcal{A}^{co}_{(u_0,v_0)}(t)$, the attainable set at t of the convexified problem. Indeed, For every $t\in[0,T]$, $\mathcal{A}_{(u_0,v_0)}(t)\subset\mathcal{A}^{co}_{(u_0,v_0)}(t)$, it is enough to show that $\mathcal{A}^{co}_{(u_0,v_0)}(t)\subset\mathcal{A}_{(u_0,v_0)}(t)$. Let $y(t)\in\mathcal{A}(u_0,v_0)^{co}(t)$, so $y(\cdot)$ is solution of $(\mathcal{ASP}_{co})$. Applying Theorem 3.3 on $[0,t]$, we find a solution $\tilde{u}(\cdot)$ to $(\mathcal{ASP})$ such that $y(t)=\tilde{y}(t)\in\mathcal{A}_{(u_0,v_0)}(t)$. Then, $\mathcal{A}^{co}_{(u_0,v_0)}(t)\subset\mathcal{A}_{(u_0,v_0)}(t)$.

Under the hypotheses on g and Z, $co(H)$ is upper semicontinuous,

$$d\Big(0,co\big(H(x,x')\big)\Big)\le d\Big(0,H(x,x')\Big)\le\kappa\big(1+\|x\|+\|x'\|\big),\ \forall(x,x')\in\mathbb{R}^d\times\mathbb{R}^d,$$

and for all x,x' and every $\beta>0$, $H(x,\beta x')=\beta H(x,x')$. Let $t_1=\inf\{\tau\in[0,t]: u_1\in A_{(u_0,v_0)}(\tau)\}$, (t_n) a sequence decreasing to t_1 and for each n, let $u_n(\cdot)$ be a solution of the problem

$$\begin{cases} \ddot{y}(t)\in -N_{D\left(y(t),\dot{y}(t)\right)}(\dot{y}(t)) + H\big(y(t),\dot{y}(t)\big),\ \text{a.e. in } [0,t_n],\\ \dot{y}(t)\in D\big(y(t),\dot{y}(t)\big),\ \forall t\in[0,t_n],\\ y(0)=u_0,\ \dot{y}(0)=v_0, \end{cases}$$

such that $u_n(t_n) = u_1$. We define the sequence $(\bar{u}_n(\cdot))$ by $\bar{u}_n(\tau) = u_n(\tau)$, for all $\tau \in [0, t_1]$. Then

$$(\bar{u}_n(\tau)) \subset A_{(u_0,v_0)}(\tau) = A^{co}_{(u_0,v_0)}(\tau).$$

Since $A^{co}_{(u_0,v_0)}(\tau)$ is compact, by extracting a subsequence, we may conclude that $(\bar{u}_n(\tau))$ converges to $\bar{u}(\tau) \in A^{co}_{(u_0,v_0)}(\tau)$, as $\bar{u}(t_1) = u_1 \in A^{co}_{(u_0,v_0)}(t_1) = A_{(u_0,v_0)}(t_1)$. So that, $\bar{u}$ is the solution of $(\mathcal{ASP}_h)$ that reaches u_1 in the minimum time and t_1 is the value of the minimum time. This completes the proof. □

4. Conclusion

In this paper, we extend our previous existence result obtained in [10] for the nonconvex perturbed second-order state-dependent sweeping process in two directions: we consider a general class of equi-uniformly subsmooth sets that contains convex sets and uniformly prox-regular sets, these sets depends jointly on time, state and velocity. The compactness of the attainable set is stated. Furthermore, for the autonomous problem, we provide a new existence result by taking a perturbation with almost convex values instead to be convex. Finally, as an application, the obtained results are used to prove the existence of solution to a minimum time problem.

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