Mathematical Analysis and its Contemporary Applications
Volume 5, Issue 2, 2023, 17–28
doi: 10.30495/maca.2023.1989709.1071
ISSN 2716-9898

Characterization of uniformly asymptotic S-Toeplitz and S-Hankel operators

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Abstract. In this paper, we show that a shift operator on a separable Hilbert space with infinite multiplicity is strongly approximated by shift operators with finite multiplicities. Moreover, for an arbitrary shift operator S, we introduce the notion of an (asymptotic) S-Hankel operator and study its relation to the class of (asymptotic) S-Toeplitz operators.

1. Introduction

Throughout this paper, the Hilbert space $\mathcal{H}$ is separable infinite dimensional, often identified with the space l^2 of square summable sequences, with the canonical basis is $\{e_n\}_{n=0}^{\infty}$. The spaces of all bounded linear operators and all compact operators on $\mathcal{H}$ are denoted by $B(\mathcal{H})$ and $K(\mathcal{H})$, respectively. The Hardy space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ is the collection of all analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the open unit disk $\mathbb{D}$ satisfying the norm condition

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

An isometric operator S on a Hilbert space $\mathcal{H}$ is called a unilateral forward shift (briefly a shift) if $\{S^{*n}\}$ tends strongly to 0. The dimension of the Hilbert space $\mathcal{H} \ominus S\mathcal{H}$ is called the multiplicity of S. It is well known that the condition $S^{*n} \to 0$ strongly is equivalent to the equality $\cap_{n=0}^{\infty} S^n(\mathcal{H}) = \{0\}$. The adjoint S^* of a shift

2020 *Mathematics Subject Classification.* Primary: 47B33; Secondary: 47B32.
Key words and phrases. Generalized Shift operator, S-Toeplitz operator, S-Hankel operator, Hardy space.
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will be referred to as a backward shift. Let $\mathcal{H}$ be l^2 endowed with the canonical basis $\{e_n\}_{n=0}^{\infty}$. One can easily check the linear operator U determined by the equations

$$Ue_n = e_{n+1} \qquad n = 0, 1, 2, \dots$$

is a shift of multiplicity 1 that is called the unilateral shift operator. The adjoint U^* is uniquely determined by the equations

$$U^* e_0 = 0$$

$$U^* e_n = e_{n-1} \qquad n = 1, 2, 3, \dots.$$

The unilateral shift on the Hardy space is the multiplication operator M_z given by $M_z f(z) = z f(z)$ for $f \in \mathcal{H}^2$. A Toeplitz operator on $\mathcal{H}$ is an operator whose matrix has constant diagonals, or equivalently an operator T satisfying $U^*TU = T$. Similarly, a Hankel operator is one whose matrix representation has constant anti-diagonals, or equivalently an operator H satisfying $U^*H = HU$.

Barria and Halmos in [2] introduced the notion of asymptotic Toeplitz operators in strong operator topology, extended by Feintuch in [3, 4] to other topologies on $B(\mathcal{H})$. An operator A is called uniformly (strongly, weakly) asymptotic Toeplitz if the sequence $\{U^{*n}AU^n\}$ is uniformly (strongly, weakly) convergent in $B(\mathcal{H})$. The commutator ideal of the Toeplitz algebra (the C^*-algebra generated by the set of all Toeplitz operators) is characterized in [2] using strongly asymptotic Toeplitz operators: an operator T in the Toeplitz algebra belongs to commutator ideal of the Toeplitz algebra if and only if the sequence $\{U^{*n}TU^n\}$ converges strongly to zero. Feintuch also studies asymptotic Hankel operators in some different operator topologies. An operator B is uniformly (strongly, weakly) asymptotic Hankel if the sequence $\{J_n BU^{n+1}\}$ converges uniformly (strongly, weakly), where J_n is the permutation operator of order n,

$$J_n e_i = \begin{cases} e_{n-i} & , 0 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

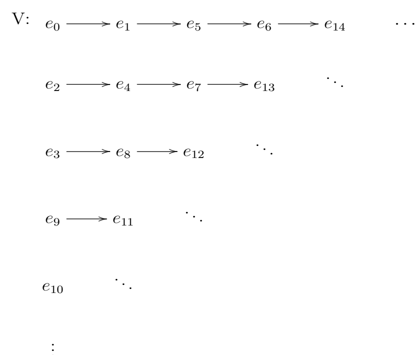
Feintuch characterized these operators and found their relation with asymptotic Toeplitz operators in [3, 4]. In Section 2, we define asymptotic Toepliz and Hankel operators with respect to an arbitrary shift operator S and give characterizations of these operators.

2. Asymptotic S-Toeplitz and S-Hankel operators

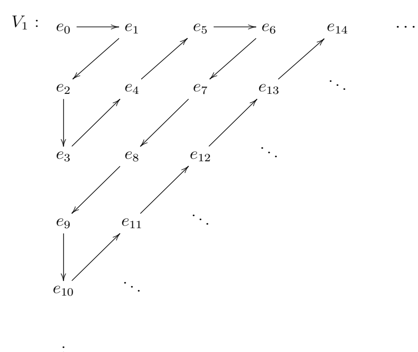
Let S be a shift operator in $B(\mathcal{H})$. If $\mathcal{K} = \ker S^*$ and $B = \{\zeta_i\}_{i\in\Lambda}$ is an orthonormal basis of $\mathcal{K}$ then by the Wold decomposition (see Chapter 1 of [7]) $\mathcal{H} = \bigoplus_{j=0}^{\infty} S^j\mathcal{K}$ with the orthonormal basis $\{S^j\zeta_i : j \ge 0, i \in \Lambda\}$ and each $f \in \mathcal{H}$ has a unique representation $f = \sum_{j=0}^{\infty} S^j k_j$ and $\|f\|^2 = \sum_{j=0}^{\infty} \|k_j\|^2$, where $k_j = P_0 S^{*j} f$ and $P_0 = I - SS^*$ is the orthogonal projection of $\mathcal{H}$ on $\mathcal{K}$ (see also [6]). The multiplicity of S is the cardinal number of the set Λ, and the Wold decomposition determines shift operators, up to unitary equivalence, by their multiplicities (see [7]). A shift operator with finite multiplicity n is unitary equivalent with U^n, where U is the unilateral shift. Let us start by showing that every shift operator with infinite multiplicity is strongly approximated by a sequence of shift operators with finite multiplicities.

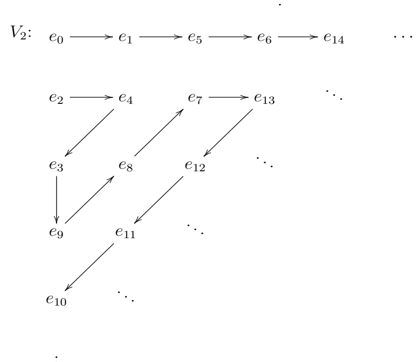
Proposition 2.1. *Let S be a shift operator on $\mathcal{H}$ with infinite multiplicity. Then there are shift operators S_n with multiplicity n, converging to S in strong* topology.*

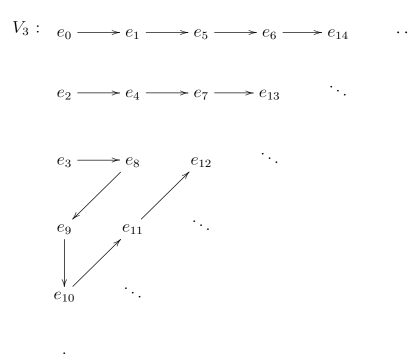
Proof. Define the shift operator V by



Then V is a shift operator with infinite multiplicity, indeed the first column of the above diagram is the basis of $\mathcal{K} = \ker V^*$ and $\bigcap_{n=0}^{\infty} V^n \mathcal{K} = \{0\}$. Define the shift operators V_n by







Each V_n is a shift operator with finite multiplicity n. To show that $\{V_n\}_{n=1}^{\infty}$ converges strongly to V, consider $f = \sum_{i=0}^{\infty} \lambda_i e_i$ in $\mathcal{H}$ and $\varepsilon > 0$, then there is a positive integer N such that $\sum_{i=n+1}^{\infty} |\lambda_i|^2 < \frac{\varepsilon}{2}$, for $n \geq N$, that is,

$$\|V_n f - Vf\| = \|(V_n - V)(\sum_{i=n+1}^{\infty} \lambda_i e_i)\| < \varepsilon.$$

Now if S is any shift operator (with infinite multiplicity), $S = a^*Va$, for some unitary operator a. Therefore, $\{a^*V_n a\}_{n=1}^{\infty}$ converges strongly to S, and each $S_n = a^*V_n a$ is a shift operator of finite multiplicity. Similarly, $\{S_n^*\}_{n=1}^{\infty}$ converges strongly to S^*. □

Throughout the rest of the paper, S is a shift operator on $\mathcal{H}$. Let $\mathcal{K}$ be the kernel of S^* and $P_0 = I - S^*S$ be the projection onto $\mathcal{K}$. Every operator $A \in B(\mathcal{H})$ has a matrix representation on $\mathcal{K}$, namely, $A \sim [A_{ij}]_{i,j=0}^{\infty}$, where $A_{ij} = P_0 S^{*i} A S^j P_0$, for $i, j \geq 0$ [7]. An operator $T \in B(\mathcal{H})$ is S-Toeplitz if $S^*TS = T$. By Theorem C in section 3.2 of [7], $T \in B(\mathcal{H})$ is S-Toeplitz if and only if its matrix representation has the following form

$$[T_{ij}]_{i,j=0}^{\infty} = [T_{i-j}]_{i,j=0}^{\infty} = \begin{bmatrix} T_0 & T_{-1} & T_{-2} & \cdots \\ T_1 & T_0 & T_{-1} & \cdots \\ T_2 & T_1 & T_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1}$$

where

$$T_j = \begin{cases} P_0 S^{*j} T P_0|_{\mathcal{K}}, & j \geq 0 \\ P_0 T S^{|j|} P_0|_{\mathcal{K}}, & j < 0. \end{cases}$$

A matrix of the form (1) is called a S-Toeplitz matrix. Then the transpose T^t of T is S-Toeplitz with matrix representation

$$[T^t_{ij}]_{i,j=0}^{\infty} = \begin{bmatrix} T_0 & T_1 & T_2 & \cdots \\ T_{-1} & T_0 & T_1 & \cdots \\ T_{-2} & T_{-1} & T_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The next result extends the well known fact that non-zero Toeplitz operators are never compact.

Proposition 2.2. *The non-zero S-Toeplitz operators are not compact.*

Proof. If K is compact and $S^{*n}KS^n = K$, $n = 1, 2, 3, \ldots$, then for an arbitrary vector v of the form $v = S^j x$, where j is a non-negative integer and x is in $\mathcal{K}$ =kerS^*, one can see that $Kv = 0$ and hence $K = 0$, since the vectors of that form span the whole space. To see $Kv = 0$, note that the sequence $\{S^n v\}$ tends weakly to zero, since, by definition, $\{S^{*n}\}$ converges strongly to 0. Hence $\{KS^n v\}$ is norm convergent to 0, for which reason, one can write

$$\|Kv\| = \|S^{*n}KS^nV\| \leq \|KS^nv\| \to 0$$

and the proof is over. □

An operator $H \in B(\mathcal{H})$ is called S-Hankel if $S^*H = HS$. In this case, $S^{*k}H = HS^k$, for each positive integer k. The matrix representation of S-Hankel operators are as follows.

Proposition 2.3. *An operator $H \in B(\mathcal{H})$ is S-Hankel if and only if*

$$[H_{ij}]_{i,j=0}^{\infty} = [H_{-(i+j+1)}]_{i,j=0}^{\infty} = \begin{bmatrix} H_{-1} & H_{-2} & H_{-3} & \cdots \\ H_{-2} & H_{-3} & & \cdots \\ H_{-3} & & \ddots & \\ \vdots & \vdots & & \end{bmatrix}, \tag{2}$$

where $H_l = P_0HS^{-(l+1)}P_0|_{\mathcal{K}}$ for $l < 0$.

Proof. Let H be a Hankel operator and $H \sim [H_{ij}]_{i,j=0}^{\infty}$. Then

$$H_{ij} = P_0S^{*i}HS^jP_0 = P_0HS^iS^jP_0 = P_0HS^{i+j}P_0$$

and we may put $H_{-(i+j+1)} = H_{ij}$. Conversely, let the matrix representation of $H \in B(\mathcal{H})$ is of the form (2) and let $[A_{ij}]$ and $[B_{ij}]$ be the matrices of S^*H and HS. Then

$$A_{ij} = P_0S^{*i}S^*HS^jP_0 = P_0S^{*(i+1)}HS^jP_0 = H_{-(i+j+2)},$$

and

$$B_{ij} = P_0S^{*i}HSS^jP_0 = P_0S^{*i}HS^{j+1}P_0 = H_{-(i+j+2)}.$$

Hence $A_{ij} = B_{ij}$, that is, $S^*H = SH$. □

A matrix of the form (2) is called a S-Hankel matrix. Define the operators J_n on $\mathcal{K} \oplus S\mathcal{K} \oplus \cdots \oplus S^n\mathcal{K}$ by $J_n(S^m\zeta_i) = S^{n-m}\zeta_i$, for $0 \leq m \leq n$ and $i \in \Lambda$. Then J_n extends by zero on $\mathcal{H}$, and is called the S-permutation operator of order n. A simple computation shows that $J_n^* = J_n$, $J_n^2 = P_n$, $\|J_n\| \leq 1$ and $J_n = J_nP_n = P_nJ_n$, where P_n is the projection onto $\mathcal{K} \oplus S\mathcal{K} \oplus \cdots \oplus S^n\mathcal{K}$.

Definition 2.1. An operator A is weak (strong, uniform) asymptotic S-Toeplitz (S-Hankel, respectively) if the sequence $\{S^{*n}AS^n\}$ (the sequence $\{J_nAS^{n+1}\}$, respectively) converges in the weak (strong, uniform) operator topology. Note that if $\{S^{*n}AS^n\}$ converges in any of these topologies, the limit is S-Toeplitz. A similar statement is true for S-Hankel operators.

Lemma 2.4. *If $A \in B(\mathcal{H})$ and the sequence $\{J_nAS^{n+1}\}$ converges weakly, then the limit is S-Hankle.*

Proof. If $\{J_nAS^{n+1}\}$ converges to B in weak operator topology then

$$\begin{aligned}\langle S^*BS^k\zeta_i, S^l\zeta_j\rangle &= \langle BS^k\zeta_i, S^{l+1}\zeta_j\rangle = \lim_n \langle J_nAS^{n+1}(S^k\zeta_i), S^{l+1}\zeta_j\rangle \\ &= \lim_n \langle AS^{n+k+1}\zeta_i, J_nS^{l+1}\zeta_j\rangle,\end{aligned}$$

for $k, l \geq 0$ and $i, j \in \Lambda$. By definition, for $n \geq l+1$, $J_nS^{l+1}\zeta_j = S^{n-(l+1)}\zeta_j$. Thus, after an appropriate relabeling,

$$\langle S^*BS^k\zeta_i, S^l\zeta_j\rangle = \lim_m \langle AS^m\zeta_i, S^{m-(k+l)-2}\zeta_j\rangle.$$

Similarly,

$$\langle BS(S^k\zeta_i), S^l\zeta_j\rangle = \lim_m \langle AS^m\zeta_i, S^{m-(k+l)-2}\zeta_j\rangle.$$

Therefore, $S^*B = BS$. □

Now let T be a S-Toeplitz operator with matrix representation (1). Then,

$$J_nTS^{n+1} = P_nH, \tag{3}$$

where H is a S-Hankel operator with matrix

$$\begin{bmatrix} T_{-1} & T_{-2} & T_{-3} & \cdots \\ T_{-2} & T_{-3} & & \cdots \\ T_{-3} & & \ddots & \\ \vdots & \vdots & & \end{bmatrix}.$$

In this case, we write $H = H(T)$. For each S-Hankel operator H, we have $H = H(T)$, for some S-Toeplitz operator T. In particular, by (3), each S-Toeplitz operator is a strongly (weakly) asymptotic S-Hankel operator. The norm closed algebra generated by all S-Teplitz and S-Hankel operators is called the S-Hankel algebra. We show the S-Hankel algebra is contained in both classes of the strongly asymptotic S-Toeplitz operators and strongly asymptotic S-Hankel operators. The next lemma is a direct consequence of definitions.

Lemma 2.5. *The set of strongly asymptotic S-Toeplitz operators is norm closed. The same is true for the class of strongly asymptotic S-Hankel operators.*

In general, the multiplication of two S-Toeplitz operators is not S-Toeplitz. However, we have the following useful formulas.

Lemma 2.6. (i) *If R and T are S-Toeplitz operators then $TR = A - H(T^t)H(R)$ and $H(T)R = B - T^tH(R)$, for a S-Toeplitz operator A and a S-Hankel operator B.*

(ii) *If $T_1, T_2, \cdots, T_n$ are S-Toeplitz operators then*

$$T_nT_{n-1}\cdots T_1 = T + A_1H_1 + \cdots + A_{n-1}H_{n-1},$$

where T is a S-Toeplitz operator, H_i's are S-Hanke and A_i's are in $B(\mathcal{H})$.

Proof. For (i), let the matrix representations of R and T be

$$R \sim \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, T \sim \begin{bmatrix} T_0 & T_{-1} & T_{-2} & \cdots \\ T_1 & T_0 & T_{-1} & \cdots \\ T_2 & T_1 & T_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

If $A = TR + H(T^t)H(R)$ then the matrix representation of A is of the form

$$[A_{ij}]_{i,j=0}^{\infty} = \begin{bmatrix} T_0 & T_{-1} & T_{-2} & \cdots \\ T_1 & T_0 & T_{-1} & \cdots \\ T_2 & T_1 & T_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$+ \begin{bmatrix} T_1 & T_2 & T_3 & \cdots \\ T_2 & T_3 & & \cdots \\ T_3 & & \ddots & \\ \vdots & \vdots & & \end{bmatrix} \begin{bmatrix} R_{-1} & R_{-2} & R_{-3} & \cdots \\ R_{-2} & R_{-3} & & \cdots \\ R_{-3} & & \ddots & \\ \vdots & \vdots & & \end{bmatrix},$$

where $A_{ij} = \sum_{k=0}^{\infty} T_{i-k}R_{k-j} + \sum_{k=0}^{\infty} T_{i+k+1}R_{-(j+k+1)}$. Thus

$$\begin{aligned} A_{(i+1)(j+1)} &= \sum_{k=0}^{\infty} T_{i-k+1}R_{k-j-1} + \sum_{k=0}^{\infty} T_{i+k+2}R_{-(j+k+2)} \\ &= \sum_{k=0}^{\infty} T_{i-k}R_{k-j} + (T_{i+1}R_{-(j+1)} + \sum_{k=0}^{\infty} T_{i+k+2}R_{-(j+k+2)}) \\ &= \sum_{k=0}^{\infty} T_{i-k}R_{k-j} + \sum_{k=0}^{\infty} T_{i+k+1}R_{-(j+k+1)} = A_{ij}. \end{aligned}$$

Thus A is S-Toeplitz. Let $B = T^t H(R) + H(T)R$, with matrix representation

$$[B_{ij}]_{i,j=0}^{\infty} = \begin{bmatrix} T_0 & T_1 & T_2 & \cdots \\ T_{-1} & T_0 & T_1 & \cdots \\ T_{-2} & T_{-1} & T_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} R_{-1} & R_{-2} & R_{-3} & \cdots \\ R_{-2} & R_{-3} & & \cdots \\ R_{-3} & & \ddots & \\ \vdots & \vdots & & \end{bmatrix}$$

$$+ \begin{bmatrix} T_{-1} & T_{-2} & T_{-3} & \cdots \\ T_{-2} & T_{-3} & & \cdots \\ T_{-3} & & \ddots & \\ \vdots & \vdots & & \end{bmatrix} \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $B_{ij}=\sum_{k=0}^{\infty}T_{-i+k}R_{-j+k-1}+\sum_{k=0}^{\infty}T_{-(i+k+1)}R_{-j+k}$. Then

$$\begin{aligned}B_{(i+1)(j-1)}&=\sum_{k=0}^{\infty}T_{-i+k-1}R_{-(j+k)}+\sum_{k=0}^{\infty}T_{-(i+k+2)}R_{-j+k+1}\\&=\sum_{k=0}^{\infty}T_{-i+k}R_{-j+k-1}+(T_{-(i+1)}R_{-j}+\sum_{k=0}^{\infty}T_{-(i+k+2)}R_{-j+k+1})\\&=\sum_{k=0}^{\infty}T_{-i+k}R_{-j+k-1}+\sum_{k=0}^{\infty}T_{-(i+k+1)}R_{-j+k}=B_{ij}.\end{aligned}$$

Then the matrix of B has constant anti-diagonals and B is S-Hankel.

Part (ii) is proved by induction. For $n=1$, the assertion is obvious. Assume that

$$T_{k-1}T_{k-2}\cdots T_1=T+A_1H_1+\cdots+A_{k-2}H_{k-2},$$

with T S-Toeplitz and H_i's S-Hankel, then, by (i), $T_kT=T^{'}+BH^{'}$, for some B, S-Toeplitz operator $T^{'}$ and S-Hankel operator $H^{'}$. Therefore

$$\begin{aligned}T_kT_{k-1}\cdots T_1&=T_kT+T_kA_1H_1+\cdots+T_kA_{k-2}H_{k-2}\\&=T^{'}+BH^{'}+T_kA_1H_1+\cdots+T_kA_{k-2}H_{k-2},\end{aligned}$$

as required. □

Theorem 2.7. *The S-Hankel algebra is contained in the class of strongly asymptotic S-Toplitz operators.*

Proof. Let H and B be bounded operators with H S-Hankel. Then BH is strongly asymptotic S-Toeplitz, since $S^{*n}BHS^n=S^{*n}BS^{*n}H\to 0$ in strong operator topology. Therefore, a multiplication of finitely many S-Toeplitz and S-Hankel operators is strongly asymptotic S-Toeplitz, by Lemma 2.6. The result follows now from Lemma 2.5. □

Theorem 2.8. *Every element of the S-Hankel algebra is a strongly asymptotic S-Hankel operator.*

Proof. If H is S-Hankel, each operator of the form BH is strongly asymptotic S-Hankel, since

$$J_nBHS^{n+1}=J_nBS^{*n+1}H\to 0,$$

in the strong operator topology. Also every S-Toeplitz operator is strongly asymptotic S-Hankel. Now, as in the proof of Theorem 2.7, the result follows from Lemmas 2.5 and 2.6. □

In [3] uniformly asymptotic Toeplitz operators are characterized as operators of the form $T+K$, where T is Toeplitz and K is compact. If S is a shift operator with finite multiplicity, Matache in [5] uses the same characterization for uniformly asymptotic S-Toeplitz operators. Here we drop the assumption on multiplicity.

Proposition 2.9. *Every uniformly asymptotic S-Toeplitz operator A is of the form $A = T + C$, where T is S-Toeplitz and C is an operator such that $\|(I - P_n)C(I - P_n)\| \to 0$.*

Proof. If T is S-Toeplitz, since $S^{*n}AS^n - T = S^{*n}(A - T)S^n$, the sequence $\{S^{*n}AS^n\}$ converges uniformly to T if and only if $\|S^{*n}(A-T)S^n\| \to 0$. The matrix representation of $S^{*n}(A - T)S^n$ is obtained from that of $A - T$ by deleting the n first block-rows and columns. Similarly the matrix of $(I - P_n)(A - T)(I - P_n)$ is obtained from that of $A-T$ by replacing the n first block-rows and columns by zero. Therefore, for each n, the operators $S^{*n}(A-T)S^n$ and $(I-P_n)(A-T)(I-P_n)$ have the same norm, and the result follows. □

If the multiplicity of S is finite, as Matache shows in [5], each uniformly asymptotic S-Toeplitz operator A is of the form $A = T + K$ such that T is S-Toeplitz and K is compact. Indeed, in this case, P_n's are finite rank projections and

$$(I - P_n)(A - T)(I - P_n) = A - T - F_n$$

for some finite rank operators F_n. By the previous Proposition, $\{S^{*n}AS^n\}$ converges uniformly to T if and only if $\{F_n\}$ converges to the compact operator $A - T$. The converse follows from the fact that compact operators are uniformly asymptotic S-Toeplitz. Next we characterize those S-Toeplitz operators which are uniformly asymptotic S-Hankel.

Lemma 2.10. *Let T be S-Toeplitz and $H = H(T)$. Then T is uniformly asymptotic S-Hankel if and only if the sequence $\{P_n(H)\}$ converges uniformly to H. When the multiplicity S is finite, T is uniformly asymptotic S-Hankel if and only if H is compact.*

Proof. The first part follows from (3). If the multiplicity of S is finite, the sequence $\{P_nH\}$ converges uniformly to H if and only if H is compact. □

For bounded operators A and B,

$$\|J_nP_nA\|^2 = \|A^*P_nJ_nJ_nP_nA\| = \|A^*P_nP_nA\| = \|P_nA\|^2,$$

and since $(I - P_n)S^{n+1} = S^{n+1}$,

$$\|J_nBS^{n+1}\| = \|J_nP_nB(I - P_n)S^{n+1}\| = \|P_nB(I - P_n)\|. \tag{4}$$

Therefore, by Lemma 2.10, uniformly asymptotic S-Toeplitz operators are not necessarily uniformly asymptotic S-Hankel. If the multiplicity of S is finite and A is uniformly asymptotic S-Toeplitz, then $A = T + K$, for some S-Teplitz operator T and compact operator K. Let A be uniformly asymptotic S-Hankel. By (4), $\|J_nKS^{n+1}\| = \|P_nK(I - P_n)\| \to 0$, hence the S-Toeplitz operator T must be uniformly asymptotic S-Hankel. By Lemma 2.10, $H(T)$ is compact.

Let $\mathcal{K} = \ker S^*$ and P_n be the finite rank projection onto $\mathcal{K} \oplus S\mathcal{K} \oplus \cdots \oplus S^n\mathcal{K}$. An operator $A \in B(\mathcal{H})$ is called quasi-triangular (relative to the sequence $\{P_n\}$) if $\|P_nA(I - P_n)\| \to 0$. The algebra of all quasi-triangular operators is a Banach algebra [1]. If alg$\{P_n\}$ consists of all operators $A \in B(\mathcal{H})$ such that $P_nA(I-P_n) = 0$, for each n, then alg$\{P_n\} + K(\mathcal{H})$ is the same as the algebra of all quasi-triangular operators [1]. The (weakly closed) algebra alg$\{P_n\}$ contains of all operators with block-lower triangular matrix representation, and it is a nest algebra. When S has finite multiplicity, uniformly asymptotic S-Hankel operators are characterized as follows.

Theorem 2.11. *Let S has finite multiplicity. Then an operator $A \in B(\mathcal{H})$ is uniformly asymptotic S-Hankel if and only if $A = T + R$, where T is a S-Toeplitz operator with $H(T)$ compact, and $R \in alg\{P_n\} + K(\mathcal{H})$.*

Proof. The S-permutations J_n are of finite rank, and if $\{J_nAS^{n+1}\}$ converges uniformly to an operator H, then H is compact. Moreover, by Lemma 2.4, H is S-Hankel with matrix representation

$$\begin{bmatrix} H_{-1} & H_{-2} & H_{-3} & \cdots \\ H_{-2} & H_{-3} & & \cdots \\ H_{-3} & & \ddots & \\ \vdots & \vdots & & \end{bmatrix}.$$

Let T be the S-Toeplitz operator with matrix representation

$$\begin{bmatrix} 0 & H_{-1} & H_{-2} & \cdots \\ 0 & 0 & H_{-1} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $J_nTS^{n+1} = P_nH$, and since H is compact, $\{J_nTS^{n+1}\}$ converges uniformly to H. Therefore, $\|J_n(A - T)S^{n+1}\| \to 0$, since $\|J_n(A - T)S^{n+1}\| \leq \|J_nAS^{n+1} - H\| + \|J_nTS^{n+1} - H\|$. Hence by (4), $A - T \in$ alg$\{P_n\} + K(\mathcal{H})$. Conversely, if $A = T + R$ with T S-Toeplitz and $H(T)$ compact, and $R \in$ alg$\{P_n\} + K(\mathcal{H})$, then $\{J_nTS^{n+1} = P_nH\}$ converges uniformly to H, $A - T \in$ alg$\{P_n\} + K(\mathcal{H})$, and $\|J_nAS^{n+1} - H\| \leq \|J_n(A - T)S^{n+1}\| + \|J_nTS^{n+1} - H\|$. Therefore A is uniformly asymptotic S-Hankel. □

The next result characterizes The block-matrix of weakly asymptotic S-Hankel and S-Toeplitz operators.

Theorem 2.12. *Let $\left[\left[t_{ij}^{k,l}\right]_{i,j\in\Lambda}\right]_{k,l=0}^{\infty}$ be the block-matrix representation of operator T with respect to the basis $\{S^n\zeta_i : n \geq 0, i \in \Lambda\}$ of $\mathcal{H}$. Then*

(i) *T is weakly asymptotic S-Hankel if and only if for $i, j \in \Lambda$ and $p \geq 1$ the sequence $\{t_{ij}^{m,m+p}\}_{m=0}^{\infty}$ converges to some complex number t_{ij}^{-p}. In this case, $\left[\left[t_{ij}^{-(k+l+1)}\right]_{i,j\in\Lambda}\right]_{k,l=0}^{\infty}$ is a S-Hankel block-matrix.*

(ii) *T is weakly asymptotic S-Toeplitz if and only if for each integer number p the sequence $\{t_{ij}^{m,m+p}\}_{m=0}^{\infty}$ converges to some t_{ij}^{-p}. In this case, $\left[\left[t_{ij}^{k-l}\right]_{i,j\in\Lambda}\right]_{k,l=0}^{\infty}$ is a S-Toeplitz block-matrix.*

Proof. For (i), let T be a weakly asymptotic S-Hankel operator. Since

$$\begin{aligned}\langle J_n T S^{n+1}(S^l\zeta_j), S^k\zeta_i\rangle &= \begin{cases} \langle T S^{n+l+1}\zeta_j, S^{n-k}\zeta_i\rangle & , n \geq k \\ 0 & , n < k \end{cases} \\ &= \begin{cases} t_{ij}^{n-k,n+l+1} & , n \geq k \\ 0 & , n < k \end{cases}\end{aligned}$$

by the change of indices $n - k = m$ and $k + l + 1 = p$, the sequence $\{t_{ij}^{m,m+p}\}_{m=0}^{\infty}$ converges to some complex number t_{ij}^{-p} for $p \geq 1$ and $i, j \in \Lambda$.

For (ii), by assumption,

$$\langle S^{*n} T S^n(S^l\zeta_j), S^k\zeta_i\rangle = \langle T S^{n+l}\zeta_j, S^{n+k}\zeta_i\rangle = t_{ij}^{n+k,n+l}.$$

and by the change indices $n + k = m$ and $l - k = p$, if T is a weakly asymptotic S-Toeplitz, the sequence $\{t_{ij}^{m,m+p}\}_{m=0}^{\infty}$ converges to some complex number t_{ij}^{-p}, for each p and $i, j \in \Lambda$. □

Corollary 2.13. *Every weakly asymptotic S-Toeplitz operator is weakly asymptotic S-Hankel.*

Acknowledgment

The authors would like to thank the referees and Editor in Chief for their comments which improved the paper.

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Received : *February* 2023
Accepted : *April* 2023